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A THEORY OF BELIEF FOR SCIENTIFIC REFUTATIONS

ABSTRACT. A probability function $\mathbb{P}$ on an algebra of events is assumed. Some of the events are scientific refutations in the sense that the assumption of their occurrence leads to a contradiction. It is shown that the scientific refutations form a a boolean sublattice in terms of the subset ordering. In general, the restriction of $\mathbb{P}$ to the sublattice is not a probability function on the sublattice. It does, however, have many interesting properties. In particular, (*i*) it captures probabilistic ideas inherent in some legal procedures; and (*ii*) it is used to argue against the commonly held view that behavioral violations of certain basic conditions for qualitative probability are indicative of irrationality. Also discussed are (*iii*) the relationship between the formal development of scientific refutations presented here and intuitionistic logic, and (*iv*) an interpretation of a belief function used in the behavioral sciences to explain empirical results about subjective, probabilistic estimation, including the Ellsberg paradox.

1. INTRODUCTION

Since the 1930s, the probability calculus of Kolmogorov (1933) has become the standard theory of probability for mathematics and physics. This article proposes an alternative calculus for empirical science. The alternative is consistent with the standard in the sense that it assigns the same probabilities to scientific events as a probability function from the Kolmogorov theory. It, however, has a different event space and employs a different logic to relate events. The difference results from platonic nature of the event space and logic of the standard, which fails account for characterizing features of scientific events, like their verifiability and refutability. Such features are accounted for in the alternative, which is based on a non-classical logic of events that intrinsically models key properties of verifiability and refutability.

An analogous situation occurred in mathematics. Once, euclidean geometry was completely dominate. Manifolds were described and worked out in an euclidean framework. Later, it was found to be far more insightful and productive to work completely "inside" the manifold, exploiting its intrinsic characteristics. Thus, for example, it became more productive for mathematicians to describe the geometry of a torus (the surface of a doughnut shaped object) intrinsically in terms of its 2-dimensional

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geometrical properties than extrinsically as a 2-dimensional surface in a 3-dimensional euclidean space. In the present case, the Kolmogorov theory is dominate. The scientific events, i.e., events of the form, "*A* occurs and is (scientifically) verifiable," or of the form, "*A* occurs and is (scientifically) refutable," of a targeted fragment of science can be viewed as a portion of a larger set of platonic events. They only form a portion, because the complement (or negation) of a scientific event need not be a scientific event; that is, the complement of a verifiable event need be neither verifiable nor refutable, and similarly, the complement of a disjunction of refutable events need be neither verifiable nor refutable. In short, the complementation operator of a boolean algebra of events does not in general preserve scientific events. In this sense, the Kolmogorov theory adds extrinsic structure to the scientific events through its use of (boolean) complementation. This article shows that the scientific events themselves have considerable structure. This *intrinsic structure* forms the basis of an alternative probabilistic theory that is applied to various examples, including American law and issues involving the behavioral characterization of rational behavior.

1.1. *Probability Functions*

Since the 1930s, probability has been defined in accordance of Kolmogorov (1933) as a finite or σ-additive measure on a boolean algebra of sets. Although σ-additivity adds important mathematical structure, the generalizations of probability theory discussed in this article do not employ the additional structure, and thus only finitely additive versions are presented. Also, to simplify notation and definitions, nonempty sets of measure 0 are excluded. If needed, the development could be extended to include the σ-additive case and nonempty sets of measure 0.

One productive way for understanding the relationship of standard probability theory to its generalizations and extensions is through lattice theory. Boolean algebras of sets form a special kind of lattice, called *boolean lattices.* The concepts of lattice and boolean lattice are defined as follows:

DEFINITION 1. $\preceq$ is said to be *partial ordering on A* if and only if A is a nonempty set and the following three conditions hold for each a, b, and c in A:

(i) $a \preceq a$;
(ii) if $a \preceq b$ and $b \preceq a$, then $a = b$; and
(iii) if $a \preceq b$ and $b \preceq c$, then $a \preceq c$. □

DEFINITION 2. $\langle A, \preceq, u, z\rangle$ is said to be a *lattice (with unit element u and zero element z)* if and only if

(i) $\preceq$ is a partial ordering on A;
(ii) for each a and b in A, there exists a unique c in A, called the *join* of a and b and denoted by $a \sqcup b$, such that $a \preceq a \sqcup b$, $b \preceq a \sqcup b$, and for all d in A,

$$\text{if } a \preceq d \text{ and } b \preceq d\,, \text{ then } a \sqcup b \preceq d;$$

(iii) for each a and b in A, there exists a unique element c in A, called the *meet* of a and b and denoted by $a \sqcap b$, such that $a \sqcap b \preceq a$, $a \sqcap b \preceq b$, and for all d in A,

$$\text{if } d \preceq a \text{ and } d \preceq b\,, \text{ then } d \preceq a \sqcap b;$$

and
(iv) for all a in A, $z \preceq a$ and $a \preceq u$. □

Let $\langle A, \preceq, u, z\rangle$ be a lattice. Then it easily follows that $\sqcup$ and $\sqcap$ are commutative and associative operations on A.

CONVENTION 1. A lattice $\langle A, \preceq, u, z\rangle$ is often written as

$$\langle A, \preceq, \sqcup, \sqcap, u, z\rangle\,. \quad \square$$

DEFINITION 3. Let $\mathfrak{A} = \langle A, \preceq, \sqcup, \sqcap, u, z\rangle$ be a lattice.

$\mathfrak{A}$ is said to be *distributive* if and only if for all a, b, and c in A,

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)\,.$$

$\mathfrak{A}$ is said to be *complemented* if and only if for each a in A, there exists an element b in A, called a *complement of a (in $\mathfrak{A}$),* such that $a \sqcup b = u$ and $a \sqcap b = z$.

$\mathfrak{A}$ is said to be *boolean* if and only if it is complemented and distributive.

Suppose $\mathfrak{A}$ is boolean. Then it is easy to show that the complement of each element of A is unique. Thus for each a in A, let $-a$ denote the complement of a (in $\mathfrak{A}$). Then it follows that $-$ is a well-defined operation on A.

$\mathfrak{A}$ is said to be *a lattice of sets (or a lattice of events)* if and only if $\preceq$ is $\subseteq$, $\sqcup$ is $\cup$, $\sqcap$ is $\cap$, u is a nonempty set, z is $\varnothing$, and each element of A is a subset of u. □

CONVENTION 2. A boolean lattice $\mathfrak{A} = \langle A, \preceq, \sqcup, \sqcap, u, z\rangle$ is often written as

$$\langle A, \preceq, \sqcup, \sqcap, -, u, z\rangle,$$

where $-$ is the complement operator of $\mathfrak{A}$.

Often when describing two or more boolean lattices, the same symbol, $-$ is used to denote the two (and, in general, different) complement operations of the two lattices. $\square$

It is easy to verify that boolean algebras of events are examples of a boolean lattice. The following is another well-known example.

EXAMPLE 1 (Boolean Lattice of Propositions). The propositional calculus from logic is an important example of a boolean lattice. Let $\rightarrow$, $\leftrightarrow$, $\vee$, $\wedge$, and neg stand for, respectively, the logical connectives of implication ("if ... then"), logical equivalence, ("if and only if"), disjunction ("or"), conjunction, ("and"), and negation ("not"). Consider two propositions α and β to be equivalent if and only if $\alpha \leftrightarrow \beta$ is a tautology. Equivalent propositions partition the the set of propositions into equivalence classes. Let A be the set of equivalence classes. The equivalence class containing a tautology is denoted by u, and the equivalence class containing a contradiction is denoted by z. By definition, equivalence class $a \leq$ equivalence class b if and only if for some elements α in a and β in b, $\alpha \rightarrow \beta$ is a tautology. It is easy to check that $\leq$ is a partial ordering on A.

Let a and b be arbitrary elements of A, α and β be respectively arbitrary elements of a and b, j, be the equivalence class of $\alpha \vee \beta$, m be the equivalence class of $\alpha \wedge \beta$, and n be the equivalence class of neg α. Then it is easy to check that j is the join of a and b, m is the meet of a and b, n is the complement of a, and

$$\mathfrak{A} = \langle A, \preceq, \sqcup, \sqcap, -, u, z \rangle ,$$

is a boolean lattice. $\square$

Although the assignments of probabilities to propositions are often the items of interest, they are usually modeled as assignments of probabilities to events. This is partially justified by the following theorem of Stone (1936).

