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A theory of belief

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Abstract

A theory of belief is presented in which uncertainty has two dimensions. The two dimensions have a variety of interpretations. The article focusses on two of these interpretations.

The first is that one dimension corresponds to probability and the other to "definiteness," which itself has a variety of interpretations. One interpretation of definiteness is as the ordinal inverse of an aspect of uncertainty called "ambiguity" that is often considered important in the decision theory literature. (Greater ambiguity produces less definiteness and vice versa.) Another interpretation of definiteness is as a factor that measures the distortion of an individual's probability judgments that is due to specific factors involved in the cognitive processing leading to judgments. This interpretation is used to provide a new foundation for support theories of probability judgments and a new formulation of the "Unpacking Principle" of Tversky and Koehler.

The second interpretation of the two dimensions of uncertainty is that one dimension of an event A corresponds to a function that measures the probabilistic strength of A as the focal event in conditional events of the form $A|B$, and the other dimension corresponds to a function that measures the probabilistic strength of A as the context or conditioning event in conditional events of the form $C|A$. The second interpretation is used to provide an account of experimental results in which for disjoint events A and B, the judge probabilities of $A|(A\cup B)$ and $B|(A\cup B)$ do not sum to 1.

The theory of belief is axiomatized qualitatively in terms of a primitive binary relation $\succsim$ on conditional events. ($A|B \succsim C|D$ is interpreted as "the degree of belief of $A|B$ is greater than the degree of belief of $C|D$.") It is shown that the axiomatization is a generalization of conditional probability in which a principle of conditional probability that has been repeatedly criticized on normative grounds may fail.

Representation and uniqueness theorems for the axiomatization demonstrate that the resulting generalization is comparable in mathematical richness to finitely additive probability theory.

1. Introduction

1.1. Introduction

That beliefs can be compared in terms of strength is intuitively compelling. The validity of such comparisons underlie the justifications of many practical methods of decision, for example, "beyond a reasonable doubt" decisions by juries, selecting treatments for medical patients, etc. de Finetti (1931, 1937) proposed that strengths of belief could be measured and compared through subjective probabilities. However, many thought this proposal unwarrantably restrictive, because it excluded important belief situations such as normative theories evidence (e.g., Shafer, 1976) or descriptive theories of how individuals evaluate uncertain propositions (e.g., Tversky & Koehler, 1994; Rottenstreich & Tversky, 1997). Various alternatives to subjective probability for measuring belief have been proposed in the literature. A major drawback to most of them was that they lacked interesting mathematical structure and effective means of calculation for understanding and manipulating degrees of belief.

Traditionally the probability calculus, as encompassed by the axioms proposed by Kolmogorov (1933), has been taken as the normative calculus for manipulating degrees of uncertainty. While the Kolmogorov calculus is arguably a very good idea for situations like casino gambling in which long random sequences exist and are easily observable, it is much more controversial for cases of uncertainty in which random sequences are either difficult to observe or are impossible by the nature of the events involved.

This article investigates an alternative to the probability calculus that is based on the idea that uncertainty is measured by probability *and* an additional dimension.

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This alternative is viewed as a very modest generalization of the probability calculus. As such, it allows for a focussed discussion about its acceptability as a generalization of various concepts based on the probability calculus.

In the literature, there are several different foundational approaches to probability that produce calculi satisfying the Kolmogorov axioms. (Various examples can be found in Fine, 1973; van Lambalgen, 1987.) In particular, de Finetti (1937) and Savage (1954) pioneered approaches designed to capture personal or subjective probability functions, and the theory of conditional belief developed here extends these and related approaches to a more general class of personal measuring functions that takes into account not only uncertainty but also possibly information about the nature of the uncertainty, and in certain psychological settings information about the processing of uncertainties. Like de Finetti and Savage, the theories of conditional belief developed in the article are axiomatic and based on a primitive qualitative preference ordering. Several axiomatizations of increasing complexity are presented. They are all motivated in part by the fact that they can be viewed as specific weakenings of corresponding axiomatic systems for related versions of conditional probability.

For events e and f, let $(e|e,f)$ stand for "likelihood of e occurring if either e or f occurs." The key axiom of conditional probability that is deleted in all the generalizations presented in this article asserts that for all distinct states of the world a, b, c, and d, if

$$(a|a,b) \sim (b|a,b) \quad \text{and} \quad (c|c,d) \sim (d|c,d), \tag{1}$$

then

$$(a|a,b) \sim (c|c,d) \tag{2}$$

and

$$(a|a,c) \sim (b|b,d). \tag{3}$$

The intuition for the desirability of the above deletion is the following: Assume

$$(a|a,b) = (b|a,b) \quad \text{and} \quad (c|c,d) = (d|c,d). \tag{4}$$

In the context of the other axioms for conditional probability, the above asserts that if a and b have equal likelihood of occurring and c and d have equal likelihood of occurring, then the conditional probabilities of $(a|a,b)$ and $(c|c,d)$ are $\frac{1}{2}$, and the conditional probabilities of $(a|a,c)$ and $(b|b,d)$ are the same.

Suppose Eq. (1) and the judgment of equal likelihood of the occurrences of a and b, given either a or b occurs, is based on much information about a and b and a good understanding of the nature of the uncertainty involved, and the judgment of equal likelihood of the occurrences of c and d, given either b or d occurs, is due to the lack of knowledge of b and d, for example, due to complete ignorance of c and d. Then, because of the differences in the understanding of the nature of the probabilities involve, a lower degree of belief may be assigned to $(c|c,d)$ than to $(a|a,b)$, thus invalidating Eq. (2).

Suppose Eq. (1) and the judgements of the likelihoods of the occurrences of a and c, given either a or c occurs, are based on much information about a and c and the nature of the uncertainty involved, and the judgment the likelihoods of the occurrences of b and d given either b or d occurs is due to the lack of knowledge of b and d. This may result in different degrees of belief being assigned to $(a|a,c)$ and $(b|b,d)$, and such an assignment would invalidate Eq. (3).

The primary goal of this article is to examine axiomatic theories of belief that result by making the above deletion. These theories will be evaluated in terms of their mathematical power, philosophical acceptability, and applicability.

Such axiomatic theories provide generalizations of subjective conditional probability. The generalizations, which are called theories of *subjective conditional belief* or *conditional belief* for short, will have similar levels of calculative power and mathematical richness as subjective conditional (finitely additive) probability. In addition, they will also provide for nonprobabilistic dimensions of belief, for example, a dimension of "ambiguity."

Convention 1.1. Let A be a subset of the nonempty set B. In this article, when discussing $(A|B)$, A will often be referred to as the *choice set*, *choice*, or *focus* (of $(A|B)$) and B as the *context* (of $(A|B)$).

In some belief situations, we have for each choice A in context B very good evidence for the strength of $(A|B)$—and in other belief situations, we have only poor evidence for evaluating the strength of $(C|D)$ for choices C in context D. In cases with good evidence for all choice and context sets, I believe it is reasonable to measure the strength of beliefs as probabilities; and in cases where there is poor evidence for some choice sets, I agree with Shafer (1976) and others that on normative grounds it is unreasonable to demand that strengths of belief be measured as probabilities. In this article, strengths of belief will be measured as "degrees of belief," which like probabilities are nonnegative real numbers, but unlike probabilities, (i) need not be less than or equal to 1, and (ii) need not be additive for beliefs of unions of disjoint events.

1.2. Belief functions

Definition 1.1. Let D be a nonempty set. Then κ_D is said to be a *belief function* on D if and only if κ_D is a function on the set of subsets of D such that for each $E \subseteq D$, (i) $0 \leqslant \kappa_D(E)$, and (ii) $\kappa_D(E) = 0$ iff $E = \emptyset$.

Let κ_D be a belief function. Then the following definitions hold:

(i) The value of κ_D on a subset E of D, $\kappa_D(E)$, is called *E's degree of belief* (*under* κ_D).
(ii) κ_D is *additive* if and only if and for all subsets E and F of D, if $E \cap F = \emptyset$, then $\kappa_D(E \cup F) = \kappa_D(E) + \kappa_D(F)$.
(iii) κ_D is *monotonic* if and only if for all subsets E and F of D, if $E \subset F$ then $\kappa_D(E) < \kappa_D(F)$.
(iv) κ_D is *modest* if and only if it is monotonic and for all subsets E of D,
$0 \leqslant \kappa_D(E) \leqslant 1$.
(v) κ_D *has norm* 1 if and only if $\kappa_D(D) = 1$.
(vi) κ_D is a *subjective probability* function if and only if it has norm 1 and is modest and additive.

Lemma 1.1. *Let κ_D be a belief function. Then the following two statements are true*:

1. *If κ_D is additive*, *then it is monotonic*.
2. *If κ_D has norm* 1 *and is additive*, *then it is a subjective probability function*.

Proof. Immediate from Definition 1.1. □

The values of a belief function are called *degrees of belief*, and when the belief function is a subjective probability function, they are often called *subjective probabilities* or simply *probabilities*. Under one interpretation, degrees of belief are distortions of subjective probabilities that take into account nonprobabilistic aspects present in the choice situation, for example, "ambiguity." Under this interpretation, it may be the case that some degrees of belief of elements of D distort probability by producing an increase and others by producing a decrease so that over D the effects of the distortions cancel and $\kappa_D(D) = 1$.

In Sections 2 and 3 axiomatic, qualitative characterizations of belief functions of the form

$$\kappa_B(A) = P_B(A)v(A)$$

are presented, where P_B is a finitely additive probability function on B and v is from a ratio scale of functions into the positive reals. For each family of subsets $\mathscr{D}$ on which v is constant, κ_B acts on $\mathscr{D}$ like a finitely additive probability function in the sense that for all E and F in $\mathscr{D}$ such that $E \cap F = \emptyset$,

$$\kappa_B(E \cup F) = \kappa_B(E) + \kappa_B(F).$$

This form of additivity need not hold for events with different v-values. These features of κ_B are use in this article to account for various empirical phenomena in observed in human probability estimation. In particular phenomena described by the Ellsberg Paradox, Tversky's and Koehler's support theory, Rottenstreich's and Tversky's generalization of support theory, and Brenner's and Rottenstreich's asymmetric support theory are accounted for.

The belief theories presented in this article may interpreted as either descriptive or normative theories. As descriptive theories, they may be useful as alternatives for existing theories and as a means for suggesting new experimentation. A separate article will argue for their acceptability as normative theories for certain kinds of probabilistic situations.

2. Basic axioms for conditional belief

In this section qualitative characterizations of various systems of conditional belief are presented and quantitative representation and uniqueness theorems are shown for them.

For purposes of presentation, the proofs of the theorems stated in this section are given in Section 7.

Definition 2.1. Throughout this article $\subseteq$ will denote the subset relation, $\subset$ the proper subset relation, $\mathbb{R}$ the real numbers, $\mathbb{R}^+$ the positive real numbers, $\mathbb{I}$ the integers, and $\mathbb{I}^+$ the positive integers.

Throughout the article, $\precsim$ will denote a binary relation that is transitive, reflexive, and connected (either $x \succsim y$ or $y \succsim x$ for all x, y in the domain of $\succsim$). Such transitive, reflexive, and connected relations on nonempty sets are called *weak orderings*. The symmetric part of $\succsim$ is denoted by $\sim$, and is defined by,

$$x \sim y \quad \text{iff } x \succsim y \text{ and } y \succsim x,$$

for all x and y in the domain of $\succsim$, and the asymmetric part of $\succsim$ is denoted by $\succ$ and is defined by,

$$x \succ y \quad \text{iff } x \succsim y \text{ and not } y \succsim x.$$

Throughout this article X will denote an infinite set of objects. By definition, a *context* is a finite subset of X that has at least two elements. $\mathscr{C}$ will denote the set of contexts. The notation $(a|C)$ will denote C is a context and $a \in C$. "$(a|C)$" will often be read as "strength of a in the context C." When $C = \{a, a_1, \ldots, a_n\}$, $(a|C)$ will often be written as $(a|a, a_1, \ldots, a_n)$. By convention, the notation $(a|a, a_1, \ldots, a_n)$ assumes the elements $a, a_1, \ldots, a_n$ are distinct.

A nonempty set $\mathscr{S}$ of functions from a nonempty set Y into $\mathbb{R}^+$ is said to be a *ratio scale* if and only if (i) for each $r \in \mathbb{R}^+$ and each $f \in \mathscr{S}$, $rf \in \mathscr{S}$, and (ii) for all f and g in $\mathscr{S}$ there exists s in $\mathbb{R}^+$ such that $f = sg$.

A nonempty set $\mathscr{S}$ of functions from a nonempty set Y into $\mathbb{R}^+$ is said to be an *interval scale* if and only if (i) for each $r \in \mathbb{R}^+$, each s in $\mathbb{R}$, and each $f \in \mathscr{S}$, $rf + s \in \mathscr{S}$, and (ii) for all f and g in $\mathscr{S}$ there exist r in $\mathbb{R}^+$ and s in $\mathbb{R}$ such that $f = rg + s$.

2.1. Basic belief axioms

The Basic Belief Axioms consist of the following 12 axioms. They provide the mathematical core of the axiom systems for this article.

Axiom 1. *$\succsim$ is a weak ordering on the set*
$\{(a,C)|C\in\mathscr{C}$ *and* $a\in C\}$.

Axiom 2. *Suppose $A\in\mathscr{C}$, $a\in A$, and B is a nonempty finite subset of X such that $B\cap A=\emptyset$. Then* $(a|A)\succ(a|A\cup B)$.

Axiom 3. *Suppose A, B are in $\mathscr{C}$ and C is a nonempty finite subset of X such that $A\cap C=B\cap C=\emptyset$, and suppose a and b are elements of X. Then*

(i) *if $a\in A$ and $a\in B$, then*
$(a|A)\succsim(a|B)$ *iff* $(a|A\cup C)\succsim(a|B\cup C)$,

(ii) *if $a\in A$ and $b\in A$, then*
$(a|A)\succsim(b|A)$ *iff* $(a|A\cup C)\succsim(b|A\cup C)$,
and
(iii) *if $a\in A$, $a\in B$, $b\in A$, and $b\in B$, then*
$(a|A)\succsim(a|B)$ *iff* $(b|A)\succsim(b|B)$.

Through the use of Axiom 3, $\succsim$ induces natural orderings on X and $\mathscr{C}$ as follows:

Definition 2.2. Define $\succsim_X$ on X by: for all a, b in X, $a\succsim_X b$ if and only if there exists a finite set C such that
$C\subseteq X-\{a,b\}$ and $(a|\{a,b\}\cup C)\succsim(b|\{a,b\}\cup C)$.

Note that it follows from Axiom 3 and the definition of $\succsim_X$ on X above that $a\succsim_X b$ if and only if *for all C*
if C is finite and $C\subseteq X-\{a,b\}$,
then $(a|\{a,b\}\cup C)\succsim(b|\{a,b\}\cup C)$.

Similarly define $\succsim_{\mathscr{C}}$ on $\mathscr{C}$ by: for all C, D in $\mathscr{C}$, $D\succsim_{\mathscr{C}} C$ if and only if
$(a|\{a\}\cup C)\succsim(a|\{a\}\cup D)$
for some a in X such that a is not in $C\cup D$. Note that it follows from Axiom 3 and the definition of $\succsim$ on $\mathscr{C}$ above that $D\succsim_{\mathscr{C}} C$ if and only if
$(a|\{a\}\cup C)\succsim(a|\{a\}\cup D)$
for all a in X such that a is not in $C\cup D$.