THEOREM 1 (Stone's Representation Theorem). Let

$$\mathfrak{A} = \langle A, \preceq, \sqcup, \sqcap, -, u, z \rangle$$

be a boolean lattice. Then there exists a boolean algebra of events $\mathfrak{B} = \langle \mathscr{B}, \subseteq, \cup, \cap, -, X, \varnothing \rangle$ and an isomorphism φ of the lattice $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\varphi(u) = X$ and $\varphi(z) = \varnothing$. $\square$

Probability functions have the formal properties of a finitely additive measure. The finite additivity comes form the condition,

(1) For all a and b in A, if $a \sqcap b = z$, then $\mathbb{P}(a \sqcup b) = \mathbb{P}(a) + \mathbb{P}(b)$.

For boolean lattices, Equation (1) is equivalent to

(2) For all a and b in A, $\mathbb{P}(\mathrm{a}) + \mathbb{P}(\mathrm{b}) = \mathbb{P}(a \sqcup b) + \mathbb{P}(a \sqcap b)$.

Obviously Equation (2) implies Equation (1). The equivalence of Equations (1) and (2) uses the existence of complements. Both Equations (1) and (2) can be used for formulating generalizations of probability functions that apply to lattices more general than the boolean ones. Because Equation (2) retains more structure of probability functions, it is used in the generalizations presented in this article.

It is well-known that lattices that have probability functions that satisfy Equation (2) must satisfy the following condition:

Modularity: for all a, b, and c in A, if $a \leq b$, then $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

Modularity is a generalization of distributivity. Birkhoff and von Neumann (1936) use non-distributive modular lattices to describe the propositional logic of quantum mechanics. This article considers only the impact of Equation (2) style probability functions on distributive lattices that generalizes boolean lattices in a particular way.

Textbook probability theory defines a and b to be independent, $a \perp b$, if and only if

(3) $\mathbb{P}(a \cap b) = \mathbb{P}(a)\mathbb{P}(b)$.

This definition of "independence" strikes me and others as rather odd: (1) In practical applications one usually *knows and uses* certain events as being "independent" as part of the process of establishing probabilities. In fact, frequency theories of probability (e.g., von Mises 1936) employ a non-numerical concept of independence at the foundational level to guarantee that the relative frequencies have the desired properties. (2) There are many situations in which there is not enough information to uniquely determine a probability function, that is, there are situations in which different probability functions describe the known facts and hypotheses. In such situations, Equation (3) is of little use for deciding independence. And (3), in situations where there is unique probability function, events a and b may be "independent" for only the accidental reason that Equation (3) holds, rather than for some principled reason involving that natures of a and b. In this article, we will assume, as is customary in some foundational theories of probability, that $\perp$ is taken as a primitive concept and require,

$$\text{if } a \perp b, \text{ then } \mathbb{P}(a)\mathbb{P}(b)\,.$$

Some of the above considerations about probability functions are summarized in the following definition.

DEFINITION 4. $\mathbb{P}$ is said to be a *probability function on the distributive lattice* $\langle A, \preceq, \sqcup, \sqcap, u, z\rangle$ *with independence relation* $\perp$ if and only if the following five conditions hold for all a and b in A:

1. $\mathbb{P}$ is a function from A into the closed interval of the reals [0, 1].
2. $\mathbb{P}(u) = 1$ and $\mathbb{P}(z) = 0$.
3. If $a \subseteq b$, then $\mathbb{P}(a) \leq \mathbb{P}(b)$.
4. $\mathbb{P}(a) + \mathbb{P}(b) = \mathbb{P}(a \sqcup b) + \mathbb{P}(a \sqcap b)$.
5. If $a \perp b$, then $\mathbb{P}(a \sqcap b) = \mathbb{P}(a)\mathbb{P}(b)$. □

1.2. *Belief Functions*

Many researchers consider traditional probability theory to be the only rational approach to uncertainty; and many others disagree, with some proposing alternative theories that they consider to be rational. The alternative theories, leaving aside the issue of whether they are truly "rational," generally suffer the deficiency of lacking the mathematical power of probability theory. Narens (2003a), however, provides a theory that generalizes probability theory and has mathematical and calculative power at the same level of finitely additive probability theory. He arrives at this theory by first providing a qualitative axiomatization of a version of conditional probability. This is done in such a way that one of the axioms captures consequences of conditional probability that some theorists believe to be invalid for a general, rational theory of belief. This axiom is then eliminated and the most general quantitative model corresponding to the remaining axioms is found. The result is a belief function $\mathbb{B}(A|B)$ having the form,

$$(4) \qquad \mathbb{B}(A|B) = \mathbb{P}(A|B)\upsilon(A)\,,$$

where $\mathbb{P}$ is a uniquely determined conditional probability function and υ is a function into $\mathbb{R}^+$ that is unique up to multiplication by a positive real. He then uses this result to give alternative accounts of various findings about psychological judgments of probability.

The axiom of qualitative conditional probability that Narens (2003a) eliminates is called *Binary Symmetry.* It is formulated as follows, where "$\sim$" stands for "has the same degree of belief" and "$(E|F)$" for the conditional event of "E occurring given F has occurred:" Suppose A, B, C, and D are mutually disjoint, nonempty events and

(5) $(A|A \cup B) \sim (B|A \cup B)$ and $(C|C \cup D) \sim (D|C \cup D)$.

Then

(6) $(A|A \cup B) \sim (C|C \cup D)$

and

(7) $(A|A \cup C) \sim (B|B \cup D)$.

Note that if degrees of belief are measured by a probability function P and Equation (5) holds, then $P(A) = P(B)$ and $P(C) = P(D)$, from which Equations (6) and (7) follow.

Narens (2003(a)) gives the following rationale for the desirability of eliminating the axiom of Binary Symmetry:

> In the context of the other axioms for conditional probability, Binary Symmetry asserts that if A and B have equal likelihood of occurring and C and D have equal likelihood of occurring, then the conditional probabilities of $(A|A \cup B)$ and $(C|C \cup D)$ are $\frac{1}{2}$, and the conditional probabilities of $(A|A \cup C)$ and $(B|B \cup D)$ are the same.
>
> Suppose Equation (5) and the judgment of equal likelihood of the occurrences of A and B, given either A or B occurs, is based on much information about A and B and a good understanding of the nature of the uncertainty involved, and the judgment of equal likelihood of the occurrences of C and D, given either C or D occurs, is due to the lack of knowledge of C and D, for example, due to complete ignorance of C and D. Then, because of the differences in the understanding of the nature of the probabilities involve, a lower degree of belief may be assigned to $(C|C \cup D)$ than to $(A|A \cup B)$, thus invalidating Equation (6).
>
> Suppose Equation (5) and the judgments of the likelihoods of the occurrences of A and C, given either A or C occurs, are based on much information about A and C and the nature of the uncertainty involved, and the judgment the likelihoods of the occurrences of B and D given either B or D occurs is due to the lack of knowledge of B and D. This may result in different degrees of belief being assigned to $(A|A \cup C)$ and $(B|B \cup D)$, and such an assignment would invalidate Equation (7).

For events A and B, let $A \succ B$ stand for "A is more likely than B." The relation $\succ$ can be observed in several ways, for example by asking subject directly, "Is A more likely than B," or by asking the subject, "Would you prefer the gamble of receiving \$100 if A occurs and receiving \$0 if A does not occur to the gamble of receiving \$100 if B occurs and receiving \$0 if B does not occur?" The following has been consistently assumed to be a basic rule of rational probabilistic behavior:

DEFINITION 5 (de Finetti's Axiom). If $A \succ B$ and $C \cap A = C \cap B = \varnothing$, then $A \cup C \succ B \cup C$.

Note that de Finetti's axiom is the qualitative version of,

$$\text{if } \mathbb{P}(A) > \mathbb{P}(B) \text{ and } C \cap A = C \cap B = \varnothing, \quad \text{then } \mathbb{P}(A \cup C) > \mathbb{P}(B \cup C)\,.$$

Ellsberg (1961) presented the following example of human behavior that contradicts de Finetti's Axiom:

EXAMPLE 2 (Ellsberg's Paradox). Suppose an urn has 90 balls that have been throughly mixed. Each ball is of one of the three colors, red, blue, or yellow. There are thirty red balls, but the number of blue balls and the number or yellow balls are unknown except that together they total 60. A ball is to be randomly chosen from the urn, and the subject will earn \$100 if he or she correctly guesses the color of the ball. Let R be the event that a red ball is chosen, B the event a blue ball is chosen, and Y the event a yellow ball is chosen. Let $U = R \cup B \cup Y$. Typically, subjects' behavior indicate,

$$\mathbb{B}(R|U) > \mathbb{B}(B|U) \text{ and } \mathbb{B}(R \cup Y|U) < \mathbb{B}(B \cup Y)|U)\,,$$

violating de Finetti's Axiom. The usual reason given for this result is that subjects perceive B as more "ambiguous" than R and $R \cup Y$ as more "ambiguous" than $B \cup Y$ and ambiguity is perceived to to increase of the riskiness of the choice. □

Ellsberg's Paradox is consistent with the belief function $\mathbb{B}$ above by the assignments to Equation (4),

$$\mathbb{P}(R|U) = \mathbb{P}(B|U)\,,\ v(R) > v(B)\,,$$
$$\mathbb{P}(R \cup Y|U) = \mathbb{P}(B \cup Y)\,, \text{ and } v(R \cup Y) < v(B \cup Y)\,.$$

Ellsberg and others have argued that probability theory is too narrow for a general theory of rational belief, with Ellsberg's Paradox being a demonstration of its limitation. If one assumes this, then the belief function $\mathbb{B}$ in Equation (4) has a normative interpretation. However, Ellsberg's Paradox is very controversial as an example of rational behavior.

One of goals of this article is to provide a non-controversial interpretation of $\mathbb{B}$ as a rational assignment of degrees of belief. This will be done by considering a lattice of events where each event can either be verified scientifically or refuted scientifically. Such a lattice will be shown to be distributive. The scientific refutations will be shown to be a boolean sublattice of it. A rational assignment of degrees of belief is given by a probability function $\mathbb{P}$ on the scientific events. When $\mathbb{P}$ is restricted to the sublattice of scientific refutations, it can be reinterpreted as a belief function $\mathbb{B}$ on the scientific refutations that satisfies Equation (4). Because $\mathbb{P}$ was chosen as a rational assignment of degrees of belief, it then follows that its restriction to the sublattice of scientific refutations, reinterpreted as $\mathbb{B}$, must also be a rational assignment.

Other issues of rationally are also considered in the article, including what sorts of behavior may be appropriately labeled "irrational."

2. LATTICES OF SCIENTIFIC EVENTS

2.1. *Basic Concepts*

In this section we start with a lattice of events $\mathfrak{S} = \langle \mathcal{S}, \subseteq, \cup, \cap, -, U, \varnothing \rangle$. In the intended interpretation, $\mathcal{S}$ is a set of events about a phenomenon under investigation. Other events are generated from $\mathcal{S}$. Some of these are through a new unary operation, $\neg$. $\neg$ applies to events to produce events. It may produce new events; that is, when applied to events in $\mathcal{S}$, it may produce an event not in $\mathcal{S}$. It also may apply to events not in $\mathcal{S}$, for example, if Q is in $\mathcal{S}$ and $\neg\ Q$ is not in $\mathcal{S}$, then $\neg$ may be applied to $\neg\ Q$ to produce $\neg\ (\neg\ Q)$. In the intended interpretation, some of the new events may be interpretable about the *phenomenon* under consideration, i.e., as subsets of U that are not in $\mathcal{S}$, while others may be about the *science of the phenomenon* under consideration, and and as such cannot be interpreted as subsets of U. The new operation is called *refutation.*

DEFINITION 6. Let Q be an event. Then the following definition holds: $\neg\ Q$ is the event that the assumption of the occurrence of the event Q leads to a contradiction. "$\neg\ Q$" is read as, "The event that the event Q has been refuted." □

Obviously Definition 6 is incomplete, because "leads to a contradiction" has not been specified. More about this will be said later.

CONVENTION 3. As usual, $\neg\ \neg\ A$ stands for $\neg\ (\neg\ A)$, $\neg\ A \cup B$ for $(\neg\ A) \cup B$, et cetera. □

$\neg$ should not confused with the boolean operation of complementation $-$. In particular, the Law of the Excluded Middle holds for $-$, i.e., $A \cup - A$ is always the sure event; however, $A \cup \neg\ A$ need not be the sure event. In this respect, $-$ behaves like negation in classical logic, whereas $\neg$ behaves like negation in intuitionistic logic (see Section 2.2).

Methods that produce scientific contradictions are many and varied. For example, an event Q may be refuted through verification of an event T such that $Q \cap T = \varnothing$. Or the assumption of the occurrence of Q may contradict a fundamental principle of the portion of science under consideration, e.g., in a portion of classical physics where the assumption of the occurrence of Q implies the existence of perpetual motion. For most rich fragments of science, we do not have complete descriptions of the methods of scientific inference. I consider it likely that such complete descriptions are incapable of formal description. Fortunately, for the kinds of issues dealt with in this article, a complete description of scientific refutation is not needed; only a few basic properties of $\neg$ are needed. These are contained in Axioms 1 to 3 of the following definition.

DEFINITION 7. Let $\mathfrak{S} = \langle \mathcal{S}, \cup, \cap, -, U, \varnothing \rangle$ be a lattice of events. Then

$$\mathfrak{E} = \langle \mathcal{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$$

is said to be the *lattice of scientific events (generated by* $\mathfrak{S}$*)* if and only if the following condition about $\mathcal{E}$ and the following three axioms about $\neg$ hold:

$\mathcal{E}$ is the smallest set of events such that

(i) $\mathcal{S} \subseteq \mathcal{E}$;
(ii) $\varnothing$ and X are in $\mathcal{E}$;
(iii) if A is in $\mathcal{E}$, then $A \subseteq X$; and
(iv) if A and B are in $\mathcal{E}$, then $A \cup B$, $A \cap B$, and $\neg A$ are in $\mathcal{E}$.

AXIOM 1. $\neg X = \varnothing$ *and* $\neg \varnothing = X$.

AXIOM 2. *For all A and B in $\mathcal{E}$, if $A \cap B = \varnothing$, then $B \subseteq \neg A$.*

AXIOM 3. *For all A in $\mathcal{E}$, $A \cap \neg A = \varnothing$.* □

Note that X is the sure event of $\mathcal{E}$ (i.e., is the largest event in $\mathcal{E}$).

DEFINITION 8. Let $\mathfrak{E} = \langle \mathcal{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be a lattice of scientific events generated by $\mathcal{S}$. Then $\mathcal{S}$ is called the set of *initial scientific events (of* $\mathfrak{E}$*)* and $\mathcal{E}$ is called the set of *scientific events (of* $\mathfrak{E}$*)*. Events of $\mathcal{E}$ of the form $\neg A$ are called *refutations.*

$\mathfrak{F}$ is said to be an *lattice of scientific events* if and only if for some $\mathcal{T}$, $\mathfrak{F}$ is the lattice of scientific events generated by $\mathcal{T}$. □

Although refutations are important in carrying out empirical science, they also have significant roles in other kinds of situations. Because Definitions 7 and 8 contain only very general properties of contradictions, refutations are also applicable to situations outside of the foundations of empirical science. The following are two very different examples of involving refutations.