Note that Definition 2.2 says that $C\succsim_{\mathscr{C}} D$ if and only if for some a in $X-(C\cup D)$ the strength of a in context $\{a\}\cup C$ is *less than or equivalent* to its strength in context $\{a\}\cup D$. Also note it immediately follows from the fact that $\succsim$ is a weak ordering that the induced orderings on X and $\mathscr{C}$ described above are also weak orderings.

Definition 2.3. An ordered pair of functions $\langle u,v\rangle$ is said to be a *basic belief representation for* $\succsim$ if and only if the following three conditions hold:

1. u and v are functions from X into $\mathbb{R}^+$.
2. For all C, D in $\mathscr{C}$, $C\succsim_{\mathscr{C}} D$ iff
$$\sum_{e\in C}u(e)\geqslant\sum_{e\in D}u(e).$$
3. For all distinct $a, a_1, \ldots, a_n$ and all distinct $b, b_1, \ldots, b_m$,
$(a|a,a_1,\ldots,a_n)\succsim(b|b,b_1,\ldots,b_m)$
if and only if
$$v(a)\frac{u(a)}{u(a)+u(a_1)+\cdots+u(a_n)}\geqslant v(b)\frac{u(b)}{u(b)+u(b_1)+\cdots+u(b_m)}.$$

Axioms 2 and 3 are necessary conditions for the existence of a basic belief representation for $\succsim$—as is easy to verify directly through Definition 2.3. Similarly, direct verification shows that the following three axioms are also necessary for the existence of a basic belief representation for $\succsim$:

Axiom 4. *For all a, b, c, e, and f in X, if*
$a\neq b$, $a\neq c$, $(e|e,a)\sim(e|e,b)$, *and* $(f|f,a)\sim(f|f,c)$,
then $\{a,b\}\sim_{\mathscr{C}}\{a,c\}$.

Axiom 5. *For all a, a', b, b' in X and all A, A', B, B' in $\mathscr{C}$, if $a\sim_X a'$, $b\sim_X b'$, and $A\cap A'=B\cap B'=\emptyset$, then the following two statements are true*:

1. *If* $(a|A)\succsim(b|B)$ *and* $(a'|A')\succsim(b'|B')$, *then* $(a|A\cup A')\succsim(b|B\cup B')$.
2. *If* $(a|A)\succ(b|B)$ *and* $(a'|A')\succsim(b'|B')$, *then* $(a|A\cup A')\succ(b|B\cup B')$.

Axiom 6. *Suppose a, b are arbitrary elements of X and A, B are arbitrary elements of $\mathscr{C}$ and $a\in A$ and $b\in B$. Then the following two statements are true*:

1. *If $A\sim_{\mathscr{C}} B$ then*
$a\succsim_X b$ *iff* $(a|A)\succsim(b|B)$.

2. *If $a\sim_X b$ then*
$A\succsim_{\mathscr{C}} B$ *iff* $(b|B)\succsim(a|A)$.

Axiom 5 in the presence of the other axioms corresponds to a well-investigated axiom of measurement theory known as "distributivity." Axiom 6 corresponds to another well–investigated axiom of measurement theory known as "monotonicity."

To obtain strong results about basic belief representations for $\succsim$, other axioms are needed. The following ones imply that the situation under consideration is rich in objects and contexts.

Axiom 7. *For all $a \in X$ and $B \in \mathscr{C}$, if $B \succsim_{\mathscr{C}} A$ for some A in $\mathscr{C}$ such that $a \in A$, then there exist $c \in X$ and $C \in \mathscr{C}$ such that $a \sim_X c$, $B \sim_{\mathscr{C}} C$, and $c \in C$.*

Axiom 8. *For all a, b in X and all A in $\mathscr{C}$, if $a \in A$ and $a \succsim_X b$, then there exists c in X and C in $\mathscr{C}$ such that $b \sim_X c$, $A \sim_{\mathscr{C}} C$, and $c \in C$.*

Axiom 9. *For all A, B in $\mathscr{C}$ and all $b \in B$, there exist $c \in X$ and C in $\mathscr{C}$ such that $b \sim_X c$, $B \sim_{\mathscr{C}} C$, $A \cap C = \emptyset$ and $(b|B) \sim (c|C)$.*

Axiom 10. *For each $A = \{a_1, \ldots, a_n\}$ in $\mathscr{C}$ there exist $A' = \{a'_1, \ldots, a'_n\}$ in $\mathscr{C}$ and e in X such that $A \cap A' = \emptyset$, $\{e\} \cap (A \cup A') = \emptyset$, and for $i = 1, \ldots, n$,*

$$(e|e, a_i) \sim (e|e, a'_i).$$

Axiom 11. *For all A, B in $\mathscr{C}$, if $A \succ_{\mathscr{C}} B$ then there exists C in $\mathscr{C}$ such that $B \cap C = \emptyset$ and $A \succ_{\mathscr{C}} B \cup C$.*

Axiom 12. *The following two statements are true*:

1. *For all a, b in X and all A in $\mathscr{C}$, if $a \in A$ and $b \succsim_X a$, then there exist c and C such that*
   $$c \sim_X b \quad \text{and} \quad (c|C) \sim (a|A).$$
2. *For all a in X and all A, B in $\mathscr{C}$, if $a \in A$, then there exist c and C such that*
   $$C \sim_{\mathscr{C}} B \quad \text{and} \quad (c|C) \sim (a|A).$$

Axiom 13 (Archimedean axiom). *For all $A, B, B_1, \ldots, B_i, \ldots$ in $\mathscr{C}$, if for all distinct i,j in $\mathbb{I}^+$ $B_i \cap B_j = \emptyset$ and $B_i \sim_{\mathscr{C}} B$, then for some $n \in \mathbb{I}^+$*

$$\bigcup_{k=1}^{n} B_k \succsim_C A.$$

Definition 2.4. Axioms 1–13 are called the *Basic Belief Axioms.*

Theorem 2.1. *Assume the basic belief axioms* (*Definition* 2.4) *are true. Then the following two statements hold*:

1. (Representation theorem) *There exists a basic belief representation for* $\succsim$ (*Definition* 2.3).
2. (Uniqueness theorem) *Let*
   $\mathscr{U} = \{u |$ *there exists v such that $\langle u, v \rangle$ is a basic belief representation for* $\succsim\}$
3. *and*
   $\mathscr{V} = \{v |$ *there exists u such that $\langle u, v \rangle$ is a basic belief representation for* $\succsim\}$.
4. *Then $\mathscr{U}$ and $\mathscr{V}$ are ratio scales.*

Proof. Similar to Theorem 8.2. □

2.2. Basic belief axioms with binary symmetry

Axiom 14 (Binary symmetry). *Let a, b, c, and d be arbitrary, distinct elements of X such that*

$$(a|a, b) \sim (b|a, b) \quad \text{and} \quad (c|c, d) \sim (d|c, d).$$

Then

$$(a|a, b) \sim (c|c, d)$$

and

$$(a|a, c) \sim (b|b, d).$$

Definition 2.5. The *basic belief axioms with binary symmetry* consists of the basic belief axioms together with the axiom of binary symmetry (Axiom 14).

Definition 2.6. A function u is said to be a *basic choice representation* for $\succsim$ if and only if the following four conditions hold:

1. u is a function from X into $\mathbb{R}^+$.
2. For all a, b in X, $a \succsim_X b$ iff $u(a) \geqslant u(b)$.
3. For all C, D in $\mathscr{C}$,
   $$C \succsim_{\mathscr{C}} D \quad \text{iff} \quad \sum_{e \in C} u(e) \geqslant \sum_{e \in D} u(e).$$
4. For all distinct $a, a_1, \ldots, a_n$ and all distinct $b, b_1, \ldots, b_m$,
   $$(a|a, a_1, \ldots, a_n) \succsim (b|b, b_1, \ldots, b_m)$$
5. if and only if
   $$\frac{u(a)}{u(a) + u(a_1) + \cdots + u(a_n)} \geqslant \frac{u(b)}{u(b) + u(b_1) + \cdots + u(b_m)}.$$

Theorem 2.2. *Assume the basic belief axioms with binary symmetry* (*Definition* 2.5). *Then the following two statements hold*:

1. (Representation theorem) *There exists a basic choice representation for* $\succsim$ (*Definition* 2.6).

2. (Uniqueness theorem) *The set of basic choice representations for* $\succsim$ *forms a ratio scale.*

Proof. Similar to Theorem 8.3. □

2.3. Comments

For intuitive purposes, the axioms presented throughout this article may be divided into three rough categories: (i) *substantive axioms* that reveal important structural relationships about conditional probability and belief; (ii) *richness axioms* that guarantee that we are dealing with rich probabilistic and belief situations; and *other axioms* that belong to neither categories (i) nor (ii). The role of the substantive axioms is to describe what conditional probability and belief are in rich settings; and the role of the richness and other axioms is to guarantee that such a description can be made easily and will work. Examples of richness axioms are Axioms 7–12 that state the existence of certain kinds of solvability relations in terms of $\sim$ and $\succ$. The Archimedean axiom (Axiom 13) is an example of an "other axiom." Throughout this article, richness axioms are freely employed to simplify exposition. In some instances this results in redundancy in the axioms. However, having nonnecessary and sometimes redundant or unneeded axioms does not impede the main objectives of the article—to formulate and evaluate generalizations of conditional probability in terms of normativeness, mathematical power, and applicability. The substantive axioms thus far are Axioms 1–6 and binary symmetry (Axiom 14).

As Theorem 2.2 shows, the basic belief axioms with binary symmetry qualitatively describes a situation of conditional probability. Taking the perspective that conditional belief may differ from conditional probability, it then follows that there may be situations in which one or more of the basic belief axioms with binary symmetry may fail to adequately characterize conditional belief. The basic belief axioms assume such a possible failure, namely the failure of binary symmetry (Axiom 14). That binary symmetry should be invalid in certain kinds of belief and choice situations has been repeatedly suggested in the literature.

The objective of this article is to give and evaluate extensions of the basic belief axioms as theoretical alternatives to conditional probability. Of course, such extensions are likely at best to produce only partial theories of conditional belief, because only binary symmetry (Axiom 14) is deleted. To obtain general theories of belief, other axioms will likely have to be changed as well. Thus, the theories of conditional belief of this paper should be viewed and evaluated as particular generalizations of conditional probability which apply to a restricted set of belief situations. Because in many ways these theories can be viewed as minimal generalizations that result from elimination of an obvious questionable principle for conditional belief (binary symmetry), they are ideal candidates for preliminary investigation.

2.4. The BTL model of choice

The basic belief axioms with binary symmetry (Definition 2.5) provide (via Theorem 2.2) a qualitative description of a widely used, important quantitative model in the behavioral sciences called the "BTL Model:"

Much behavioral science research involves the modeling of the probabilistic choice of objects from a set of alternatives. A particularly important model is one in which objects are assigned positive numbers by a function u so that the probability p that object a is chosen from the set of alternatives $\{a|a_1, \ldots, a_n\}$ is given by the equation,

$$p = \frac{u(a)}{u(a) + u(a_1) + \cdots + u(a_n)}. \tag{5}$$

In the literature, this model is often called the *Bradley–Terry–Luce model*, which is often abbreviated to the *BTL model*.[1]

There are many ways in which the ordering $\succsim$ on choice strengths of objects in contexts can be established empirically. In particular, by letting "$(a|A) \succsim (b|B)$" stand for "The conditional probability of a being chosen from A is at least as great as the conditional probability of b being chosen from B," a model of the basic belief axioms with binary symmetry results. However, other interpretations can be given to $\succsim$ that also yield the basic belief axioms with binary symmetry. The added flexibility of multiple interpretations of primitives is one of the great strengths of the qualitative approach to axiomatization. This extends to the basic belief axioms. For example, the characteristic property of the BTL model of choice is that context plays no role in determining the odds of alternatives (Luce's Choice Axiom). Smith and Yu (1982) formulate a quantitative generalization where the odds of alternatives may vary with the contexts in which they occur. In this generalization, the function

$$\frac{\mathbb{P}(x|C)}{\eta(x)},$$

[1] A special case of the BLT model,

$$p = \frac{u(a)}{u(a) + u(a_1)}. \tag{6}$$

was used by the famous set-theorist E. Zermelo to describe the power of chess players (Zermelo, 1929). The choice models implicit in Eqs. (5) and (6) have been used by Bradley and Terry (1952), Luce (1959), and many others in behavioral applications.

which is mathematically equivalent to $\mathbb{P}(x|C)v(x)$, is used as a measure of the "context sensitivity" of x in context C. ($\mathbb{P}(x|C)$ denotes the probability of x given C.) Thus, by Theorem 2.1, the basic belief axioms provide a qualitative account of "context sensitivity." Smith and Yu apply their theory to choice situations in which there is a similarity structure on the alternatives.

3. Belief axioms

Definition 3.1. Let

$\mathscr{F} = \{F|F$ is a finite subset of $X\}$.

Elements of $\mathscr{F}$ are called *finite events* (*of X*). Elements of $\mathscr{F} - \{\emptyset\}$ are called *context events* (*of X*). In the notation "$(A|B)$," where A is in $\mathscr{F}$, B is in $\mathscr{F} - \{\emptyset\}$, and $A \subseteq B$, A is called the *focal event* of $(A|B)$ and B is called the *context event* of $(A|B)$. By definition, the *finite conditional events* (*of X*) consists of all $(A|B)$ where $A \in \mathscr{F}$, B in $\mathscr{F} - \{\emptyset\}$, and $A \subseteq B$. (In terms of the earlier notation, in "$(a|a,b)$," a is focal event $\{a\}$ and a, b is the context event $\{a,b\}$, and thus $(a|a,b)$ is the same as $(\{a\}|\{a,b\})$.

The basic belief axioms and basic belief axioms with binary symmetry are concerned with finite conditional events $(A|B)$, where B has at least two elements and A has the special form $A = \{b\}$ for some b in B. This section extends these axiom systems to all finite conditional events. To accomplish this, a new primitive relation $\succsim_{\mathscr{E}}$ is introduced and additional axioms involving $\succsim_{\mathscr{E}}$ are assumed.

Axiom 15. *The following two statements are true*:

(1) $\succsim_{\mathscr{E}}$ *is a weak order on the set of finite conditional events of X.*
(2) $\succsim_{\mathscr{E}}$ *is an extension of* $\succsim$ (*where* $\succsim$ *is as in Axiom* 1).

Finite conditional events of the forms $(\emptyset|B)$, $(A|B)$ with A having at least two elements, and $(B|B)$ are not involved in the $\succsim$-ordering. The following four axioms specify, with respect to the $\succsim$-ordering, the placement within the $\succsim_{\mathscr{E}}$-ordering of these three kinds of events.

Axiom 16. *The following two statements are true for all finite conditional events* $(A|B)$ *of X and all context events C of X*:

(1) $(A|B) \sim_{\mathscr{E}} (\emptyset|C)$ *iff* $A = \emptyset$.

(2) *if* $A \neq \emptyset$ *Then* $(A|B) \succ_{\mathscr{E}} (\emptyset|C)$.

Axiom 17. *For each nonempty finite event A of X there exists e in X − A such that*

(1) *for each finite event B of X, if* $\emptyset \subset A \subset B$ (*and therefore B has at least two elements*), *then there exists E in* $\mathscr{C}$ *such that* $B \sim_{\mathscr{C}} E$ (*Definition* 2.2) *and* $(A|B) \sim_{\mathscr{E}} (e|E)$; *and*

(2) *there exists f in X such that* $f \neq e$, $f \notin A$, *and* $(f|e,f) \sim_{\mathscr{E}} (f|A \cup \{f\})$.

Axiom 18. *Suppose* $\emptyset \subset A \subset B$, $\emptyset \subset A \subset B'$, $e \in E$, $e' \in E'$, $B \sim_{\mathscr{C}} E$, $B' \sim_{\mathscr{C}} E'$, *and*

$(A|B) \sim_{\mathscr{E}} (e|E)$ *and* $(A|B') \sim_{\mathscr{E}} (e'|E')$.

Then

$(e|e,e') \sim (e'|e,e')$.

Axiom 17 associates with each finite conditional event $(A|B)$, with $A \neq \emptyset$ and B having at least two elements, a finite conditional event of the form $(e|E)$, where $e \in X$ and $E \in \mathscr{C}$, such that

$(A|B) \sim_{\mathscr{E}} (e|E)$.

Since $\sim_{\mathscr{E}}$ is an extension of $\succsim$, this allows the placement of $(A|B)$ in the $\succsim_{\mathscr{E}}$-ordering to be determined by the placement of $(e|E)$ in the $\succsim$-ordering.

Axioms 17 and 18 play a critical role in extending a basic belief representation $\langle u, v \rangle$ for $\succsim$ to elements of the domain of $\succsim_{\mathscr{E}}$. ($u(A)$ is defined to be $u(e)$ and $v(A)$ is defined to be $v(e)$, where e is as in condition (1) of Axiom 17.)

Axiom 19. *The following two statements are true*:

1. *For each a in X there exists E in* $\mathscr{C}$ *such that* $(\{a\}|\{a\}) \sim_{\mathscr{E}} (E|E)$.

2. *Let* $(A|B)$ *and* $(C|D)$ *be arbitrary finite conditional events such that B and D are in* $\mathscr{C}$. *Then there exist finite events B′ and D′ such that* $B \cap B' = \emptyset$, $B \sim_{\mathscr{C}} B'$, $D \cap D' = \emptyset$, $D \sim_{\mathscr{C}} D'$, *and* $(A|B) \succsim_{\mathscr{E}} (C|D)$ *iff* $(A|B \cup B') \succsim_{\mathscr{E}} (C|D \cup D')$.

Definition 3.2. The *belief axioms* consist of the basic belief axioms (Definition 2.4) together with Axioms 15–19.

Definition 3.3. $\mathbb{B}$ is said to be a *belief representation for* $\succsim_{\mathscr{E}}$ *with context function u and definiteness function v* if and only if the following three conditions hold:

(i) u and v are functions from $\mathscr{F}$ into, respectively, $\mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}^+$.
(ii) For each C in $\mathscr{F}$,

$$C = \emptyset \quad \text{iff} \quad u(C) = 0,$$

and

$$\text{if } C \neq \emptyset, \text{ then } u(C) = \sum_{c \in C} u(\{c\}).$$

(iii) $\mathbb{B}$ is a function on the finite conditional events of X such that for all finite conditional events $(A|B)$ and $(C|D)$ of X,

$$(A|B) \succsim_E (C|D) \quad \text{iff } \mathbb{B}(A|B) \geqslant \mathbb{B}(C|D)$$

and

$$\mathbb{B}(A|B) = v(A)\frac{u(A)}{u(B)}.$$

Belief representations for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v are useful in applications where one wants to characterize events in terms of a probabilistic dimension and another dimension. For the purposes of this article, the other dimension is called "definiteness." Definiteness may split into additional dimensions. In the intended interpretations, u measures the probabilistic dimension and v measures definiteness. One naturally encounters various kinds of "definiteness," and the intended interpretations of v may vary with the kinds of definiteness.

Consider the example of evaluating evidence in criminal cases. Here we assume that "guilty beyond a reasonable doubt" is determined by having a sufficiently high "degree of belief." Consider the following two situations: (1) where the evidence is purely circumstantial, and (2) where the evidence is almost entirely physical. Suppose the subjective probabilities for the two situations are the same, e.g., 0.998. I believe it is reasonable in this case to assign (2) a higher degree of belief than (1). This reflects the idea that degree of belief depends not only on probabilities, but also on the kinds of evidence that the probabilities are based on. A particular version of this idea is presented in the concept of "belief representation of $\succsim_{\mathscr{E}}$ with context function u and definiteness function v."

Definiteness may also be viewed as an ordinal opposite of the following "ambiguity" concept of Ellsberg (1961, pp. 659–660):

> Let us assume, for purposes of discussion, that an individual can always assign relative weights to alternative probability distributions reflecting the relative support given by his information, experience and intuition to these rival hypotheses. This implies that he can always assign relative likelihoods to the states of nature. But how does he *act* in the presence of his uncertainty? The answer to that may depend on another sort of judgment, about the reliability, credibility, or adequacy of his information (including his relevant experience, advice and intuition) as a whole: not about the relative support it may give to one hypothesis as opposed to another, but about its ability to lend support to any hypothesis at all.
>
> If all the information about the events in a set of gambles were in the form of sample-distributions, the ambiguity might be closely related, inversely to the size of the sample. But sample-size is not a universally useful index of this factor. Information about many events cannot be conveniently described in terms of a sample distribution; moreover, sample-size seems to focus mainly on the quantity of information. "Ambiguity" may be high (and the confidence in any particular estimate of probabilities low) even where there is ample quantity of information, when there questions of reliability and relevance of information, and particularly where there is *conflicting* opinion and evidence.
>
> This judgment of the ambiguity of one's information, of the over-all credibility of one's composite estimates, of one's confidence in them, cannot be expressed in terms of relative likelihoods or events (if it could, it would simply affect the final, compound probabilities). Any scrap of evidence bearing on relative likelihood should already be represented in those estimates. But having exploited knowledge, guess, rumor, assumption, advice to arrive at a final judgment that one event is more likely than another or that they are equally likely, one can still stand back from this process and ask: "How much, in the end, is all this worth? How much do I really know about the problem? How firm a basis for choice, for appropriate decision and action, do I have?" The answer, "I don't know very much, and I can't rely on that," may sound familiar, even in connection with markedly unequal estimates of relative likelihood. If "complete ignorance" is rare or nonexistent, "considerable" ignorance is surely not.

The next theorem, which extends Theorem 2.1 from singleton focal events to finite events, provides existence and uniqueness results concerning belief representations for $\succsim_{\mathscr{E}}$.

Theorem 3.1. *Assume the belief axioms (Definition 3.2). Then the following two statements are true:*

1. *There exists a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v.*
2. *Let $\mathbb{B}$ be a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v. Then the*

following two statements are true:

(i) *For all positive reals r and s there exists a belief representation for* $\succsim_{\mathscr{E}}$ *with context function ru and definiteness function sv.*

(ii) *Let* $\mathbb{B}_1$ *be a belief representation for* $\succsim_{\mathscr{E}}$ *with context function* u_1 *and definiteness function* v_1*. Then for some positive real numbers r and s,*

$$u_1 = ru \quad \textit{and} \quad v_1 = sv.$$

Proof. Similar to Theorem 8.4. □

Let u be as in statement 1 of Theorem 3.1, and for finite events E and F of X with $E \subset F$ let

$$P_F(E) = \frac{u(E)}{u(F)}.$$

Then the following theorem, which is a restatement of Theorem 3.1 in terms of belief functions (Definition 1.1), is an immediate consequence of Theorem 3.1.

Definition 3.4. Suppose $\mathbb{B}$ is a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v (Definition 3.3). Then for all finite conditional events $(A|B)$,

$$\mathbb{B}(A|B) = v(A)\frac{u(A)}{\sum_{b \in B} u(b)}. \tag{7}$$

In the *definiteness interpretation* of Eq. (7), u is interpreted as a measure of probabilistic strength, and v is interpreted as a measure of something that is the opposite of ambiguity or vagueness, which is called *definiteness.* With these interpretations in mind, the right-hand side of Eq. (7) is interpreted as a subjective probability $\mathbb{P}(A|B)$ of A occurring when B is presented, where

$$\mathbb{P}(A|B) = \frac{u(A)}{\sum_{b \in B} u(b)},$$

weighted by the definiteness factor $v(A)$, i.e.,

$$\mathbb{B}(A|B) = v(A)\mathbb{P}(A|B). \tag{8}$$

Then when Eq. (8) is used to interpret $\mathbb{B}$, $\mathbb{B}$ is called a *definiteness representation for* $\succsim_{\mathscr{E}}$ *with probability function* $\mathbb{P}$ *and definiteness function v.*

4. Belief axioms with binary symmetry

Definition 4.1. $\mathbb{B}$ is said to be a *choice representation for* $\succsim_{\mathscr{E}}$ (*with support u*) if and only if $\mathbb{B}$ is a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v and for all finite events $(A|B)$ of X,

$$\mathbb{B}(A|B) = \frac{u(A)}{u(B)}.$$

Note that by Definition 4.1, each choice representation for $\succsim_{\mathscr{E}}$ is a conditional probability function on the finite conditional events of X.

Definition 4.2. The *belief axioms with binary symmetry* consist of the belief axioms together with the axiom of binary symmetry (Axiom 14).

Theorem 4.1. *Assume the belief axioms with binary symmetry* (*Definition* 4.2). *Then there exists a choice representation for* $\succsim_{\mathscr{E}}$.

Proof. Let $\langle u, v \rangle$ be a belief representation for $\succsim_E$ with context function u and definiteness function v. It needs to be only shown that for all nonempty finite events A and B of X, $v(A) = v(B)$. Let A and B be arbitrary finite events. If A has at least two elements, then it follows from Axiom 17 that e_A in X can be found so that

$$u(A) = u(e_A) \quad \text{and} \quad v(A) = v(e_A).$$

If $A = \{a\}$, let $e_A = a$ and thus again,

$$u(A) = u(e_A) \quad \text{and} \quad v(A) = v(e_A).$$

Similarly an element e_B of X can be found so that

$$u(B) = u(e_B) \quad \text{and} \quad v(B) = v(e_B).$$

Thus to show $v(A) = v(B)$, it is sufficient to show $v(e_A) = v(e_B)$. But because the belief axioms with binary symmetry include the basic belief axioms with binary symmetry, it follows from Theorem 1.2 that $v(x) = v(y)$ for all x and y in X, and thus that $v(e_A) = v(e_B)$.

Definition 4.3. Let $\mathbb{B}$ be a belief representation for $\succsim_{\mathscr{E}}$ with context and definiteness functions. Then $\mathbb{B}$ is said to be *additive* if and only if for all finite conditional events $(A|C)$ and $(B|C)$ of X, if $A \cap B = \emptyset$, then

$$\mathbb{B}(A \cup B|C) = \mathbb{B}(A|C) + \mathbb{B}(B|C).$$

Additive belief representations provide a quantitative theory of belief that is very close in mathematical form to that of the probability calculus. However, in several applications involving uncertainty, additivity is often difficult to justify, and often in such applications normative or intuitive arguments can be given for nonadditivity.

There are interesting cases of additive belief representations that are not trivial variants of choice representations. Examples of these can be constructed using the following idea: Let u and w be functions from X into $\mathbb{R}^+$. Extend u and w to nonempty finite subsets A of X as follows:

$$u(A) = \sum_{a \in A} u(a) \quad \text{and} \quad w(A) = \sum_{a \in A} w(a).$$

Then in terms of u and w, the functions v and $\mathbb{B}$ are defined so that u, v, and $\mathbb{B}$ have the algebraic characteristics of a belief representation: For nonempty finite subsets A of X and conditional events $(A|B)$ of X, let

$$v(a) = \frac{w(A)}{u(A)},$$

and by the above equation, let

$$\mathbb{B}(A|B) = \frac{w(A)}{u(B)} = v(A)\frac{u(A)}{u(B)}.$$

It is easy to show that $\mathbb{B}$ is additive.

When $\mathbb{B}$ is additive, the above process can be reversed: Let $\mathbb{B}$ be an additive belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v. For each A in $\mathscr{F}$, let

$$w(A) = u(A)v(A).$$

Then for all finite conditional events $(A|B)$ of X,

$$\mathbb{B}(A|B) = \frac{w(A)}{u(B)}, \tag{9}$$

and (because $\mathbb{B}$ is additive) for all finite conditional events $(C|E)$ and $(D|E)$ of X such that $C \cap D = \emptyset$,

$$w(C \cup D) = w(C) + w(D).$$

An intended interpretation of Eq. (9) (when $\mathbb{B}$ is additive or nonadditive) is that for conditional beliefs $(A|B)$, w is a measure of the probabilistic strength of the focal event A, u is a measure of the probabilistic strength of the context event B, and $\mathbb{B}$ is the measure of the strength of belief of $(A|B)$. This interpretation is employed later in the article.

5. Belief support probability

Many probabilists and decision analysts believe that the degree of belief of event E is properly measured by a probability p, and that the same probability, p, is the proper weight to assign to E in normative models of utility under uncertainty. In the theory of belief developed here, the two different kinds of measurements of E—as (i) a degree of belief and as (ii) a weight in a model of utility under uncertainty—are kept separate, and in general, assign different values to E.

Definition 5.1. Let A and B be disjoint finite events of X such that $B \neq \emptyset$, and let $\mathbb{B}$ be a belief representation for $\succsim_{\mathscr{E}}$ with context and definiteness functions. By definition, let

$$\mathbb{O}_{\mathbb{B}}(A, B) = \frac{\mathbb{B}(A|A \cup B)}{\mathbb{B}(B|A \cup B)}.$$

$\mathbb{O}_{\mathbb{B}}(A, B)$ is called the *belief support odds* (*induced by* $\mathbb{B}$) of A over B. By definition, let $P_{\mathbb{B}}$ be a function on finite conditional events of X such that for each finite conditional event $(E|F)$ of X,

$$P_{\mathbb{B}}(E|F) = \frac{\mathbb{B}(E|F)}{\mathbb{B}(E|F) + \mathbb{B}(F - E|F)}.$$

$P_{\mathbb{B}}$ is called the *belief support probability function* (*induced by* $\mathbb{B}$). $P_{\mathbb{B}}(E|F)$ is called the *belief support probability* of $(E|F)$.

Note by part (ii) of statement 2 of Theorem 3.1, that the definitions of $\mathbb{O}_{\mathbb{B}}$ and $P_{\mathbb{B}}$ in Definition 5.1 are independent of the choice of $\mathbb{B}$, i.e., if $\mathbb{B}'$ is another belief representation with context and definiteness functions, then $\mathbb{O}_{\mathbb{B}} = \mathbb{O}_{\mathbb{B}'}$ and $P_{\mathbb{B}} = P_{\mathbb{B}'}$.