EXAMPLE 3 (Proof Theory). Let Γ be formal theory of arithmetic used by Gödel (1931) in his famous incompleteness theorem, and let $\Gamma \vdash \alpha$ stand for the first-order sentence α is a formal consequence of Γ. Let $\equiv$ be the equivalence relation defined on the set of first-order arithmetic sentences, $\mathbf{F}$, such that for all ϕ and ψ in $\mathbf{F}$, $\phi \equiv \psi$ if and only if $(\Gamma \vdash \phi \leftrightarrow \psi)$. Let $\mathcal{S}$ be the set of $\equiv$-equivalence classes. Define $\preceq$, $\vee$, $\wedge$, u, and z as follows: For all A and B in $\mathcal{S}$,

(i) $A \preceq B$ if and only if for some α in A and β in B, if $\Gamma \vdash \alpha$, then $\Gamma \vdash \beta$,
(ii) $A \sqcup B$ if and only if for some α in A and β in B, $\Gamma \vdash \alpha$ or $\Gamma \vdash \beta$,
(iii) $A \sqcap B$ if and only if for some α in A and β in B, $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$,
(iv) u is the set of α in $\mathbf{F}$ such that $\Gamma \vdash \alpha$.
(v) z is the set of α in $\mathbf{F}$ such that $\Gamma \vdash \mathsf{neg}\,\alpha$, where **neg** is the negation operator of the language of Γ.

Then $\langle \mathcal{S}, \preceq, \sqcup, \sqcap, u, z \rangle$ is a distributive lattice. For A in $\mathcal{S}$, let $\neg A$ stand for, "For some α in A, $\Gamma \vdash \alpha$ leads to a contradiction." Let γ be the critical sentence ("I am not a theorem") that Gödel used to establish his incompleteness theorem, and let G be the element of $\mathcal{S}$ to which γ belongs. Gödel showed that the assumption of $\Gamma \vdash \gamma$ contradicted a fundamental principle of the metatheory of arithmetic, ω-consistency, which in the current setup yields $\neg G$.

Note the following analogies between refutation in science and refutation in this mathematical example: (1) Verifiability is analogous to provability from Γ. (2) Refuting A by verifying a B such that $A \cap B = \varnothing$ is analogous to assuming $\Gamma \cup \{\delta\}$ and showing $\Gamma \cup \{\delta\} \vdash \kappa$, where κ is a contradiction of $\mathbf{F}$, to conclude $\neg D$, where D is the element of $\mathcal{S}$ to which δ belongs. And (3), assuming the occurrence of A to contradict a fundamental principle of the portion of science under consideration is analogous to assuming $\Gamma \vdash \theta$ and showing that a fundamental principle of the metatheory of arithmetic is contradicted. □

EXAMPLE 4 (Psychology of Judgment). In the psychology of subjective judgments of probability, Narens (2003b) presents a model in which cognitive representations of linguistic descriptions of events are given a topological structure. It is assumed that when a subject is asked to judge the conditional probability of $\mathbf{A}|\mathbf{B}$, $\mathbb{P}(\mathbf{A}|\mathbf{B})$, the subject does this by first forming cognitive representations A for $\mathbf{A}$ and B for $\mathbf{B}$. The subject then uses cognitive heuristics to determine the support for A, $S^+(A)$ (modeled as a non-negative real number), and the support against A, $S^-(A)$ (modeled a non-negative real number), and produces a probability judgment consistent with the formula,

$$\mathbb{P}(\mathbf{A}|\mathbf{B}) = \frac{S^+(A)}{S^+(A) + S^-(A)} .$$

$S^-(A)$ is determined as follows: The subject constructs a cognitive complement of A, $\neg\ A$ (taking B to be the universe in which the complementation takes place), uses cognitive heuristics to find the support for $\neg\ A$, $S^+(\neg\ A)$, and takes $S^-(A)$ to be $S^+(\neg\ A)$. Narens assumes that the cognitive representations B, A, and $\neg\ A$ are open sets in a topology with B as the universal set in the topology. The presentation of $\mathbf{A}|\mathbf{B}$ triggers cognitively clear and ambiguous instances for judgment. The clear instances triggered by $\mathbf{A}$ form A. The ambiguous instances triggered by $\mathbf{A}$ form the (topological) boundary of A. Similarly the cognitive construction of $\neg\ A$ leads to clear and ambiguous instances, with the clear ones forming $\neg\ A$ and the ambiguous ones the boundary of $\neg\ A$. Thus in computing $S^+(A)$ and $S^+(\neg\ A)$, the triggered boundaries are ignored. This leads to a failure of the Law of the Excluded Middle, that is, $A \cup \neg\ A$ is a proper subset of B. Narens assumes that for open sets Q, $\neg\ Q$ is the interior of the closure of $B - Q$, i.e., $\neg\ Q$ is the largest open set in B such that $Q \cap (\neg\ Q) = \varnothing$. With this definition,

$$\langle \mathcal{B}, \subseteq, \cup, \cap, \neg, B, \varnothing \rangle ,$$

where $\mathcal{B}$ is the set of open subsets of B, is formally a lattice of scientific events. Narens (2003b) shows that this topological model accounts for many puzzling empirical findings in the extensive literature of human probability judgments. □

Other examples of lattices of scientific events can be constructed from applications of intuitionistic logic for reasons discussed next.

2.2. *Relationship to Intuitionistic Logic*

Intuitionism was introduced by the mathematician L. L. J. Brouwer as an alternative form of mathematics. It followed from Brouwer's philosophy of mathematics that the methods of derivation used in intuitionistic mathematical theorems could not be formalized. However, the intuitionistic, mathematical theorems that Brouwer produced displayed sufficient regularity in their proofs that an axiomatic approach to his methods of proof appeared feasible. Such an axiomatization was accomplished by Heyting (1930). Today logics that are equivalent to Heyting's axiomatization are called *intuitionistic logic.* Although Heyting designed his logic for intuitionistic mathematics, it was shown to have other applications. For example, Kolmogorov (1932) showed that Heyting's logic had the correct formal properties to provide a theory for mathematical constructions; and Gödel (1933) showed that it can be interpreted as a foundational logic for concepts that naturally arise in proof theory of mathematical logic. Kolmogorov and Gödel achieved their results by giving interpretations to the logical primitives that were different from Heyting's.

In classical propositional logic, the operation of "if and only if" yields an equivalence relation on propositions, and the induced algebra on the resulting equivalence classes is a boolean lattice (see Example 1). In a similar manner, in intuitionistic propositional logic, the intuitionistic "if and only if" operation, yields an equivalence relation on intuitionistic propositions, and the induced algebra on its equivalence classes also produces a lattice, called a *pseudo boolean algebra.* Like boolean lattices, each pseudo boolean algebra is isomorphic to a lattice of subsets of the form,

$$\mathfrak{P} = \langle \mathscr{P}, \subseteq, \cup, \cap, X, \varnothing \rangle \,,$$

(Stone 1937; McKinsey and Tarski 1946).[1] Such lattices of subsets are called "pseudo boolean algebras of subsets," and are defined formally as follows:

DEFINITION 9. $\mathfrak{P} = \langle \mathscr{P}, \subseteq, \cup, \cap, \Rightarrow, X, \varnothing \rangle$, where $\Rightarrow$ is a binary operation on $\mathscr{P}$, is said to be a *pseudo boolean algebra of subsets* if and only if the following three conditions hold for all A and B in $\mathscr{P}$:

(1) $\mathfrak{P} = \langle \mathscr{P}, \subseteq, \cup, \cap, X, \varnothing \rangle$ is a lattice of subsets.
(2) $A \cap (A \Rightarrow B) \subseteq B$.
(3) For all C in $\mathscr{P}$, if $A \cap C \subseteq B$ then $C \subseteq (A \Rightarrow B)$.

$\Rightarrow$ is called the operation of *relative pseudo complementation.* □

$\Rightarrow$ is the pseudo boolean algebra of subsets version of the implication operation of intuitionistic logic.

Let $\mathfrak{B} = \langle \mathscr{B}, \subseteq, \cup, \cap, -, X, \varnothing \rangle$ be a boolean algebra of subsets. Then it is not difficult to show that

$$\langle \mathscr{B}, \subseteq, \cup, \cap, \longrightarrow, X, \varnothing \rangle$$

is pseudo boolean algebra of subsets, where for all A and B in $\mathscr{B}$,

$$A \longrightarrow B \;=\; -A \cup B,$$

i.e., $\longrightarrow$ is the boolean algebra of subsets version of the implication operation of classical logic. Thus each boolean algebra of subsets is a pseudo boolean algebra with $-A \cup B \;= \longrightarrow \;= \;\Rightarrow$.