Also note that $P_{\mathbb{B}}$ behaves like a probability function in that for $A \cap B = \emptyset$,

$$P_{\mathbb{B}}(A|A \cup B) + P_{\mathbb{B}}(B|A \cup B) = 1.$$

However, unlike a probability function, $P_{\mathbb{B}}$ may not be *additive*, i.e., situations with finite events C, D, and E can be found such that $C \cap D = \emptyset$ and

$$P_{\mathbb{B}}(C \cup D|E) \neq P_{\mathbb{B}}(C|E) + P_{\mathbb{B}}(D|E).$$

The notion of a "fair bet" relates the strengths of beliefs of the events in the bet to the value of the outcomes of the event. The following is one reasonable notion of "fair bet:"

Definition 5.2. Let $\mathbb{B}$ be an individual's belief representation for $\succsim$ with context and definiteness functions, and A and B be nonempty finite events such that $A \cap B = \emptyset$. Consider the gamble of gaining something that has value $a > 0$ to the individual if A occurs and losing something that has value $b > 0$ to the individual if B occurs. Call this gamble a *belief odds fair bet* for this individual if and only if

$$\mathbb{O}_{\mathbb{B}}(A, B) = \frac{b}{a}.$$

Note that in terms of the belief support probability function, the above gamble is a belief odds fair bet if and only if

$$aP_{\mathbb{B}}(A|A \cup B) - bP_{\mathbb{B}}(B|A \cup B) = 0.$$

The formulation of "belief odds fair bet" starts with degrees of belief for conditional events. When these conditional events occur in evaluation of gambles, other notions based on degrees of belief are needed to capture key concepts inherent in gambling such as a "belief odds fair bet." For "belief odds fair bet" this is accomplished very naturally in terms of belief odds. But it is also accomplished in a logically equivalent and natural manner through belief support probabilities. For a task where an individual is asked to judge numerically the probabilities of conditional events, both $\mathbb{B}$ and $P_{\mathbb{B}}$ are natural candidates for modeling the judgments. Empirical results discussed in Section 6 suggest that $P_{\mathbb{B}}$ is better in this regard.

Assume $\mathbb{B}$ is a belief representation for $\succsim$ with context function u and definiteness function v. Let A, B, C, and D be nonempty finite events such that

$$(A|A\cup B)\sim_{\mathscr{E}}(B|A\cup B) \quad \text{and} \\ (C|C\cup D)\sim_{\mathscr{E}}(D|C\cup D). \tag{10}$$

Assume that a high definiteness value, say 1, is assigned to A and B because much is known about them; and assume a low definiteness value, $\alpha<1$, is assigned to C and D, because little is known about them. Then by Eq. (10),

$$u(A)=u(B) \quad \text{and} \quad u(C)=u(D).$$

Thus,

$$\mathbb{B}(A|A\cup B)=\mathbb{B}(B|A\cup B)=\tfrac{1}{2} \quad \text{and} \\ \mathbb{B}(C|C\cup D)=\mathbb{B}(D|C\cup D)=\tfrac{1}{2}\alpha.$$

Therefore,

$$P_{\mathbb{B}}(A|A\cup B)=P_{\mathbb{B}}(C|C\cup D)=\tfrac{1}{2}.$$

Thus, although the conditional events $(A|B)$ and $(C|D)$ differ in degree of belief and definiteness, they are given the same value by the belief support probability function $\mathbb{P}_{\mathbb{B}}$. This value is the same as $\mathbb{P}(A|A\cup B)=\mathbb{P}(C|C\cup D)$, when $\mathbb{B}$ is interpreted as a definiteness representation for $\succsim_{\mathscr{E}}$ with probability function $\mathbb{P}$ and definiteness function v (Definition 3.4).

In general for conditional events $(E|F)$ with $v(E)=v(F)$,

$$P_{\mathbb{B}}(E|F)=\mathbb{P}(E|F).$$

Thus, to interestingly differentiate $P_{\mathbb{B}}$ from $\mathbb{P}$ (and therefore, from $\mathbb{B}$), one needs to consider situations where the definitenesses of the focal events differ from the definitenesses of their context events.

Such a situation is suggested in Ellsberg (1961). Suppose an urn has 90 balls that have been thoroughly mixed. Each ball is of one of the three colors, red, blue, or yellow. There are thirty red balls, but the number of blue balls and the number of yellow balls are unknown except that together they total 60. A ball is to be randomly chosen from the urn. Let R be the event that a red ball is chosen, B the event a blue ball is chosen, and Y the event a yellow ball is chosen. Let $U=R\cup B\cup Y$. Assume that this situation is part of the domain of a definiteness representation $\mathbb{B}$ with probability function $\mathbb{P}$ and definiteness function v (Definition 3.4). For this situation I consider the following to be reasonable probability and definiteness assignments:

$$\mathbb{P}(R|U)=\mathbb{P}(B|U)=\mathbb{P}(Y|U)=\tfrac{1}{3}, \\ \mathbb{P}(R\cup Y|U)=\mathbb{P}(B\cup Y|U)=\mathbb{P}(R\cup B|U)=\tfrac{2}{3},$$

and

$$v(B)=v(Y)<v(R\cup Y)=v(R\cup B)<v(B\cup Y)=v(R).$$

Then it is easy to verify that the following three statements are true:

1. $\mathbb{B}\neq\mathbb{P}$,
2. $\mathbb{B}(B|U)<\mathbb{B}(R|U)$ and $\mathbb{B}(B\cup Y)|U)>\mathbb{B}(R\cup Y|U)$.
3. $P_{\mathbb{B}}(B|U)<P_{\mathbb{B}}(R|U)$ and $P_{\mathbb{B}}(B\cup Y)|U)<P_{\mathbb{B}}(R\cup Y|U)$.

Note that by statement 2, $\mathbb{B}$ is not additive (Definition 4.3). Similarly, by statement 3 $P_{\mathbb{B}}$ is not additive.

The second inequality of statement 3 yields the following conclusion for belief odds fair bets: Suppose U is presented. Consider the gamble

$$g_1=(a,B\cup Y;-b,R)$$

of gaining something that has value $a>0$ to an individual if $B\cup Y$ occurs and losing something that has value $b>0$ to the individual if R occurs. Then for this individual this gamble is a belief odds fair bet if and only if

$$b=2a.$$

Similarly, consider the gamble

$$g_2=(a,R\cup Y;-c,B)$$

of gaining something that has value $a>0$ to the individual if $R\cup Y$ occurs and losing something that has value $c>0$ to the individual if B occurs. Then the latter gamble is a fair bet if and only if

$$c=2\frac{v(R\cup Y)}{v(B)}a>2a.$$

This result makes intuitive sense for "ambiguity adverse" individuals: If B had less ambiguity (and therefore more definiteness) to the extent that $v(B)=v(R\cup Y)$, then intuitively, c would equal $2a$. Therefore, to the extent that $v(B)<v(R\cup Y)$, c is greater than $2a$. It is reasonable to suppose that ambiguity adverse individuals would prefer g_1 to g_2.

The upshot of the above is that even though g_1 and g_2 are belief odds fair bets, g_1 is preferable to g_2 by ambiguity adverse individuals. This implies that in general, the values of g_1 and g_2 should not be computed by the formulas,

$$aP_{\mathbb{B}}(B\cup Y|U)-bP_{\mathbb{B}}(R|U) \quad \text{and} \\ aP_{\mathbb{B}}(R\cup Y|U)-bP_{\mathbb{B}}(B|U),$$

because the first formula would then yield g_1 as having value 0 and the second formula g_2 as having value 0, thus implying that the individual would be indifferent between g_1 and g_2.

6. Support theory

6.1. Tversky's and Koehler's theory

Belief support probability provides a theory of subjective probability judgment that is similar in many respects to the "support theory" of Tversky and Koehler (1994).

Support theory is a descriptive theory in which probability judgments are assigned to descriptions of events, called *hypotheses*, instead of to events. It is assumed that there is a finite set T with at least two elements that generates an event space and a set of hypotheses $\mathscr{H}$ such that each hypothesis A in $\mathscr{H}$ describes an event, called the *extension of* A and denoted by A', that is a subset of T. It is allowed that different hypotheses describe the same event, e.g., for a roll of a pair of dice, the hypotheses "the sum is 3" and "the product is 2" describe the same event, namely one die shows 1 and the other 2.

Hypotheses that describe an event $\{t\}$, where $t \in T$, are called *elementary*; those that describe $\emptyset$ are called *null*; and nonnull hypotheses A and B in $\mathscr{H}$ such that the conjunction of A and B describe $\emptyset$ are called *exclusive* (*with respect to* $\mathscr{H}$). C in $\mathscr{H}$ is said to be an *explicit disjunction* (*with respect to* $\mathscr{H}$)—or for short, an *explicit hypotheses* (*of* $\mathscr{H}$)—if and only if there are exclusive A and B in $\mathscr{H}$ such that $C = A \vee B$, where "$\vee$" stands for the logical disjunction of A and B, "A or B." D in $\mathscr{H}$ is said to be *implicit* (*with respect to* $\mathscr{H}$) if and only if D is not $\emptyset$, is not elementary, and is not explicit with respect to $\mathscr{H}$.

$\mathscr{H}$ may have implicit and explicit hypotheses that describe the same event, e.g., C: "Ann majors in a natural science," A: "Ann majors in a biological science," and B: "Ann majors in a physical science." Then C and $A \vee B$ describe the same event, i.e., have the same *extension*, or letting H' stand for the extension of a hypothesis H,

$$C' = (A \vee B)' = A' \cup B'.$$

It is assumed that whenever exclusive A and B belong to $\mathscr{H}$, then their disjunction $A \vee B$ also belong to $\mathscr{H}$.

Tversky and Koehler (1994) provide empirical data for many situations in which subjects judge explicit hypotheses E to be more likely than implicit ones I with same extensions ($E' = I'$). They suggest that this empirical result reflects a basic principle of human judgment. They explain it in terms of an intuitive theory of information processing involving (i) the formation of a "global impression that is based primarily on the most representative or available cases" and modulated by factors such as memory and attention, and (ii) the making of judgments that are mediated by heuristics such as representativeness, availability, and anchoring and adjusting.

Formally, support theory is formulated in terms of "evaluation frames" and "support functions:"

> An *evaluation frame* (A, B) consists of a pair of exclusive hypotheses: the first element A is the focal hypothesis that the judge evaluates, and the second element B is the alternative hypothesis. We assume that when A and B are exclusive the judge perceives them as such, but we do not assume that the judge can list all the constituents of an implicit disjunction. Thus, the judge recognizes the fact that "biological sciences" and "physical sciences" are disjoint categories, but he or she may be unable to list all their disciplines. This is a form of bounded rationality; we assume recognition of exclusivity, but not perfect recall.
>
> We interpret a person's probability judgment as a mapping P from an evaluation frame to the unit interval. To simplify matters we assume that $P(A, B)$ equals 0 if and only if A is null and it equals 1 if and only if B is null; we assume that A and B are not both null. Thus, $P(A, B)$ is the judged probability that A rather than B holds, assuming that one and only one of them is valid. …
>
> Support theory assumes that there is a ratio scale s (interpreted as degree of support) that assigns to each hypothesis in $\mathscr{H}$ a nonnegative real number such that for any pair of exclusive hypotheses A, B in $\mathscr{H}$,
>
> $$P(A, B) = \frac{s(A)}{s(A) + s(B)}, \qquad (11)$$
>
> [and] for all hypotheses A, B, and C in $\mathscr{H}$, if B and C are exclusive, A is implicit, and $A' = (B \vee C)'$, then
>
> $$s(A) \leqslant s(B \vee C) = s(B) + s(C). \qquad (12)$$
>
> (Quoted from preprint of Tversky and Koehler, 1994).

Tversky and Koehler show that conditions 11 and 12 above imply the following four principles for all A, B, C, and D in $\mathscr{H}$:

1. *Binary complementarity*. $P(A, B) + P(B, A) = 1$.
2. *Proportionality*. If A, B, and C are mutually exclusive and B is not null, then
$$\frac{P(A, B)}{P(B, A)} = \frac{P(A, B \vee C)}{P(B, A \vee C)}.$$
3. *Product rule*. Let $R(A, B)$ be the odds of A against B, i.e, let
$$R(A, B) = \frac{P(A, B)}{P(B, A)}.$$

Then

$$R(A,B)R(C,D) = R(A,D)R(C,B), \tag{13}$$

provided A, B, C, D are not null, and the four pairs of hypotheses in Eq. (13) are pairwise exclusive.

4. *Unpacking principle*. Suppose B, C, and D are mutually exclusive, A is implicit, and $A' = (B \vee C)'$. Then

$$P(A,D) \leqslant P(B \vee C, D) = P(B, C \vee D) + P(C, B \vee D).$$

Tversky and Koehler show the following theorem:

Theorem 6.1. *Suppose $P(A,B)$ is defined for all exclusive $A, B \in \mathscr{H}$, and that it vanishes if and only if A is null. Then binary complementarity, proportionality, the product rule, and the unpacking principle hold if and only if there exists a ratio scale s on $\mathscr{H}$ that satisfies Eqs.* (11) *and* (12).

Proof. Theorem 1 of Tversky and Koehler (1994). □

Belief-support probabilities can account for phenomena that are the basis of Koehler's and Tversky's support theory. The key idea for this is to interpret "definiteness" as a measure of unpackedness. To carry this out, a few minor modifications are needed.

Instead of the set T of elementary hypotheses, an infinite set X of elementary hypotheses will be assumed. It will also be assume that each hypothesis A in $\mathscr{H}$ has an extension A' that is a finite subset of X. To avoid extra notation and extra conditions, it will be assumed that each element of $\mathscr{H}$ is nonnull. Also instead of the evaluation frame notation (A,B), the conditional hypothesis notation $(A|A \vee B)$ (where, of course, $A \vee B$ is explicit) will be employed to describe the kinds of situations that support theory is concerned with. The purpose of these changes is to make the discussion coordinate to the discussions and results given earlier in the article. They are not essential for the points made throughout this section.

Let u be a function from X into the positive reals. Extend u to $\mathscr{H}$ as follows: For each nonempty finite subset A of X, let

$$u(A) = \sum_{a \in A} u(a). \tag{14}$$

For each H in $\mathscr{H}$, let $u(H) = u(H')$ (where, of course, H' is the extension of H). u is to be interpreted as a measure of probabilistic strength.

Let v be a function from $\mathscr{H}$ into the positive reals such that for all A and B in $\mathscr{H}$, if A is implicit, B is explicit, and $A' = B'$, then

$$v(A) \leqslant v(B). \tag{15}$$

v is to be interpreted as a "distortion factor" due to specific kinds of cognitive processing, and Eq. (15) captures the important characteristic of the distortion that is due to "unpacking."

Let $\mathbb{B}$ be a function from conditional hypotheses to the positive reals such that for each conditional hypothesis $(A|A \vee B)$,

$$\mathbb{B}(A|A \vee B) = v(A)\frac{u(A)}{u(A \vee B)}. \tag{16}$$

Eqs. (14) and (16) give $\mathbb{B}$ the same algebraic form as the belief representations considered in Section 3. $\mathbb{B}(A|A \vee B)$ is intended to be interpreted as a distortion (by a factor of $v(A)$) of the probabilistic strength,

$$\frac{u(A)}{u(A \vee B)},$$

of $(A|A \vee B)$. The distortion of interest for the kinds of situations covered by Koehler–Tversky theory is due to the unpacking principle, which is captured in large part by Eq. (15).

Let $P_{\mathbb{B}}$ be the belief-support probability function determined by $\mathbb{B}$, i.e., let

$$P_{\mathbb{B}}(A|A \vee B) = \frac{\mathbb{B}(A|A \vee B)}{\mathbb{B}(A|A \vee B) + \mathbb{B}(B|A \vee B)}.$$

Then it is easy to verify that $P_{\mathbb{B}}$ satisfies Binary complementarity, proportionality, and the product rule. The following theorem is also immediate:

Theorem 6.2. *Suppose A and D are exclusive and B, C, and D are mutually exclusive, A is implicit, and $A' = (B \vee C)'$. Then the following three statements are true*:

1. (Ordinal unpacking) $P_{\mathbb{B}}(A|A \vee D) \leqslant P_{\mathbb{B}}(B \vee C|B \vee C \vee D)$.
2. (Definiteness unpacking) If $v(B) = v(C)$, *then*
$$P_{\mathbb{B}}(A|A \vee D) \leqslant P_{\mathbb{B}}(B \vee C|B \vee C \vee D) = P_{\mathbb{B}}(B|B \vee C \vee D) + P_{\mathbb{B}}(C|C \vee B \vee D).$$
3. *The following two statements are equivalent*:
   (i) $\mathbb{B}$ *is* additive, *i.e.*,
   $$\mathbb{B}(B \vee C|B \vee C \vee D) = \mathbb{B}(B|B \vee C \vee D) + \mathbb{B}(C)|B \vee C \vee D).$$
   (ii) *The unpacking principle holds*, *i.e.*,
   $$P_{\mathbb{B}}(A|A \vee D) \leqslant P_{\mathbb{B}}(B \vee C|B \vee C \vee D) = P_{\mathbb{B}}(B|B \vee C \vee D) + P_{\mathbb{B}}(C|C \vee B \vee D).$$

Observe that the inequalities in Theorem 6.2 become strict if the inequality in Eq. (15) becomes strict.

The above shows that belief-support probabilities, when generated by an additive $\mathbb{B}$, is a form of support theory. This form of support theory generalizes phenomena outside of support theory when generated by a nonadditive $\mathbb{B}$. The nonadditive case still retains much of the flavor of support theory, particularly its empirical basis, because ordinal unpacking and definiteness unpacking hold. Of possible empirical importance is the consideration that it may be possible to find many natural situations in which definiteness unpacking holds but the unpacking principle fails.

Let $(A|A \vee B)$ be a conditional hypothesis. Then in the formula,

$$\mathbb{B}(A|A \vee B) = v(A)\frac{u(A)}{u(A \vee B)},$$

may be viewed as "distorting" the fraction,

$$\frac{u(A)}{u(A \vee B)}.$$

By the definition of u,

$$\frac{u(A)}{u(A \vee B)} = \frac{u(A')}{u((A \vee B)')} = \frac{u(A')}{u(A' \cup B')}.$$

However, by Eq. (14),

$$\frac{u(A')}{u(A' \cup B')} = \frac{\sum_{a \in A'} u(a)}{\sum_{c \in A' \cup B'} u(c)},$$

may be interpreted as a subjective conditional probability of the conditional, extensional hypothesis $(A'|A' \cup B')$. With this interpretation in mind, $v(A)$ is the amount that the subjective conditional probability of the extension of $(A|A \vee B)$ needs to be distorted to achieve $\mathbb{B}(A|A \vee B)$.

Tversky and Koehler (1994) provided many examples of their theory. However, latter empirical studies showed their theory to be inadequate. To accomodate these additional studies, generalizations of Tversky's and Koehler's theory were developed, and two of these are discussed next.

6.2. Rottenstreich's and Tversky's support theory

Rottenstreich and Tversky (1997) noted empirical examples in which probabilities for explicit disjunctions $G \vee H$ were subadditive, i.e., empirical situations where

$$P(G \vee H) \leqslant P(G) + P(H).$$

To accommodate such subadditive situations, they provided the following generalization of Tversky and Koehler (1994): For all hypotheses E and F, where F is nonnull, let

$$R(E, F) = \frac{P(E, F)}{P(F, E)}.$$

Then the following three assumptions hold for all hypotheses A, A_1, A_2, B, C, and D:

1. (*Binary complementarity*) $P(A, B) + P(B, A) = 1$.
2. (*Product rule*) (i) If (A, B), (B, D), (A, C), and (C, D) are exclusive, then
   $R(A, B)R(B, D) = R(A, C)R(C, D)$, and
   (ii) if (A, B), (B, D), and (A, D) are exclusive, then
   $R(A, B)R(B, D) = R(A, D)$.
3. (*Odds inequality*) Suppose A_1, A_2, and B are mutually exclusive, A is implicit, and the judge recognizes $A_1 \vee A_2$ as a partition of A. That is, $(A_1 \vee A_2)' = A'$ and the judge recognizes that $A_1 \vee A_2$ has the same extension as A. Then
   $R(A, B) \leqslant R(A_1 \vee A_2, B) \leqslant R(A_1, B) + R(A_2, B)$.

Rottenstreich and Tversky (1997) show the following theorem:

Theorem 6.3. *Suppose $P(A, B)$ is defined for all exclusive hypotheses A and B and that it vanishes if and only if A is null. Then the above three assumptions hold if and only if there exists a nonnegative function s on the set of hypotheses such that for all exclusive hypotheses C and D,*

$$P(C, D) = \frac{s(C)}{s(C) + s(D)}.$$

Furthermore, if A_1 and A_2 are exclusive, A is implicit, and $(A_1 \vee A_2)$ is recognized as a partition of A, then

$$s(A) \leqslant s(A_1 \vee A_2) \leqslant s(A_1) + s(A_2).$$

Binary complementarity is the same as in Tversky's and Koehler's theory. Rottenstreich's and Tversky's product rule is slightly stronger than the product rule of Tversky and Koehler, since it contains the additional product condition $R(A, B)R(B, D) = R(A, D)$. Odds inequality is a replacement for the unpacking condition. Tversky and Rottenstreich (1997) note that in Odds inequality, "The recognition requirement, which restricts the assumption of implicit subadditivity, was not explicitly stated in the original (Tversky & Koehler, 1994) version of the theory, although it was assumed in its applications."

In Section 6.1 it is shown that belief support probabilities, when generated by an additive $\mathbb{B}$, is a form of Tversky's and Koehler's support theory. When the belief support probabilities are generated by a subadditive $\mathbb{B}$, this form of support theory naturally generalizes to a form of Rottenstreich's and Tversky's support theory, with the product rule and odds inequality holding. For nonadditive $\mathbb{B}$, much of the flavor of Tversky and Koehler's original support theory is still maintained, particularly its empirical basis, because ordinal unpacking and definiteness unpacking hold. Of possible empirical importance for Tversky's and Koehler's theory is the consideration that it may be possible to find many natural situations in which definiteness unpacking holds but the unpacking principle fails.

For subaddtive $\mathbb{B}$, the belief support probabilities add for hypotheses of the same definiteness, and thus the support form corresponding to Tversky's and Koehler's theory apply to hypotheses that have the same v-value. This suggests the following for the form of support corresponding to Rottenstreich's and Tversky's theory (i.e., for subaddtive $\mathbb{B}$): (1) The hypotheses naturally partition into families, with the hypotheses in each family having the same definiteness and hypotheses from different families having different definitenesses. (2) For each family, the support form of Tversky's and Koehler's theory holds for hypotheses from the family. And (3) the support form of Rottenstreich's and Tversky's theory holds for hypotheses from different families.

6.3. Asymmetric support theory

Recently, Brenner and Rottenstreich (manuscript) developed a version of support theory in which binary complementarity may fail. They call their theory *asymmetric support theory*, and this section presents their model and some of the examples and results contained in their manuscript.

The empirical bases for support theory are experimental studies of probability judgments that consistently find sums of probabilities greater than 1 for partitions consisting of more than two elements and sums equal to 1 for binary partitions. For example, Fox, Rogers, and Tversky (1996) asked professional option traders to judge the probability that the closing price of Microsoft stock would fall within a particular interval on a specific future date. When four disjoint intervals that spanned the set of possible prices were presented for evaluation, the sums of the assigned probabilities were typically about 1.50. However, when binary partitions were presented, the sums of the assigned probabilities were very close to 1, e.g., 0.98. Tversky and Fox also observed in probability judgments involving future temperature in San Francisco, the point-spread of selected NBA and NFL professional sports games, and many other quantities, sums of assigned probabilities greater than 1 for partitions consisting of more than two elements and sums nearly equal 1 for binary partitions. This pattern of results were also replicated by Redelmeier, Koehler, Liberman, and Tversky (1995) in a study of practicing physicians making judgments of patient longevity. Other researchers, e.g., Wallsten, Budescu, and Zwick (1992) have also observed binary complementarity in experimental settings.

Asymmetric support theory is based on judgments of probability $P(A, B)$ of propositions of the form "A holds rather than B," where A and B are exclusive hypotheses. In the above, A is called the *focal* hypothesis and B the *alternative* hypothesis. The theory assumes two support functions are used in evaluating $P(A, B)$, s_f for focal hypotheses s for alternative hypotheses. $P(A, B)$ is then determined by the formula

$$P(A, B) = \frac{s_f(A)}{s_f(A) + s(B)}.$$

The special case $s_f = s$ yields support theory. According to Brenner and Rottenstreich this special case arose in the above mentioned studies because "all earlier studies involved hypotheses that were especially well-defined and left no room for variation in their representation …:"

> Fox et al. (1996), for example, studied hypotheses such as "the price of Microsoft stock will be above seventy dollars." There is little ambiguity in such hypotheses and consequently little room for variability or asymmetry in their representations. We suggest that earlier researchers failed to observe [failures of Binary Complementarity] because they investigated only such especially well-defined hypotheses which essentially left on room for representational asymmetry [i.e., $s_f \neq s$]. (p. 16)

Macchi, Osherson, and Krantz (1999) conducted studies involving ultra-difficult general information questions. For example subjects were presented one of the following:

> The freezing point of gasoline is not equal to that of alcohol. What is the probability that the freezing point of gasoline is greater than that of alcohol?

> The freezing point of alcohol is not equal to that of gasoline. What is the probability that the freezing point of alcohol is greater than that of gasoline?

The typical sum of probabilities over all such binary partition was about 0.90, relatively far from 1, indicating a failure of binary complementarity. Macchi et al. reasoned that given the ultra-difficulty of the questions there is relatively little evidence in favor of the focal hypothesis. If the subjects attended relatively more to the focal rather than the alternative hypothesis, then they might not appreciate the fact that the alternative hypothesis also has little support, leading to having a sum of judged probabilities less than 1.

In a follow-up study, Macchi et al. attempted to equalize the amounts of attention paid to focal and alternative hypotheses by explicit mention of both hypotheses. Subjects were presented one of the following:

> The freezing point of gasoline is not equal to that of alcohol. Thus, either the freezing point of gasoline is greater than that of alcohol, or the freezing point of

alcohol is greater than that of gasoline. What is the probability that the freezing point of gasoline is greater than that of alcohol?

The freezing point of alcohol is not equal to that of gasoline. Thus, either the freezing point of alcohol is greater than that of gasoline, or the freezing point of gasoline is greater than that of alcohol. What is the probability that the freezing point of alcohol is greater than that of gasoline?

In this study, the typical sum of judged probabilities was about 1.01. Apparently, mentioning both the focal and alternate hypotheses made it natural for subjects to evaluate them in similar ways.

Brenner and Rottenstreich (manuscript) conducted empirical studies where evaluations of category size underlay the likelihood judgment and consistently found that the sums of percentage judgments for binary partitions were less than 1. They also conducted empirical studies where evaluations of similarity underlay the likelihood judgment and consistently found that the sums of probability judgments for binary partitions were greater than 1.

Brenner and Rottenstreich presents the following axiomatization and theorem for the representation

$$\frac{s_f(A)}{s_f(A)+s(B)}$$

for partitions (A, B) in terms of the probability or percentage function P:

Definition 6.1. Let

$$Q(A,B)=\frac{P(A,B)}{1-P(A,B)},$$

where B is nonnull. Then the following two definitions obtain:

1. The *asymmetric product rule* is said to hold if and only if
$$Q(A,B)Q(C,D)=Q(A,D)Q(C,B),$$
whenever the arguments of Q are exclusive.
2. The *asymmetric triple product rule* is said to hold if and only if
$$Q(A,B)Q(B,C)Q(C,A)=Q(C,B)Q(B,A)Q(A,C),$$
whenever the arguments of Q are exclusive.

Note that whenever there exists a hypothesis exclusive of A, B, and C, the asymmetric triple product rule for A, B, and C, follows from the asymmetric product rule.

Theorem 6.4. *Assume the notation and concepts of support theory of the previous section. Let $\mathscr{H}$ be a set of hypotheses. Suppose $P(A, B)$ is defined for all exclusive A and B in $\mathscr{H}$ and it vanishes if and only if A is null. Then the following two statements are equivalent*:

1. *The asymmetric product rule and the asymmetric triple product rule.*
2. *There exist functions s_f and s on $\mathscr{H}$ such that for all exclusive A and B in $\mathscr{H}$,*
$$P(A,B)=\frac{s_f(A)}{s_f(A)+s(B)}.$$

6.4. A few comments about support theories

1. *Judging probabilities degrees of belief*: Let u, v, and $\mathbb{B}$ be as in Eqs. (14), (15), and (16). In particular,

$$\mathbb{B}(A|A\vee B)=v(A)\frac{u(A)}{u(A\vee B)}=v(A)\frac{u(A)}{u(A)+u(B)}.$$

Letting $w(A)=u(A)v(A)$, we then obtain the formula

$$\mathbb{B}(A|A\vee B)=\frac{w(A)}{u(A)+u(B)}, \tag{17}$$

which is similar in many ways to Brenner and Rottenstreich formula,

$$P(A,B)=\frac{s_f(A)}{s_f(A)+s(B)}.$$

In Eq. (17), w is interpreted as a measure the probabilistic strength of the focal hypothesis, u as a measure of the probabilistic strength of the alternative hypothesis, and $\mathbb{B}(A|A\vee B)$ is the probability judgment given by the subject. Using this interpretation, explanations can be provided for the qualitative shifts in probabilistic judgments occurring in the empirical examples of the previous subsection. Brenner and Rottenstreich (manuscript) also show that their formula quantitatively fits the data. My guess is that it would be difficult to distinguish the two formulas in terms of goodness of fit using the kind of data collected by Brenner and Rottenstreich.

2. *Dual belief support representations*: In the previous subsection it was shown that belief support probabilities provided an adequate theory for support theory. There, the definiteness function $v(A)$ was interpreted as a factor that accounted for the amount that a hypothesis A was distorted with respect to its extension through cognitive processing. In particular, distortion to due unpacking (as well as other distortions) could be incorporated in v. However, the kinds of distortions considered in asymmetrical support theory cannot be included in v, because $v(A)$ depends only on the hypothesis A and thus not on whether A is a appearing as a focal or an alternative hypothesis. Thus to extend the belief support development to the kinds of empirical situations considered by Brenner and Rottenstreich, two belief support functions are needed: one, $\mathbb{B}_f$ for focal hypotheses, and the other, $\mathbb{B}$, for alternative hypotheses. The following is a natural

way of accomplishing this: Let u, u_f, v, and v_f be functions from nonnull hypotheses into the positive reals and P be such that for each conditional hypothesis $(A|A \vee B)$,

$$P(A|A \vee B) = \frac{\mathbb{B}_f(A|A \vee B)}{\mathbb{B}_f(A|A \vee B) + \mathbb{B}(B|A \vee B)} = \frac{v_f(A)v(A)\frac{u_f(A)}{u_f(A \vee B)}}{v_f(A)v(A)\frac{u(A)}{u(A \vee B)} + v(B)\frac{u(B)}{u(A \vee B)}}. \quad (18)$$

Note that in Eq. (18) v_f is *not* the definiteness function for $\mathbb{B}_f$; rather the product $v_f(A)v(A)$ gives the definiteness of A in $\mathbb{B}$ for $\mathbb{B}_f$.