The following theorem show the relationship between the refutation operator $\neg$ of an empirical algebra of events and the relative pseudo complementation operator $\Rightarrow$ of a pseudo boolean algebras of subsets.

THEOREM 2. Suppose $\mathfrak{P} = \langle \mathscr{P}, \subseteq, \cup, \cap, \Rightarrow, X, \varnothing \rangle$ is a pseudo boolean algebra of subsets. For each A in $\mathscr{P}$, let

$$\neg\, A \;=\; A \Rightarrow \varnothing.$$

Then $\langle \mathscr{P}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ is an lattice of scientific events.

Proof. Axioms 1 to 3 are immediate consequences of Definition 9. □

In intuitionistic logic, $\neg$ in Theorem 2 is called *intuitionistic negation.*

Lattices of scientific events are lattices of subsets with a weakened form of complementation that has the fundamental properties of intuitionistic negation. They differ from pseudo boolean algebras in that the operation of relative pseudo complementation, $\Rightarrow$, is not used. While in some applications of lattices of scientific events $\Rightarrow$ has cogent interpretations, this is not always the case. For all the applications of Intuitionistic logic of which I am aware, $\neg$ has cogent interpretations.

2.3. *Properties of Lattices of Scientific Events*

THEOREM 3. Let $\mathfrak{E} = \langle \mathscr{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be a lattice of scientific events and A and B be arbitrary events in $\mathscr{E}$. Then the following five statements are true:

1. If $B \subseteq A$, then $\neg\, A \subseteq \neg\, B$.
2. $A \subseteq \neg\,\neg\, A$.
3. $\neg\, A = \neg\,\neg\,\neg\, A$.
4. $\neg\,(A \cup B) = \neg\, A \;\cap\; \neg\, B$.
5. $\neg\, A \cup \neg\, B \subseteq \neg\,(A \cap B)$.

Proof. 1. Suppose $B \subseteq A$. By Axiom 3, $A \cap \neg\, A = \varnothing$. Thus $B \cap \neg\, A = \varnothing$. Therefore, by Axiom 2, $\neg\, A \subseteq \neg\, B$.

2. By Axiom 3, $\neg\, A \cap A = \varnothing$. Thus by Axiom 2, $A \subseteq \neg\,\neg\, A$, showing Statement 2.

3. By Statement 2, $A \subseteq \neg\neg A$. Thus by Statement 1,

$$\neg\neg\neg A \subseteq \neg A.$$

However, by Statement 2,

$$\neg A \subseteq \neg\neg(\neg A) = \neg\neg\neg A.$$

Therefore, $\neg A = \neg\neg\neg A$.

4. By Axiom 3, (*i*) $\neg A \subseteq -A$ and (*ii*) $\neg B \subseteq -B$. Thus

$$\neg A \cap \neg B \subseteq -A \cap -B = -(A \cup B).$$

Therefore $(A \cup B) \cap (\neg A \cap \neg B) = \varnothing$, and thus by Axiom 2,

$$\neg A \cap \neg B \subseteq \neg(A \cup B). \quad (8)$$

Because $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows from Statement 1 that

$$\neg(A \cup B) \subseteq \neg A \text{ and } \neg(A \cup B) \subseteq \neg B.$$

Therefore

$$\neg(A \cup B) \subseteq \neg A \cap \neg B. \quad (9)$$

Equations (8) and (9) show that

$$\neg(A \cup B) = \neg A \cap \neg B.$$

5. From

$$A \cap B \subseteq A \text{ and } A \cap B \subseteq B,$$

it follows from Statement 1 that

$$\neg A \subseteq \neg(A \cap B) \text{ and } \neg B \subseteq \neg(A \cap B),$$

and thus

$$\neg A \cup \neg B \subseteq \neg(A \cap B). \quad \square$$

2.4. *Boolean Lattice of Refutations*

DEFINITION 10. Let $\mathfrak{E} = \langle \mathcal{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be a lattice of scientific events. By definition, let

$$\mathcal{R} = \{\neg B | B \in \mathcal{E}\} = \{A | A \text{ is a refutation in } \mathcal{E}\}.$$

By definition, for each A and B in $\mathcal{R}$, let

$$A \Cup B = \neg\neg(A \cup B).$$

Then $\mathfrak{R}$ is said to be the *boolean lattice of refutations (of* $\mathfrak{E}$*)* if and only if

$$\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle. \quad \square$$

Let $\mathfrak{E}$ be a lattice of scientific events and $\mathfrak{R}$ be the boolean lattice of refutations in $\mathfrak{E}$. The following lemmas and theorem establish facts about $\mathfrak{R}$, including that $\mathfrak{R}$ is a boolean lattice.

Hypotheses for Lemmas 1 to 6: Let $\mathfrak{E} = \langle \mathscr{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be a lattice of scientific events and $\mathfrak{R} = \langle \mathscr{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$ be the boolean lattice of refutations of $\mathfrak{E}$. Let *A, B, and C be arbitrary elements of $\mathscr{R}$, and A' and B' be elements of $\mathscr{E}$ such that $A = \neg\, A'$ and $B = \neg\, B'$.*

LEMMA 1. $A = \neg\,\neg\, A$.

Proof. Because $A = \neg\, A'$, it follows from Statement 3 of Theorem 3 that

$$\neg\,\neg\, A = \neg\,\neg\,\neg\, A' = \neg\, A' = A. \quad \square$$

LEMMA 2. *$A \cap B$ is in $\mathfrak{R}$.*

Proof. By Statement 4 of Theorem 3,

$$A \cap B = \neg\, A' \;\cap\; \neg\, B' = \neg\,(A' \cup B'). \quad \square$$

LEMMA 3. *If $B \subseteq A$, then $\neg\,\neg\, B \subseteq A$.*

Proof. By Lemma 1, $B = \neg\,\neg\, B$. $\square$

LEMMA 4. $\mathfrak{L} = \langle \mathscr{R}, \subseteq, X, \varnothing \rangle$ is a lattice with $A \cap B$ as the meet of A and B, and $A \Cup B$ as the join of A and B.

Proof. By Lemma 2, $A \cap B$ is in $\mathscr{R}$. Because $A \cap B$ is the meet in $\mathfrak{E}$ and $\mathscr{R} \subseteq \mathscr{E}$, it follows that $A \cap B$ is the meet in $\mathfrak{L}$.

Let D in $\mathscr{R}$ be such that

$$A \subseteq D \text{ and } B \subseteq D.$$

(Such a D exists, because the above expression holds for $D = X$.) Then $A \cup B \subseteq D$. By two applications of Statement 1 of Theorem 3,

$$\neg\,\neg\,(A \cup B) \subseteq \neg\,\neg\, D.$$

By Statement 2 of Theorem 3, $A \cup B \subseteq \neg\,\neg\,(A \cup B)$, and by Lemma 1, $\neg\,\neg\, D = D$. Thus,

$$A \cup B \subseteq \neg\,\neg\,(A \cup B) = A \Cup B \subseteq D.$$

Therefore,

$$A \subseteq A \Cup B \subseteq D \quad \text{and} \quad B \subseteq A \Cup B \subseteq D.$$

Thus $A \Cup B$ is the join of A and B in $\mathfrak{L}$. $\square$

LEMMA 5. $(A \cap B) \Cup (A \cap C) = A \cap (B \Cup C)$.

Proof.

$$\begin{aligned}
(A \cap B) \Cup (A \cap C) &= \neg\neg [(A \cap B) \cup (A \cap C)] \\
&= \neg\neg [A \cap (B \cup C)] \\
&\supseteq \neg [\neg A \cup \neg (B \cup C)] \\
&\quad \text{(Statement 5 of Theorem 3)} \\
&= [\neg\neg A \cap \neg\neg (B \cup C)] \\
&\quad \text{(Statement 4 of Theorem 3)} \\
&= [\neg\neg A \cap (B \Cup C)] \\
&= A \cap (B \Cup C)] \text{ (Lemma 1)},
\end{aligned}$$

that is,

$$(10) \qquad (A \cap B) \Cup (A \cap C) \supseteq A \cap (B \Cup C)\,.$$

Because,

$$\begin{aligned}
(A \cap B) \cup (A \cap C) &= A \cap (B \cup C) \\
&\subseteq A \cap \neg\neg (B \cup C) \\
&\quad \text{(Statement 2 of Theorem 3)} \\
&= A \cap (B \Cup C),
\end{aligned}$$

it follows that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \Cup C)\,.$$

Then

$$A \cap B \subseteq A \cap (B \Cup C) \text{ and } A \cap C \subseteq A \cap (B \Cup C)\,.$$

Thus, because by Lemma 4 $(A \cap B) \Cup (A \cap C)$ is the $\subseteq$-smallest element in $\mathcal{R}$ such that

$$A \cap B \subseteq A \cap (B \Cup C) \text{ and } A \cap C \subseteq A \cap (B \Cup C)\,,$$

it then follows that

$$(11) \qquad (A \cap B) \Cup (A \cap C) \subseteq A \cap (B \Cup C).$$

Equations (10) and (11) show

$$(A \cap B) \Cup (A \cap C) = A \cap (B \Cup C)\,. \quad \Box$$

LEMMA 6. $A \Cup \neg A = X$.

Proof.

$$\begin{aligned} A \Cup \neg A &= \neg\neg(A \cup \neg A) \\ &= \neg(\neg A \cap \neg\neg A) \quad \text{(Statement 4 of Theorem 3)} \\ &= \neg(\neg A \cap A) \quad \text{(Lemma 1)} \\ &= \neg\varnothing \quad \text{(Axiom 3)} \\ &= X. \quad \text{(Axiom 1)} \quad \square \end{aligned}$$

THEOREM 4. $\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$ is a boolean lattice.

Proof. $\subseteq$ partially orders $\mathcal{R}$, and X and $\varnothing$ are respectively the maximal and minimal elements of $\mathcal{R}$ with respect to $\subseteq$. By Lemma 4, $\cap$ and $\Cup$ are respectively the meet and join operations of $\mathfrak{R}$. By Lemma 5, $\cap$ distributes over $\Cup$, and by Lemma 6 and Axiom 3, $\neg$ is the complement operator on $\mathcal{R}$. Thus $\mathfrak{R}$ is a boolean lattice. $\square$

3. PROBABILITY THEORY FOR SCIENTIFIC EVENTS

One way to produce probability functions on an lattice of scientific events $\mathfrak{E}$ is to find a probability function $\mathbb{P}$ on a boolean lattice of events that contains $\mathfrak{E}$ and restrict $\mathbb{P}$ to the domain of $\mathfrak{E}$.

DEFINITION 11. Let $\mathfrak{E} = \langle \mathcal{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be an lattice of scientific events. Let $\mathcal{B}$ be the smallest set such that (*i*) $\mathcal{E} \subseteq \mathcal{B}$, and (*ii*) for all A and B in $\mathcal{E}$, $A \cup B$, $A \cap B$, and $-A$ are in $\mathcal{B}$. Then

$$\mathfrak{B} = \langle \mathcal{B}, \subseteq, \cup, \cap, -, X, \varnothing \rangle$$

is called the *outer boolean lattice generated by* $\mathfrak{E}$. $\square$

Let $\mathfrak{E}$ be an lattice of scientific events and $\mathfrak{B}$ be the outer boolean lattice generated by $\mathfrak{E}$. Then under usual assumptions about rationality and probability, subjective probabilities can be assigned to the events in $\mathfrak{B}$ through a probability function $\mathbb{P}$. Then the restriction of $\mathbb{P}$ to the domain of $\mathfrak{E}$, $\mathbb{P}_{\mathcal{E}}$ is a probability function (in the sense of Definition 4) on the lattice $\mathfrak{E}$.

Let $\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$ be the boolean lattice of refutations of $\mathfrak{E}$ (Definition 10). Let $\mathbb{B}$ be the restriction of $\mathbb{P}_{\mathcal{E}}$ to $\mathcal{R}$. Then $\mathbb{B}$ is, in general, not a probability function of $\mathcal{R}$. In fact, $\mathbb{B}$ is *superadditive;* that is, for all A and B in $\mathcal{R}$ such that $A \cap B = \varnothing$,

$$\mathbb{B}(A) + \mathbb{B}(B) \leq \mathbb{B}(A \Cup B),$$

and there may be elements C and D in $\mathcal{R}$ such that $C \cap D = \varnothing$ and

$$\mathbb{B}(C) + \mathbb{B}(D) < \mathbb{B}(C \Cup D)\,.$$

The supperadditivity of $\mathbb{B}$ results from the fact that $\mathcal{R}$ may have elements A and B such that $A \cap B = \varnothing$ and $A \cup B$ is not a refutation, and therefore,

$$A \Cup B = \neg\neg\,(A \cup B) \supset A \cup B.$$

Let $\mathfrak{L} = \langle A, \leq, \sqcup, \sqcap, u, z\rangle$ be a lattice and $\preceq$ be a binary relation on A. The intended interpretation of "$a \preceq b$" is "the degree of belief for a is less than or equal to the degree of belief for b." For boolean lattices, the following condition has been considered as necessary for rationality:

DEFINITION 12. Let $\mathfrak{L}$ and $\preceq$ be as above. Then $\preceq$ is said to be *qualitatively additive* if and only if $\preceq$ is a total ordering on A and de Finetti's Axiom (Definition 5) holds, that is, for all a, b, and c in A, if $a \sqcap c = z$ and $b \sqcap c = z$, then

$$a \prec b \ \text{ iff } \ a \sqcup c \prec b \sqcup c. \quad \square$$

Suppose $\mathfrak{L}$ is boolean. In the behavioral sciences, particularly in economics and experimental psychology, it is often considered appropriate to formulate "rationality" in terms of an individual's behavior rather than in terms of the individual's interpretation of what is giving rise to his or her behavior. Thus it is said that a person's behavior is *consistent with rationality* if it satisfies certain rationality conditions. It is recognized that a person can display rational behavior by employing nonrational subjective methods of evaluation. This is part of the reason of why the word "consistent" is used in the phrase "consistent with rationality." In the literature, the qualitative additivity of $\preceq$ has been repeatedly taken as a condition used in showing behavior is consistent with rationality. However, as discussed below, I find it problematic to assert that the failure of qualitative additivity is sufficient to establish that a person's behavior is *inconsistent with rationality.*

Consider the case where $\mathbb{P}$ is a probability function on the outer boolean lattice $\mathfrak{B}$ of an scientific lattice of events, $\mathfrak{E}$. Let

$$\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing\rangle$$

be the boolean lattice of refutations of $\mathfrak{E}$, and let $\mathbb{B}$ be the restriction of $\mathbb{P}$ to $\mathcal{R}$. Define $\prec$ on $\mathcal{R}$ as follows: For all S and T in $\mathcal{R}$,

$$S \prec T \ \text{ iff } \ \mathbb{B}(S) < \mathbb{B}(T).$$

Because there may exist A, B, and C in $\mathfrak{R}$ such that $A \cap C = B \cap C = \varnothing$, $B \prec A$, and

$$A \Cup C = A \cup C \text{ and } B \Cup C = \neg\neg(B \cup C) \supset B \cup C,$$

it may be the case that

$$\mathbb{P}(A \Cup C) < \mathbb{P}(B \Cup C),$$

and thus, because $\mathbb{P}(A \Cup C) = \mathbb{B}(A \Cup C)$ and $\mathbb{P}(B \Cup C) = \mathbb{B}(B \Cup C)$,

$$A \Cup C \prec B \Cup C,$$

i.e., de Finetti's Axiom, and therefore qualitative additivity, may fail in $\mathfrak{R}$. Such a failure should not be considered to be inconsistent with rationality, for if the probability assignments via $\mathbb{P}$ to the events in the outer boolean lattice $\mathfrak{B}$ of $\mathcal{E}$ are to be considered to be "consistent with rationality," then so should assignments via $\mathbb{P}$ with respect to the boolean lattice $\mathfrak{R}$ (which is a substructure of $\mathfrak{B}$).