In the second equality in Eq. (18), $u_f(A)$ is interpreted as the probabilistic strength of the extension of A when A is the focal hypothesis, and similarly, $u(A)$ is interpreted as the probabilistic strength of the extension of A when A is the alternative hypothesis. The following assumption is within the spirit of support theory:

$$u_f = u. \quad (19)$$

Also, because $\mathbb{B}_f$ and $\mathbb{B}$ are intended to be belief support functions, it is assumed that

$$u(A) = u(A') \quad \text{and} \quad u(A \vee B) = u(A) + u(B), \quad (20)$$

where as usual, A' is the extension of A and $(A \vee B)' = A' \vee B'$.

$v(A)$ is intended as a factor accounting for the probabilistic distortion of A with respect to the probabilistic strength, $u(A')$ $(= u(A))$, of its extension A'. $v_f(A)$ is intended as a factor that accounts for the additional distortion resulting from A being a focal rather than alternative hypothesis.

Applying Eqs. (19) and (20) to Eq. (18) then yields,

$$P(A|A \vee B) = \frac{v_f(A)v(A)\frac{u(A)}{u(A)+u(B)}}{v_f(A)v(A)\frac{u(A)}{u(A)+u(B)} + v(B)\frac{u(B)}{u(A)+u(B)}} = \frac{v_f(A)v(A)u(A)}{v_f(A)v(A)u(A) + v(B)u(B)}. \quad (21)$$

Note that by letting for each hypothesis H in $\mathscr{H}$,

$$s_f(H) = v_f(H)v(H)u(H) \quad \text{and} \quad s(H) = v(H)u(H),$$

Eq. (21) becomes

$$P(A|A \vee B) = \frac{s_f(A)}{s_f(A) + s(B)},$$

which is Brenner's and Rottenstreich's asymmetric support representation. The representation given by Eq. (21) differs from Brenner's and Rottenstreich's in that the correspondents to s_f and s have an inner structure to them. This "inner structure" allows for the formulation of unpacking principles.

3. *Extending the dual belief support representation to include unpacking*: Although Brenner and Rottenstreich allude to unpacking studies and use the same formal setup involving implicit and explicit hypotheses and their extensions as Tversky and Koehler (1994), they do not extend their axiomatization or mathematical model to include a version of the unpacking principle. In Section 6.1 unpacking was characterized in terms of the belief support probabilities representation. This characterization generalizes easily to the dual belief support representation:

Let A and B be arbitrary elements of $\mathscr{H}$ such that A is implicit, B is explicit, and $A' = B'$. Assume

$$v(A) \leqslant v(B) \quad (22)$$

and

$$v_f(A) \leqslant v_f(B). \quad (23)$$

The the following generalizes Theorem 6.2.

Theorem 6.5. *Suppose the above assumptions, notation, and conventions. Suppose A and D are exclusive, B, C, and D are mutually exclusive, A is implicit, and $A' = (B \vee C)'$. Then the following two statements are true*:

1. (Ordinal unpacking) $P_{\mathbb{B}}(A|A \vee D) \leqslant P_{\mathbb{B}}(B \vee C|B \vee C \vee D)$.
2. *Suppose* $\mathbb{B}_f$ *is* additive, *i.e.*,
$$\mathbb{B}_f(B \vee C|B \vee C \vee D) = \mathbb{B}_f(B|B \vee C \vee D) + \mathbb{B}_f(C|B \vee C \vee D).$$
Then the unpacking principle *holds*, *i.e.*,
$$P_{\mathbb{B}}(A|A \vee D) \leqslant P_{\mathbb{B}}(B \vee C|B \vee C \vee D) = P_{\mathbb{B}}(B|B \vee C \vee D) + P_{\mathbb{B}}(C|C \vee B \vee D).$$

Proof. By the hypothesis $A' = (B \vee C)'$ and Eq. (20),

$$u(A) = u(B \vee C) = u(B) + u(C). \quad (24)$$

By Eq. (21),

$$P(A|A \vee D) = \frac{v_f(A)v(A)u(A)}{v_f(A)v(A)u(A) + v(D)u(D)} = \frac{1}{1 + \frac{v(D)u(D)}{v_f(A)v(A)u(A)}}. \quad (25)$$

Similarly, by Eq. (21), the hypothesis $A' = (B \vee C)'$, and Eq. (20),

$$P(B \vee C|B \vee C \vee D) = \frac{v_f(B \vee C)v(B \vee C)u(B \vee C)}{v_f(B \vee C)v(B \vee C)u(B \vee C) + v(D)u(D)} = \frac{1}{1 + \frac{v(D)u(D)}{v_f(B \vee C)v(B \vee C)u(B \vee C)}} = \frac{1}{1 + \frac{v(D)u(D)}{v_f(B \vee C)v(B \vee C)u(A)}}. \quad (26)$$

Then statement 1 follows from Eqs. (22) and (23).

To show statement 2, suppose $\mathbb{B}_f$ is additive. Then the unpacking principle follows from statement 1 and the first equality in Eq. (18).

Observe that all the inequalities in statements 1 and 2 of Theorem 6.5 become strict if the inequality in Eq. (22) becomes strict; and the inequality in statement 1 becomes strict if the inequality in Eq. (23) becomes strict.

7. Conclusions

The belief axioms (Definition 3.2) yield the existence of a definiteness representation $\mathbb{B}$ with probability function $\mathbb{P}$ and definiteness function v (Definition 3.4). When the value of v is constant, $\mathbb{B}$ may be looked at as a ratio-scaled version of $\mathbb{P}$, i.e., $\mathbb{B}$ is essentially a conditional probability function. Thus the notion of a definiteness representation generalizes the notion of a probability function. More importantly, in the multifarious uses of probability in mathematical modeling, $\mathbb{B}$ or other probabilistic concepts generated by it are often substitutable for probability functions, yielding new and more general models.

As an example consider utility theory. Let $\mathbb{U}$ be an individual's function from objects of value (both positive and negative) into the real numbers, and let A and B be nonempty, disjoint finite events. Let $g = (x, A; y, B)$ be the gamble of receiving object of value x if A occurs and receiving y if B occurs. In the behavioral sciences, models of utility have the form

$$\mathbb{U}(g) = \mathbb{U}(x)W_1 + \mathbb{U}(y)W_2,$$

where W_1 and W_2 are *weights* satisfying the conditions,

$$0 \leqslant W_1 \leqslant 1, \quad 0 \leqslant W_2 \leqslant 1, \quad \text{and} \quad W_1 + W_2 = 1.$$

Then theories of utility result by specifying W_1 and W_2.

One of the most important of these theories is subjective expected utility (SEU), which assumes that the individual has a conditional probability function P such that

$$W_1 = P(A|A \cup B) \quad \text{and} \quad W_2 = P(B|A \cup B).$$

Thus, SEU assumes that only one dimension of uncertainty is relevant for specifying W_1 and W_2—the dimension that is measured by probability. The belief representation $\mathbb{B}$ can be interpreted as the combined measurement of two dimensions—a dimension of probability measured by $\mathbb{P}$, and a dimension of definiteness of focal events measured by v. Let $F_{\mathbb{P},v}$ denote a function from the set conditional events into the real interval $[0, 1]$ that is determined by $\mathbb{P}$ and v. Then consideration of

$$W_1 = F_{\mathbb{P},v}(A|A \cup B) \quad \text{and} \quad W_2 = F_{\mathbb{P},v}(B|A \cup B)$$

provides a starting point for generating utility theories that generalizes SEU. (Current utility theories would also have W_1 and W_2 depend on features of $\mathbb{U}(x)$ and $\mathbb{U}(y)$, e.g., *rank-dependent* theories on whether $\mathbb{U}(x) \geqslant \mathbb{U}(y)$ or whether $\mathbb{U}(y) \geqslant \mathbb{U}(x)$, and *sign-dependent* theories on whether $\mathbb{U}(x) \geqslant 0$ or whether $\mathbb{U}(y) \geqslant 0$ or whether both $\mathbb{U}(x)$ and $\mathbb{U}(y)$ are $\leqslant 0$.)

In Section 6, $\mathbb{B}$ was used as a structured support function to produce generalizations of support theories of probability judgment. The structured nature of $\mathbb{B}$ allowed for a different kind of formulation of the unpacking principle of Tversky and Koehler (1994) as well as for generalizations of it. Then with insights gained from theses formulations to unpacking for support theory, it was easy to extend the unpacking principle to the more general and complicated versions of support theory.

Traditional probability theory has two components to it: A probability function that is a finitely additive measure and an independence relation on events. While the development of the analog of independence for belief theory is outside the scope of this article, a few observations will be made about it.

In the Kolmogorov theory of probability, independence is a concept defined in terms of probabilities, i.e., the events A and B are by definition "independent" if and only if $P(A \cap B) = P(A)P(B)$. I and others find this notion of independence problematic: In the most important applications of probabilistic independence, one has a very good idea of which key events are independent of one another without resort to calculation or often even without knowing their probabilities and the probability of their intersections. Also in most applications where probabilities are used, they are admittedly inexact; but the Kolmogorov definition of independence require the probabilities be exact. In foundations of probability founded on relative frequencies, independence of trials are assumed before the probability of events are defined.

Because of these considerations, I find it preferable to introduce a primitive binary relation $\perp$ on events representing probabilistic independence and have it linked to probability by the following:

$$\text{if } A \perp B \text{ then } \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

In intended interpretations, $\perp$ is often known independently of the underlying probability function, often through intuition or theory about the nature of reality being modeled, e.g., that "red" coming up on a future turn of a particular roulette wheel is independent of the sum of points scored being even by the teams involve in a particular future football game.

Let $\mathbb{B}$ be a belief representation for $\succsim$ with probability function $\mathbb{P}$ and definiteness function v (Definition 3.4). Let $\perp$ be the relation of causal independence between two events. Interpreting $\mathbb{P}$ as conditional probability then reasonably yields

$$\text{if } A \perp B \text{ then } \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

as a desirable condition for belief. However, the specification of $v(A \cap B)$ is difficult and may depend on the kind of definiteness one is considering. For example, even though A is causally independent of B, the definiteness of A need not be causally independent of the definiteness of B. An extreme example of this is where the definiteness of A and B ($A \cap B \neq \emptyset$) is due to the same kind of lack of information or unreliability of information, and thus the definiteness A and B (and therefore $A \cap B$) are the same and completely causally dependent. The other extreme is when the definiteness of A is causally independent of the definiteness of B. What is needed to advance further the theories of belief considered in this article is a means of sensibly classifying different kinds of definiteness. Notions of "independence" may prove to be useful in such an effort.

The theory of belief developed in this article arose out of generalizing probability theory to apply to situations where the axiom of binary symmetry (Axiom 14) may fail. These generalizations, which are captured by the belief axioms, use belief representations instead of probability functions. When the belief representations are additive, they provide a theory that is very close to that of probability theory. But for many kinds of belief situations additivity is unwarranted and unwanted. Nevertheless, belief representations without additivity are still rich in mathematical structure and provide fertile ground for generating probabilistic-like concepts.

8. Additional lemmas, theorems, and proofs

8.1. Preliminary lemmas and theorems

Convention 8.1. Throughout this subsection the Basic Belief Axioms (Definition 2.4) will be assumed.

Lemma 8.1. *$\sim$, $\sim_{\mathscr{C}}$, and $\sim_X$ are equivalence relations.*

Proof. Since $\precsim$, $\succsim_{\mathscr{C}}$, and $\succsim_X$ are weak orderings, it easily follows that $\sim$, $\sim_{\mathscr{C}}$, and $\sim_X$ are equivalence relations. □

Definition 8.1. Let $\mathbf{C}$ be the set of $\sim_{\mathscr{C}}$-equivalence classes of $\mathscr{C}$ and $\mathbf{X}$ be the set of $\sim_X$-equivalence classes of X. Let $\succcurlyeq_{\mathbf{C}}$ be $\succsim_{\mathscr{C}}/\sim_{\mathscr{C}}$ on $\mathbf{C}$; that is, let $\succcurlyeq_{\mathbf{C}}$ be the binary relation on $\mathbf{C}$ such that for all α, β in $\mathbf{C}$, $\alpha \succcurlyeq_{\mathbf{C}} \beta$ if and only if there exist A in α and B in β such that $A \succsim_{\mathscr{C}} B$. Similarly let $\succcurlyeq_{\mathbf{X}}$ be $\succsim_X/\sim_X$ on $\mathbf{X}$; that is, let $\succcurlyeq_{\mathbf{X}}$ be the binary relation on $\mathbf{X}$ such that for all x, y in $\mathbf{X}$, $x \succcurlyeq_{\mathbf{X}} y$ if and only if there exist a in x and b in y such that $a \succsim_X b$.

Lemma 8.2. *$\succcurlyeq_{\mathbf{C}}$ and $\succcurlyeq_{\mathbf{X}}$ are total orderings.*

Proof. Left to reader. □

Definition 8.2. $\mathfrak{Y} = \langle Y, \succcurlyeq', \bigcirc \rangle$ is said to be an *extensive structure* if and only if the following seven conditions hold:

1. $Y \neq \emptyset$, $\bigcirc$ is a binary operation on Y, and $\succcurlyeq'$ is a total ordering on Y.
2. $\bigcirc$ is *commutative*, i.e., $x \bigcirc y = y \bigcirc x$ for all x and y in Y.
3. $\bigcirc$ is *associative*, i.e., $(x \bigcirc y) \bigcirc z = x \bigcirc (y \bigcirc z)$ for all x, y, and z in Y.
4. $\mathfrak{Y}$ is *positive*, i.e., $x \bigcirc y \succ' x$ for all x and y in Y.
5. $\mathfrak{Y}$ is *monotonic*, i.e., for all x, y, and z in Y, $x \succcurlyeq' y$ iff $x \bigcirc z \succcurlyeq' y \bigcirc z$.
6. $\mathfrak{Y}$ is *restrictedly solvable*, i.e., for all x and y in Y, if $x \succ' y$ then for some z in Y, $x \succ' y \bigcirc z$.
7. $\mathfrak{Y}$ is *Archimedean*, i.e., for all x and y in Y, there exists a positive integer n such that $nx \succ' y$, where $1x = x$, and for all positive integer k, $(k+1)x = (kx) \bigcirc x$.

Definition 8.3. Define $\oplus$ on $\mathbf{C}$ as follows: For all α, β, γ in $\mathbf{C}$, $\alpha \oplus \beta = \gamma$ if and only if there exist $A \in \alpha$, $B \in \beta$, and $C \in \gamma$ such that $A \cap B = \emptyset$ and $A \cup B = C$.

Theorem 8.1. *$\oplus$ is an operation on $\mathbf{C}$ and $\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$ is an extensive structure (Definition 8.2).*

Proof. Note that if $E_1, \ldots, E_i, \ldots$ are elements of $\mathscr{C}$, then by repeated use of axiom 9 elements $F_1, \ldots, F_i, \ldots$ of $\mathscr{C}$ can be found so that $E_i \sim_{\mathscr{C}} F_i$ and for all distinct i,j in $\mathbb{I}^+$, $E_i \cap E_j = \emptyset$. From this and the fact that $\cup$ is commutative and associative, it easily follows that $\oplus$ is a commutative and associative operation on $\mathbf{C}$. From the commutativity of $\oplus$ and Axiom 2 it follows that $\mathfrak{C}$ is positive. From the commutativity and Axioms 3 and 9 it follows that $\mathfrak{C}$ is monotonic. From Axiom 11 it follows that $\mathfrak{C}$ is restrictedly solvable, and from Axioms 9 and 13 that $\mathfrak{C}$ is Archimedean. By Lemma 8.2 $\succcurlyeq_{\mathbf{C}}$ is a total ordering on $\mathfrak{C}$.

Definition 8.4. Define the binary relation $\succsim_{\bigstar}$ on $\mathbf{X} \times \mathbf{C}$ as follows: For all $x\alpha, y\beta$ in $\mathbf{X} \times \mathbf{C}$, $x\alpha \succsim_{\bigstar} y\beta$ if and only if there exist $a \in x$, $A \in \alpha$, $b \in y$, $B \in \beta$, such that $(a|A) \succsim (b|B)$.

Lemma 8.3. *The following five statements are true*:

1. *$\succsim_{\bigstar}$ is a weak ordering.*
2. *For all x, y in $\mathbf{X}$ and all α in $\mathbf{C}$, if $x\alpha$ is in the domain of $\succsim_{\bigstar}$ and $x \succcurlyeq_{\mathbf{X}} y$, then $y\alpha$ is in the domain of $\succsim_{\bigstar}$ and $x\alpha \succsim_{\bigstar} y\alpha$.*
3. *For all α, β in $\mathbf{C}$ and all x in $\mathbf{X}$, if $x\alpha$ is in the domain of $\succsim_{\bigstar}$ and $\beta \succcurlyeq_{\mathbf{C}} \alpha$, then $x\beta$ is in the domain of $\succsim_{\bigstar}$ and $x\alpha \succsim_{\bigstar} x\beta$.*
4. *For all x, y in $\mathbf{X}$, the following two propositions are true*: (i) *if $x\alpha \succsim_{\bigstar} y\alpha$ for some α in $\mathbf{C}$, then $x \succcurlyeq_{\mathbf{X}} y$ and*

$x\beta \precsim_\star y\beta$ for all β in **C** *such that $\beta \succcurlyeq_{\mathbf{C}} \alpha$; and* (ii) *if $x\alpha \succ_\star y\alpha$ for some α in* **C**, *then $x \succ_X y$ and $x\beta \succ_\star y\beta$ for all β in* **C** *such that $\beta \succcurlyeq_{\mathbf{C}} \alpha$.*

5. *For all α, β in* **C**, *the following two propositions are true*: (i) *if $x\alpha \succsim_\star x\beta$ for some x in* **X**, *then $\beta \succcurlyeq_{\mathbf{C}} \alpha$ and $y\alpha \succsim_\star y\beta$ for all y in* **X** *such that $x \succcurlyeq_{\mathbf{X}} y$; and* (ii) *if $x\alpha \succ_\star x\beta$ for some x in* **X**, *then $\beta \succ_{\mathbf{C}} \alpha$ and $y\alpha \succ_\star y\beta$ for all y in* **X** *such that $x \succcurlyeq_{\mathbf{X}} y$.*

Proof. 1. It is immediate that $\succsim_\star$ is a transitive relation on $\mathbf{X} \times \mathbf{C}$. To show reflexivity, let $x\alpha$ be an arbitrary element of the domain of $\succsim_\star$, and let a be in X and A be in α. Then by Axiom 7, let $c \in X$ and $C \in \mathscr{C}$ be such that $a \sim_X c$, $A \sim_{\mathscr{C}} C$, and $c \in C$. Then from $(c|C) \sim (c|C)$, it follows that $x\alpha \sim_\star x\alpha$. It will next be shown that $\succsim_\star$ is connected. Because $\succsim_\star$ is reflexive, the domain of $\succsim_\star$ = the range of $\succsim_\star$. Let $x\alpha$ and $y\beta$ be arbitrary elements of the domain of $\succsim_\star$. Let a, A, y, and B be such that $a \in A$, $A \in \mathscr{C}$, $y \in B$, $B \in \mathscr{C}$. Then because $\succsim$ is a weak ordering,

either $(a|A) \succsim (b|B)$ or $(b|B) \succsim (a|A)$,

which by Definition 8.4 yields,

either $x\alpha \succsim_\star y\beta$ or $y\beta \succsim_\star x\alpha$.

Because $\succsim_\star$ is transitive, reflexive, and connected, it is a weak ordering.

2. Suppose x and y are in **X**, α is in **C**, $x\alpha$ is in the domain of $\succsim_\star$, and $x \succcurlyeq_{\mathbf{X}} y$. By Axiom 8 and Definitions 8.1 and 8.4, $y\alpha$ is in the domain of $\succsim_\star$. Then by Definitions 8.1 and 8.4 and statement 1 of Axiom 6, $x\alpha \succsim_\star y\alpha$.

3. Suppose α and β are in **C**, x is in **X**, and $x\alpha$ is in the domain of $\succsim_\star$, and $\beta \succcurlyeq_{\mathbf{C}} \alpha$. Then by Definitions 8.1 and 8.4 and Axiom 7, $x\beta$ is in the domain of $\succsim_\star$. Then by Definitions 8.1 and 8.4 and statement 2 of Axiom 6, $x\alpha \succsim_\star x\beta$.

4. (i) Suppose x, y in **X** and α in **C** are such that $x\alpha \succsim_\star y\alpha$. It then follows from Definitions 8.4 and 8.1 and Statement 2 of Axiom 12 that $a \in x$, $A \in \alpha$, $b \in y$, and $B \in \alpha$ can be found such that $(a|A) \succsim (b|B)$ and $A \sim_{\mathscr{C}} B$. Thus by statement 1 of Axiom 6, $a \succsim_X b$. Therefore by Definition 8.1, $x \succcurlyeq_{\mathbf{X}} y$. Let β in **C** be such that $\beta \succcurlyeq_{\mathbf{C}} \alpha$. Then it follows from Definition 8.1 that C and D in β can be found such that $C \sim_{\mathscr{C}} D$, $C \succsim_{\mathscr{C}} A$, and $D \succsim_{\mathscr{C}} B$. It then follows from Axiom 7 that e and f in X and E and F in $\mathscr{C}$ can be found such that $a \sim_X e$, $b \sim_X f$, $C \sim_{\mathscr{C}} D \sim_{\mathscr{C}} E \sim_{\mathscr{C}} F$, $e \in E$, and $f \in F$. Then since $e \sim_X a \succsim_X b \sim_X f$, it follows by statement 1 of Axiom 6, that $(e|E) \succsim (f|F)$. Thus by Definition 8.4, $x\beta \succsim_\star y\beta$. Proposition (ii) follows by a similar argument.

5. (i) Suppose α, β in **C** and x in **X** are such that $x\alpha \succsim_\star x\beta$. Then it follows from Definitions 8.4 and 8.1 that $a \in x$, $A \in \alpha$, $b \in x$, and $B \in \beta$ can be found such that $(a|A) \succsim (b|B)$ and $a \sim_X b$. Therefore by statement 2 of Axiom 6, $B \succsim_{\mathscr{C}} A$. Thus by Definition 8.1, $\beta \succcurlyeq_{\mathbf{C}} \alpha$. Let y in **X** be such that $x \succcurlyeq_{\mathbf{X}} y$. Then it follows from Definition 8.1 that c and d in y can be found such that $a \succsim_X c$, $b \succsim_X d$, and $c \sim_X d$. Since $(a|A)$ and $(b|B)$ and $a \succsim_X c$ and $b \succsim_X d$, it easily follows that elements h and k in X can be found so that $A \succsim_{\mathscr{C}} \{c, h\}$ and $B \succsim_{\mathscr{C}} \{d, k\}$. It then follows from Axiom 7 that e and f in X and E and F in $\mathscr{C}$ can be found such that $c \sim_X e$, $d \sim_X f$, $A \sim_{\mathscr{C}} E$, and $B \sim_{\mathscr{C}} F$, $e \in E$, and $f \in F$. Since $c \sim_X e$ and $d \sim_X f$, it follows that $e \in y$ and $f \in y$. From $B \succsim_{\mathscr{C}} A$ it follows that $F \succsim_{\mathscr{C}} E$. Therefore by statement 2 of Axiom 6, $(e|E) \succsim (f|F)$. Thus, by Definition 8.4, $y\alpha \succsim_\star y\beta$. (ii) Proposition (ii) follows by a similar argument. □

Lemma 8.4. *The following five statements are true*:

1. *For all x, y in* **X** *and all α in* **C**, *if $y \succcurlyeq_{\mathbf{X}} x$ and $x\alpha$ is in the domain of $\succsim_\star$, then there exists β in* **C** *such that $x\alpha \sim_\star y\beta$.*
2. *For all x in* **X** *and all α, β in* **C**, *if $x\alpha$ is in the domain of $\succsim_\star$, then there exists y in* **X** *such that $x\alpha \sim_\star y\beta$.*
3. *The following two propositions are true for all x, y in X and all $\alpha, \beta, \alpha', \beta'$ in $\mathscr{C}$*: (i) *if $x\alpha \succsim_\star y\beta$ and $x\alpha' \succsim_\star y\beta'$, then*

   $x(\alpha \oplus \alpha') \succsim_\star y(\beta \oplus \beta')$, *and*

   (ii) *if $x\alpha \succ_\star y\beta$ and $x\alpha' \succsim_\star y\beta'$, then*

   $x(\alpha \oplus \alpha') \succ_\star y(\beta \oplus \beta')$.

4. *For all x, y in X and all α, β in $\mathscr{C}$, if*

   $x\alpha \sim_\star y\beta$

   then

   $x \succcurlyeq_{\mathbf{X}} y$ *iff* $\alpha \succcurlyeq_{\mathbf{C}} \beta$.

5. *For all x, y in* **X** *and all α, β in* **C**, *if*

   $x\alpha \sim_\star y\beta$ *and* $\gamma \succcurlyeq_{\mathbf{C}} \beta$,

   then there exists α' and such that

   $x\alpha' \sim_\star y\gamma$.

Proof. Statements 1 and 2 follow from Definitions 8.1 and 8.4 and Axiom 12.

To show statement 3, note that by Theorem 8.1 and its proof that $\alpha \oplus \alpha' \succcurlyeq_{\mathbf{C}} \alpha$ and $\beta \oplus \beta' \succcurlyeq_{\mathbf{C}} \beta$, which by statement 3 of Lemma 8.3 yields that $x(\alpha \oplus \alpha')$ and $y(\beta \oplus \beta')$ are in the domain of $\succsim_\star$. Statement 3 then follows from Axioms 9 and 5 and Definitions 8.1, 8.3, and 8.4.

To show statement 4, we need only show (i) that $x \succcurlyeq_{\mathbf{X}} y$ and $\beta \succ_{\mathbf{C}} \alpha$ leads to a contradiction and (ii) that $y \succ_{\mathbf{X}} x$ and $\alpha \succcurlyeq_{\mathbf{C}} \beta$ leads to a contradiction.

(i) $x \succcurlyeq_{\mathbf{X}} y$ and $\beta \succ_{\mathbf{C}} \alpha$. By statement 2 of Lemma 8.3,

$$x\alpha \gtrsim_{\bigstar} y\alpha,$$

and thus by statement 4 of Lemma 8.3,

$$x\beta \gtrsim_{\bigstar} y\beta.$$

Since by hypothesis $x\alpha \sim_{\bigstar} y\beta$, it then follows from statement 1 of Lemma 8.3 that

$$x\beta \gtrsim_{\bigstar} x\alpha,$$

which by statement 5 of Lemma 8.3 yields $\alpha \succcurlyeq_{\mathbf{C}} \beta$, contradicting the hypothesis $\alpha \succcurlyeq_{\mathbf{C}} \beta$.

(ii) $y \succ_X x$ and $\beta \succcurlyeq_{\mathbf{C}} \alpha$. By statement 3 of Lemma 8.3,

$$y\beta \gtrsim_{\bigstar} y\alpha,$$

and thus by statement 5 of Lemma 8.3,

$$x\alpha \gtrsim_{\bigstar} x\alpha.$$

Since by hypothesis, $x\alpha \sim_{\bigstar} y\beta$, it then follows from statement 1 of Lemma 8.3,

$$x\beta \gtrsim_{\bigstar} y\beta,$$

which by statement 4 of Lemma 8.3 yields $x \succcurlyeq_{\mathbf{X}} y$, contradicting the hypothesis $y \succ_{\mathbf{X}} x$.

Statement 5 is immediate from Axiom 12 and Definition 8.4. □

Definition 8.5. Suppose $x_i\alpha_i$ is in the domain of $\gtrsim_{\bigstar}$ and β, and γ are arbitrary elements of $\mathscr{C}$, $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$, and $\gamma \succcurlyeq_{\mathbf{C}} \alpha_i$. Then by statement 3 of Lemma 8.3, $x_i\beta$ is in the domain of $\gtrsim_{\bigstar}$. By statement 2 of Lemma 8.4, let $\tau_i(\beta)$ be such that

$$\tau_i(\beta)\alpha_i \sim_{\bigstar} x_i\beta. \tag{27}$$

By statement 4 of Lemma 8.4,

$$\tau_i(\beta) \preccurlyeq_{\mathbf{X}} x_i. \tag{28}$$

Since by Eq. (27) $\tau_i(\beta)\alpha_i$ is in the domain of $\gtrsim_{\bigstar}$, it follows from statement 3 of Lemma 8.3 that $\tau_i(\beta)\gamma$ is in the domain of $\gtrsim_{\bigstar}$. By statement 1 of Lemma 8.4, let $\Delta_{i,\gamma}(\beta)$ be such that

$$x_i\Delta_{i,\gamma}(\beta) \sim_{\bigstar} \tau_i(\beta)\gamma. \tag{29}$$

Lemma 8.5. *Suppose x_i is an arbitrary element of X, and α_i, β, and γ are arbitrary elements of $\mathscr{C}$ such that $x_i\alpha_i$ is in the domain of $\gtrsim_{\bigstar}$ and $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$, and $\gamma \succcurlyeq_{\mathbf{C}} \alpha_i$. Then the following two statements are true*:

1. $\tau_i(\beta) \preccurlyeq_{\mathbf{X}} x_i$.
2. $\Delta_{i,\gamma}(\beta) \succcurlyeq_{\mathbf{C}} \gamma \succcurlyeq_{\mathbf{C}} \alpha_i$.