Let $\mathfrak{B}$, $\mathfrak{E}$, $\mathfrak{R}$, $\mathbb{P}$, and $\mathbb{B}$ be as above. As discussed previously, it is often the case that a person believes in the independence of two events A and B without recourse to checking whether $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Let $A \perp B$ stand for A and B exhibiting such "belief independence." I will assume—although many would insist that rationality demands—that if A and B are elements of $\mathcal{R}$ and when considered as elements of $\mathfrak{B}$, $A \perp B$, then when considered as elements of $\mathfrak{R}$, it is also the case that $A \perp B$. That is, $A \perp B$ does not depend on the boolean lattice to which they belong. Consistency between $\perp$ and $\mathbb{P}$ requires that for all elements C and D of $\mathfrak{B}$ such that $C \perp D$, $\mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D)$. The following condition, stated for $\mathfrak{B}$ and called *union independence,* is often taken as a rational qualitative axiom about probability with an independence relation $\perp$: For all A, B, and C in $\mathcal{B}$, if $A \perp B$, $A \perp C$, and $B \cap C = \varnothing$, then $A \perp (B \cup C)$. Union independence does not hold for $\mathfrak{R}$, because it may be the case that for some A, B, and C in $\mathcal{R}$,

$$A \perp B, \quad A \perp C, \quad B \cap C = \varnothing, \quad B \Cup C = \neg\neg(B \cup C) \supset B \cup C,$$

and not $A \perp B \Cup C$.

DEFINITION 13. Let $\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$ be the boolean lattice of refutations of an lattice of scientific events.

$\mathbb{B}'$ is said to be a *belief function* on $\mathfrak{R}$ if and only if $\mathbb{B}'$ is a function from $\mathcal{R}$ into the closed interval of reals [0, 1] such that $\mathbb{B}'(\varnothing) = 0$, $\mathbb{B}'(X) = 1$, and for all A in $\mathcal{R}$, if $A \neq \varnothing$ and $A \neq X$, then $0 < \mathbb{B}'(A) < 1$. Similar definitions hold for *belief function* on lattices of scientific events and on boolean lattices of events.

Suppose $\mathbb{B}'$ is a belief function on $\mathfrak{R}$. Then (*i*) $\mathbb{B}'$ is said to be *monotonic* if and only if for all A and B in $\mathcal{R}$,

$$\text{if } A \subset B \quad \text{then} \quad \mathbb{B}'(A) < \mathbb{B}'(B),$$

and (*ii*) $\mathbb{B}'$ is said to be *superadditive* if and only if for all C and D in $\mathcal{R}$ such that $C \cap D = \varnothing$,

$$\mathbb{B}'(C) + \mathbb{B}'(D) \leq \mathbb{B}'(C \Cup D). \quad \square$$

Let $\mathfrak{R} = \langle \mathcal{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$ be the boolean lattice of refutations of an lattice of scientific events $\mathfrak{E}$, and let $\mathfrak{B}$ be the outer boolean lattice of $\mathfrak{E}$. Then not every monotonic, superadditive belief function $\mathbb{B}'$ on $\mathfrak{R}$ can be extended to a probability function on $\mathfrak{B}$. A necessary condition for such extensions is that for all A, B, C, and D in $\mathcal{R}$, if

$$A \Cup B \subseteq C \Cup D\,,$$

then

$$\mathbb{B}'(A) + \mathbb{B}'(B) - \mathbb{B}'(A \cap B) \leq \mathbb{B}'(C) + \mathbb{B}'(D) - \mathbb{B}'(C \cap D)\,.$$

This condition is behaviorally testable in probability estimation experiments. It appears to me to be sensible to test this and related conditions on superadditive belief functions before asserting such functions exhibit "irrational behavior," even for researchers committed to textbook probability theory as a criterion for rationality.

4. CONDITIONAL EVENTS

Let $\mathfrak{B} = \langle \mathcal{B}, \subseteq, \cup, \cap, -, X, \varnothing \rangle$ be a boolean algebra of events, $\mathbb{B}$ be a belief function on $\mathfrak{B}$, and $\mathbb{P}$ be any probability function on $\mathfrak{B}$. For each nonempty A in $\mathcal{B}$, let

$$\upsilon(A) = \frac{\mathbb{B}(A)}{\mathbb{P}(A)}.$$

Then $\mathbb{B}(A) = \mathbb{P}(A)\upsilon(A)$. Thus a belief function having the form "$\mathbb{P}(A)\upsilon(A)$," puts no restriction on the belief function; that is, given a probability function $\mathbb{P}$ on $\mathfrak{B}$, each belief function is obtainable through an appropriate choice of υ. However, such is not the case for conditional belief representations of the form,

$$\mathbb{B}(C|B) = \frac{\mathbb{P}(C)\upsilon(C)}{\mathbb{P}(B)} = \mathbb{P}(C|B)\upsilon(C),$$

where C and B are arbitrary events in $\mathscr{B}$ such that $C \subseteq B$ and $B \neq \varnothing$. This is because $\upsilon(C)$ does not vary with the conditioning event B in such representations—a powerful restriction. In this section $\upsilon(C)$ is viewed as a factor that describes the amount the probability $\mathbb{P}(C)$ is distorted. Note that this way of viewing υ requires that the distortion of $\mathbb{P}(C)$ be the same as the distortion of $\mathbb{P}(C|B)$, for all B such that $C \subseteq B$.

Let $\mathfrak{E} = \langle \mathscr{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$ be a lattice of scientific events,

$$\mathfrak{R} = \langle \mathscr{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$$

be the boolean lattice of refutations of $\mathfrak{E}$, and $\mathbb{P}$ be a probability function on $\mathfrak{E}$. For each nonempty A in $\mathscr{E}$, let

$$\upsilon(A) = \frac{\mathbb{P}(\neg \neg A)}{\mathbb{P}(A)},$$

and for each B and C in $\mathfrak{E}$ such that B is nonempty and $C \subseteq B$, let

$$\mathbb{B}(C|B) = \frac{\mathbb{P}(C)\upsilon(C)}{\mathbb{P}(B)} = \mathbb{P}(C|B)\upsilon(C).$$

The above equation may be interpreted as an individual's belief in C is his or her subjective probability that given an scientific demonstration of the event B, a scientific refutation can be given to the claim that C is scientifically refutable. Although complicated, such conditional beliefs and their associate degrees of belief make sense. They are especially useful in situations where the demonstration of the double refutation of an event A (i.e., $\neg \neg A$) is sufficient to take the same action, or have the same effect, as the event A.

EXAMPLE 5 (American Law). In American law there is a presumption of innocence; that is, in a criminal proceeding, a lawyer needs only to accomplish the double refutation demonstrating innocence in order to free his or her client; that is, the lawyer needs only to refute the claim that it can be demonstrated that the innocence of the client is refutable. For legal purposes, this has the same effect as the demonstration of innocence. Let p_i be the probability of demonstrating innocence, I, and p the probability of demonstrating the double refutation of innocence, $\neg \neg I$. Then $p_i \leq p$, because $I \subseteq \neg \neg I$. For various legal decisions and actions, for example, plea bargaining, it is p rather than p_i that is used. □

5. DISCUSSION

Narens (2003a) produced a qualitative axiomatization of conditional probability in which one axiom – Binary Symmetry – contains principles that have been challenged in the literature (e.g., Ellsberg 1961) as necessary for the rational measurement of uncertainty. He explored the mathematical consequences of deleting this axiom and found that the most general representation of the remaining axioms measured uncertainty in terms of a conditional belief function $\mathbb{B}$ of the form,

$$\mathbb{B}(A|B) = \mathbb{P}(A|B)v(A),$$

where $\mathbb{P}$ is a uniquely determined conditional probability function and v is a function that is unique determined up to multiplication by positive reals.

Narens (2003a) investigates the applicability of the conditional belief representation as a descriptive theory for various empirical results in the literature involving subjective probability. Except for the Ellsberg Paradox, these empirical results are about "irrational" probabilistic phenomena. And the Ellsberg Paradox is highly controversial as a proper application of rationality. Thus Narens (2003a) does not provide an example where the conditional belief function has a clear, rational application. Such examples are provided here by considering the restriction of rational probability assignments to boolean lattices of refutations.

The approach to rationality in this article is a limited one: Only observable rational behavior is considered. Such an approach deliberately ignores the motivations and internal decision processes resulting in behavior. In experimental psychology and economics this version of rationality has proven to be very productive.

CONVENTION 4. Throughout the rest of this section let

$$\mathfrak{E} = \langle \mathscr{E}, \subseteq, \cup, \cap, \neg, X, \varnothing \rangle$$

be a lattice of scientific events,

$$\mathfrak{B} = \langle \mathscr{B}, \subseteq, \cup, \cap, -, X, \varnothing \rangle$$

the outer boolean algebra of $\mathfrak{E}$, and

$$\mathfrak{R} = \langle \mathscr{R}, \subseteq, \Cup, \cap, \neg, X, \varnothing \rangle$$

the boolean lattice of refutations of $\mathfrak{E}$. Also let $\mathbb{P}$ be a probability function on $\mathfrak{B}$, and $\mathbb{P}_{\mathscr{R}}$ be the belief function that is the restriction of $\mathbb{P}$ to $\mathscr{R}$. □

EXAMPLE 6 (Behavioral Irrationality/Rationality). Suppose that $\mathbb{P}$ is a person P's subjective probability function on $\mathfrak{B}$. In Section 3, it was shown that $\mathbb{P}_{\mathcal{R}}$ fails both standard quantitative and qualitative properties generally attributed to rational belief. In particular, $\mathbb{P}_{\mathcal{R}}$ is superadditive and fails qualitative additivity. It was argued that the assignments of degrees of belief to elements of $\mathfrak{R}$ by $\mathbb{P}_{\mathcal{R}}$ are consistent with rationality, because the assignments of probabilities to elements of $\mathfrak{B}$ by $\mathbb{P}$ are consistent with rationality. Note that in this situation, we have a complete view of the relationship of $\mathfrak{R}$ to $\mathfrak{B}$ without entering into P's mind. Because of this, it might well be argued that an eccentric interpretation is being given to the join operator $\Cup$ in $\mathfrak{R}$—that is, for P, $\Cup$ is not the set-theoretic correlate of "or," (the latter being, of course, $\cup$). However, other examples can be given where this latter argument does not hold without entering into a person's mind.[2]

Consider the case where the Experimenter E presents events to person Q for evaluation of subjective probability. E can check behaviorally that Q is treating the events as if they were events from a boolean algebra of events. For example, E can give (from E's perspective) the events $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ and ask Q if they are "identical." Suppose Q passes this test by treating the events "rationally;" that is, treating them from the Experimenter's perspective like they were from a boolean algebra of events. Of course Q could have an idiosyncratic perspective about the events. For example, through odd socialization Q may be what E would consider to be a rabid empiricist who interprets the stimuli different than E: Whereas E considers events to be platonic – either they occur or they do not occur, Q interprets them as empirical and identifies an occurrence of each event A with its double empirical refutation, $\neg\neg A$. Suppose for each event A given by E to Q for "probabilistic evaluation," $\mathbb{P}_{\mathcal{R}}(\neg\neg A)$ is Q's response. Based on this behavior, should Q's behavior be considered as "inconsistent with rationality," because $\mathbb{P}_{\mathcal{R}}$ is superadditive on $\mathfrak{R}$ and violates qualitative additivity? I think not. I believe a different version of "consistent with rationality" is called for: Q's behavior should be considered as consistent with rationality as long as we can conceive of an *isomorphic situation of events* in which, *under isomorphism,* another person could give the same probability estimates to event stimuli as Q and still be considered "rational." Under this version, the demonstration of the non-rationality of Q's behavior would consist of showing that no such isomorphism exists. □

For all A and B in $\mathscr{E}$ such that $A \subseteq B$ and $B \neq \varnothing$, let

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)},$$

and

$$v(A) = \frac{\mathbb{P}(\neg \neg A)}{\mathbb{P}(A)},$$

and

$$\mathbb{B}(\neg \neg A|B) = \frac{\mathbb{P}(\neg \neg A)}{\mathbb{P}(B)} = \mathbb{P}(A|B)v(A).$$

Let $\mathcal{S}$ be the set of initial events that generates $\mathcal{E}$ (Definitions 8 and 7).

EXAMPLE 7 (Updating Due to New Evidence). In the following, elements of $\mathcal{S}$ are considered as possible new pieces of evidence (e.g., the outcome of a particular and ongoing DNA test; a new, and very reliable, witness is discovered who provides testimony asserting the proposition Θ; et cetera). Suppose, like in law, the success (= the occurrence) a double refutation depends on evidence. For the sake of argument, assume that (*i*) the evidence are indisputable facts, (*ii*) only elements of $\mathcal{S}$ can be presented as additional evidence, and (*iii*) the success of a double refutation depends only on the evidence presented. Suppose $\mathbb{P}$ is P's probability function on $\mathfrak{B}$. $\mathbb{P}$ describes P's current degree of beliefs of events in $\mathfrak{R}$. As new evidence comes in, P will update his or her event space and probability function. Because the success of a double refutation depends only on evidence, P's probability function will only be updated when new evidence E in $\mathcal{S}$ is presented and the updating of an event D in $\mathcal{E}$ will be of the form

$$\mathbb{P}(D|E) = \frac{\mathbb{P}(E \cap D)}{\mathbb{P}(E)}.$$

Suppose $\neg \neg A$ and $\neg \neg B$ are in $\mathcal{R}$ and

$$\varnothing \subset \neg \neg A \subset \neg \neg B \subset X,$$

where X is the sure event.

Given the above, When is

$$\frac{\mathbb{P}(\neg \neg A)}{\mathbb{P}(\neg \neg B)}$$

a proper updating of $\mathbb{P}$? The answer is "rarely." The reason is that for $\neg \neg B$ to occur, new evidence needs to be presented, and by hypothesis, this evidence, E, is in $\mathcal{S}$. Because, by assumption, the occurrence of $\neg \neg B$ completely depends on the occurrence of E, $E \subseteq \neg \neg B$. The probability theory updating of $\mathbb{P}$ for $\neg \neg A$ in this circumstance is given by

$$\frac{\mathbb{P}(E \cap \neg \neg A)}{\mathbb{P}(E)} = \mathbb{P}((E \cap \neg \neg A)|E),$$

and only rarely is this same as

$$\frac{\mathbb{P}(\neg\neg A)}{\mathbb{P}(\neg\neg B)}.$$

However, $E \cap \neg\neg A$ need not be in $\mathcal{R}$, and, as discussed in Section 4, it is double refutations that are relevant. Therefore, unlike in traditional probability theory, the double refutation event space $\mathcal{R}$ cannot in general be properly updated to a different double refutation event space, because from the point of view of $\mathfrak{B}$, $\mathfrak{R}$ should be updated to $\{E \cap F | F \in \mathcal{R}\}$, which generally contains non-refutations. I do not see this presenting serious difficulties, because for taking actions only updated degrees of belief are needed, and these can always be computed for events in the event space $\mathfrak{R}$, conditionalized on conjunctions of events from $\mathcal{S}$. □

The last two examples establish that conditional belief representations of the form

$$\mathbb{B}(A|B) = \mathbb{P}(A|B)\upsilon(A)$$

are in some circumstances a basis for a rational assignment of degrees of belief. The argument for rationality rests on conceiving a circumstance where for all *relevant* events A and B, the conditional degrees of belief $\mathbb{B}(A|B)$ agree with the rational assignments of conditional probabilities in a structurally isomorphic belief situation. The argument for the existence of appropriate applications of such belief representations rests on the effectiveness and correctness of such representations for boolean lattices of refutations of lattices of scientific events.

NOTES

[1] See Rasiowa and Sikorski 1968 for a complete development of intuitionistic logics and pseudo boolean algebras.

[2] Of course, a person can enter his own mind and provide an explanation of his behavior. However, experimental psychologists have found that such verbal reports of subjects to be highly suspect and often unreliable as veridical accounts of how the behaviors were produced – see, for example, Nisbet and Wilson (1977). For this and related reasons, behavioral scientists choose to model directly the behavior of subjects rather than their verbal reports of it or a combination of the two.

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