Proof. Statement 1 follows from Eq. (28). To show statement 2, note that by hypothesis $\gamma \succcurlyeq_{\mathbf{C}} \alpha_i$ and from Eq. (29), statement 1 of this lemma, and statement 4 of Lemma 8.4 that $\Delta_{i,\gamma}(\beta) \succcurlyeq_{\mathbf{C}} \gamma$. □

Lemma 8.6. *Suppose $x_i\alpha_i$ is in the domain of $\gtrsim_{\bigstar}$, $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$, $\gamma \succcurlyeq_{\mathbf{C}} \alpha_i$, and $\delta \succcurlyeq_{\mathbf{C}} \alpha_i$. Then the following three statements are true*:

1. $\beta \succcurlyeq_{\mathbf{C}} \delta$ *if and only if* $\Delta_{i,\gamma}(\beta) \succcurlyeq_{\mathbf{C}} \Delta_{i,\gamma}(\delta)$.
2. $\Delta_{i,\gamma}(\beta \oplus \delta) = \Delta_{i,\gamma}(\beta) \oplus \Delta_{i,\gamma}(\delta)$.
3. *For all ζ in* $\mathbf{C}$, *if $\zeta \succcurlyeq_{\mathbf{C}} \gamma$ then there exists θ such that $\theta \succcurlyeq_{\mathbf{C}} \alpha_i$ and $\Delta_{i,\gamma}(\theta) = \zeta$.*

Proof. 1. (i) Suppose $\beta \succcurlyeq_{\mathbf{C}} \delta$. Then by statement 3 of Lemma 8.3, $x_i\delta$ and $x_i\beta$ are in the domain of $\gtrsim_{\bigstar}$, and by statement 3 of Lemma 8.3,

$$x_i\delta \gtrsim_{\bigstar} x_i\beta,$$

which by Definition 8.5 yields

$$\tau_i(\delta)\alpha_i \gtrsim_{\bigstar} \tau_i(\beta)\alpha_i,$$

which by statement 4 of Lemma 8.3 yields

$$\tau_i(\delta)\gamma \gtrsim_{\bigstar} \tau_i(\beta)\gamma,$$

which by Definition 8.5 yields,

$$x_i\Delta_{i,\gamma}(\delta) \gtrsim_{\bigstar} x_i\Delta_{i,\gamma}(\beta),$$

and which by statement 5 of Lemma 8.3 yields

$$\Delta_{i,\gamma}(\beta) \succcurlyeq_{\mathbf{C}} \Delta_{i,\gamma}(\delta).$$

(ii) Suppose $\Delta_{i,\gamma}(\beta) \succcurlyeq_{\mathbf{C}} \Delta_{i,\gamma}(\delta)$. By Definition 8.5, $x_i\Delta_{i,\gamma}(\beta)$ and $x_i\Delta_{i,\gamma}(\delta)$ are in the domain of $\gtrsim_{\bigstar}$, and thus by statement 3 of Lemma 8.3,

$$x_i\Delta_{i,\gamma}(\delta) \gtrsim_{\bigstar} x_i\Delta_{i,\gamma}(\beta),$$

which by Definition 8.5 yields

$$\tau_i(\gamma)\delta \gtrsim_{\bigstar} \tau_i(\gamma)\beta,$$

which by statement 5 of Lemma 8.3 yields

$$\beta \succcurlyeq_{\mathbf{C}} \delta.$$

2. Since $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$ and $\delta \succcurlyeq_{\mathbf{C}} \alpha_i$, it follows from Theorem 8.1 that $\oplus$ is positive (Definition 8.2), and thus that $\beta \oplus \delta \succ_{\mathbf{C}} \beta \succcurlyeq_{\mathbf{C}} \alpha_i$. By Definition 8.5,

$$x_i\Delta_{i,\gamma}(\beta) \sim_{\bigstar} \tau_i(\gamma)\beta$$

and

$$x_i\Delta_{i,\gamma}(\delta) \sim_{\bigstar} \tau_i(\gamma)\delta.$$

Thus by statement 3 of Lemma 8.4,

$$x_i(\Delta_{i,\gamma}(\beta) \oplus \Delta_{i,\gamma}(\delta)) \sim_{\bigstar} \tau_i(\gamma)(\beta \oplus \delta). \tag{30}$$

But by Definition 8.5,

$$\tau_i(\gamma)(\beta \oplus \delta) \sim_{\bigstar} x_i\Delta_{i,\gamma}(\beta \oplus \delta).$$

Thus by Eq. (30),

$$x_i(\Delta_{i,\gamma}(\beta)\oplus\Delta_{i,\gamma}(\delta))\sim_\star x_i\Delta_{i,\gamma}(\beta\oplus\delta),$$

which by statement 5 of Lemma 8.3 yields

$$\Delta_{i,\gamma}(\beta)\oplus\Delta_{i,\gamma}(\delta)) = \Delta_{i,\gamma}(\beta\oplus\delta).$$

3. Suppose $\zeta\succcurlyeq_{\mathbf{C}}\gamma$. Then by statement 3 of Lemma 8.3,

$$x_i\gamma\gtrsim_\star x_i\zeta. \tag{31}$$

By statement 2 of Lemma 8.4, let y in X be such that

$$y\gamma\sim_\star x_i\zeta. \tag{32}$$

Then by statement 4 of Lemma 8.4,

$$x_i\succcurlyeq_{\mathbf{C}}y. \tag{33}$$

Thus by statement 2 of Lemma 8.3, $y\alpha_i$ is in the domain of $\gtrsim_\star$. By statement 1 of Lemma 8.4, let θ be such that

$$x_i\theta\sim_\star y\alpha_i. \tag{34}$$

Then by Definition 8.5,

$$y = \tau_i(\theta), \tag{35}$$

and by Eq. (32), $\zeta = \Delta_{i,\gamma}(\theta)$. Thus to complete the proof of statement 3, we need to only show that $\theta\succcurlyeq_{\mathbf{C}}\alpha_i$. This follows from Eqs. (33), (34), and statement 4 of Lemma 8.4. □

Definition 8.6. A function L from $\mathbf{C}$ into $\mathbb{R}^+$ is said to be an *additive representation* for $\langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$ if and only if all α and α' in $\mathbf{C}$,

$$\alpha\succcurlyeq\alpha' \quad \text{iff} \quad L(\alpha)\geqslant L(\alpha') \tag{36}$$

and

$$L(\alpha\oplus\alpha') = L(\alpha)+L(\alpha'). \tag{37}$$

Lemma 8.7. *The following two statements are true*:

1. *There exists an additive representation for* $\langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$.
2. *For all additive representations* φ *and* φ' *of* $\langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$, *there exists* r *in* $\mathbb{R}^+$ *such that* $\varphi = r\varphi'$.

Proof. Theorem 2.8.1 of Narens (1985) □.

Definition 8.7. $x_i\alpha_i,\ i = 1,\ldots,$ is said to be an *unbounded* **X–C** *sequence* if and only if the following three statements are true:

1. $x_i\alpha_i$ is in the domain of $\gtrsim_\star$ for all $i\in I^+$.
2. For all α in $\mathbf{C}$, there exists j in I^+ such that $\alpha\succ_C\alpha_j$.
3. For all x in $\mathbf{X}$, there exists k in $\mathbb{I}^+$ such that $x_k\succ_X x$.

Lemma 8.8. *There exists an unbounded* **X–C** *sequence*.

Proof. By Lemma 8.7, let φ be an additive representation for

$$\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle.$$

It then easily follows from the fact that $\mathfrak{C}$ is an extensive structure (Definition 8.2) and φ is an additive representation for $\mathfrak{C}$ that $\varphi(\mathbf{C})$ has an infinite sequence of elements, $\alpha_1, \alpha_3, \alpha_5, \ldots,$ such that for all α in $\mathbf{C}$, $\alpha\succ_{\mathbf{C}}\alpha_{2k-1}$ for some positive integer k. For each positive integer k, let A_{2k-1} be an element of α_{2k-1} and a_{2k-1} be an element of A_{2k-1} and x_{2k-1} be the element of $\mathbf{X}$ such that a_{2k-1} is in x_{2k-1}. Then, it follows from Definitions 8.1 and 8.4 that $x_{2k-1}\alpha_{2k-1}$ is in the domain of $\gtrsim_\star$ for each positive integer k.

It easily follows from the fact that $\mathfrak{C}$ is a concatenation structure and φ is an additive representation for $\mathfrak{C}$ that $\varphi(\mathbf{C})$ has an infinite sequence of elements, $\alpha_2, \alpha_4, \alpha_6, \ldots,$ such that for all α in $\mathbf{C}$, $\alpha_{2k}\succ_{\mathbf{C}}\alpha$ for some positive integer k. For each positive integer k, let A_{2k} be an element of α_{2k}. Let a_2 be an element of A_2. By Axiom 12 let for each positive integer k, b_{2k} and B_{2k} be elements of respectively X and $\mathscr{C}$ such that

$$A_{2k}\sim_{\mathscr{C}}B_{2k} \quad \text{and} \quad (a_2|A_2)\sim(b_{2k}|B_{2k}).$$

We will show by contradiction that for each b in X there exists a positive integer k such that $b_{2k}\succ_X b$. For suppose not. Let b in X be such that $b\gtrsim_X b_{2k}$ for all positive integers k. By Axiom 12, let B in $\mathscr{C}$ and b' in X be such that $b\sim_X b'$ and $(a_2|A_2)\sim(b'|B)$. By the choice of $\alpha_2, \alpha_4, \alpha_6$, let k be a positive integer such that $A_{2k}\succ_{\mathscr{C}}B$. Then $B_{2k}\succ_{\mathscr{C}}B$. Thus since

$$(b_{2k}|B_{2k})\sim(a_2|A_2)\sim(b'|B),$$

it follows from statement 4 of Lemma 8.4 that $b_{2k}\succ_X b'$. Thus, because $b\sim_X b'$, it follows that $b_{2k}\succ_X b$, a contradiction. For each positive integer k, let x_{2k} be the element of $\mathbf{X}$ of which b_{2k} is an element, and let α_{2k} be the element of $\mathbf{C}$ of which B_{2k} is an element. Then for each positive integer k, $x_{2k}\alpha_{2k}$ is in the domain of $\gtrsim_\star$, and for each x in $\mathbf{X}$, there exists an integer n such that $x_{2n}\succ_{\mathbf{X}}x$.

The sequence defined above by $x_i\alpha_i$, i a positive integer, is an unbounded **X–C** sequence. □

Lemma 8.9. *Suppose* $x_i\alpha_i$ *is an unbounded* **X–C** *sequence*, j *and* k *are positive integers*, $\alpha_j\succ_{\mathbf{C}}\alpha_k$, *and* β, γ, *and* δ *are elements of* $\mathbf{C}$ *such that*

$$\beta\succcurlyeq_{\mathbf{C}}\alpha_k, \quad \gamma\succcurlyeq_{\mathbf{C}}\alpha_j, \quad \text{and} \quad \delta\succcurlyeq_{\mathbf{C}}\alpha_j,$$

and $\Delta_{k,\beta}(\delta) = \Delta_{j,\gamma}(\delta)$. *Then for all* ζ *such that* $\zeta\succcurlyeq_{\mathbf{C}}\alpha_j$, $\Delta_{k,\beta}(\zeta)$ *and* $\Delta_{j,\gamma}(\zeta)$ *are defined and*

$$\Delta_{k,\beta}(\zeta) = \Delta_{j,\gamma}(\zeta).$$

Proof. Suppose ζ is an arbitrary element of $\mathbf{C}$ such that $\zeta \succsim_{\mathbf{C}} \alpha_j$. Since $\alpha_j \succ_{\mathbf{C}} \alpha_k$, it follows that $\zeta \succsim_{\mathbf{C}} \alpha_k$. Thus by statement 2 of Lemma 8.5, $\Delta_{k,\beta}(\zeta)$ and $\Delta_{j,\gamma}(\zeta)$ are defined. Suppose $\Delta_{k,\beta}(\zeta) \neq \Delta_{j,\gamma}(\zeta)$. A contradiction will be shown. There are two cases to consider.

Case 1: $\Delta_{j,\gamma}(\zeta) \succ_{\mathbf{C}} \Delta_{k,\beta}(\zeta)$. By Lemma 8.7, let L be an additive representation for $\langle \mathbf{C}, \succsim_{\mathbf{C}}, \oplus \rangle$. Then

$$L(\Delta_{j,\gamma}(\zeta)) > L(\Delta_{j,\beta}(\zeta)).$$

By properties of the real number system, let m in $\mathbb{I}^+$ be such that

$$L(\Delta_{j,\gamma}(\zeta)) - L(\Delta_{j,\beta}(\zeta)) > 2^{-m} L(\Delta_{j,\gamma}(\delta)).$$

Then, again by using properties of the real number system, let p in $\mathbb{I}^+$ be such that

$$L(\Delta_{j,\gamma}(\zeta)) > p2^{-m} L(\Delta_{j,\gamma}(\delta)) \geq L(\Delta_{k,\beta}(\zeta)). \tag{38}$$

Since L is an additive representation, it follows from a hypothesis of the lemma that $L(\Delta_{j,\gamma}(\delta)) = L(\Delta_{k,\beta}(\delta))$, which by Eq. (38) yields

$$L(\Delta_{j,\gamma}(\zeta)) > p2^{-m} L(\Delta_{j,\gamma}(\delta)) \tag{39}$$

and

$$p2^{-m} L(\Delta_{k,\beta}(\delta)) \geq L(\Delta_{k,\beta}(\zeta)). \tag{40}$$

By statement 2 of Lemma 8.6 and by the property of additive representations expressed in Eq. (37), it easily follows from Eqs. (39) and (40) that

$$L(\Delta_{j,\gamma}(2^m\zeta)) > L(\Delta_{j,\gamma}(p\delta))$$

and

$$L(\Delta_{k,\beta}(p\delta)) \geq L(\Delta_{k,\beta}(2^m\zeta)),$$

which by the property of additive representations expressed in Eq. (37) and statement 1 of Lemma 8.6 yields

$$2^m\zeta \succ_{\mathbf{C}} p\delta$$

and

$$p\delta \succsim_{\mathbf{C}} 2^m\zeta,$$

which contradicts that $\succsim_{\mathbf{C}}$ is a total ordering.

Case 2: $\Delta_{k,\beta}(\zeta) \succ_{\mathbf{C}} \Delta_{j,\gamma}(\zeta)$, follows by a similar argument. □

Lemma 8.10. *Let $x_i\alpha_i$ be an unbounded* **X–C** *sequence. Then*

$$\Delta_{j,\gamma}(\alpha_j) = \gamma.$$

Proof. By Definition 8.5,

$$x_j\Delta_{j,\gamma}(\alpha_j) \sim_{\star} \tau_j(\alpha_j)\gamma \sim_{\star} x_j\gamma,$$

and thus by statement 4 of Lemma 8.4, $\Delta_{j,\gamma}(\alpha_j) = \gamma$. □

Definition 8.8. Let $x_i\alpha_i$ be an unbounded **X–C** sequence. Then a set $\mathscr{S}$ of elements of the form $\Delta_{j,\gamma}$, where $j \in \mathbb{I}^+$ and $\gamma \succsim_{\mathbf{C}} \alpha_j$, is said to be a *Δ-set* (*dependent on the sequence* $x_i\alpha_i$) if and only if the following four statements are true:

1. $\mathscr{S} \neq \emptyset$.
2. For all $\Delta_{j,\gamma}$ and $\Delta_{k,\beta}$ in $\mathscr{S}$, if $\Delta_{j,\gamma}(\delta) = \Delta_{k,\beta}(\delta)$ for some $\delta \in \mathbf{C}$, then $\Delta_{j,\gamma}(\sigma) = \Delta_{k,\beta}(\sigma)$ for all σ common to the domains of the functions $\Delta_{j,\gamma}$ and $\Delta_{k,\beta}$.
3. For all $j \in \mathbb{I}^+$ and all $\gamma \succsim_{\mathbf{C}} \alpha_j$, if there exist $\Delta_{k,\beta}$ in $\mathscr{S}$ and δ in $\mathbf{C}$ such that $\Delta_{i,\gamma}(\delta) = \Delta_{k,\beta}(\delta)$, then $\Delta_{i,\gamma}$ is in $\mathscr{S}$.
4. For all $\Delta_{j,\gamma}$ and $\Delta_{k,\beta}$ in $\mathscr{S}$, there exits δ in $\mathbf{C}$ such that $\Delta_{i,\gamma}(\delta) = \Delta_{k,\beta}(\delta)$.

Definition 8.9. f is said to be a **C**-*function* if and only if for some Δ-set $\mathscr{S}$,

$$f = \bigcup \mathscr{S}.$$

Lemma 8.11. *Let $x_i\alpha_i$ be an unbounded* **X–C** *sequence*, *m be a positive integer*, *and $\lambda \succsim_{\mathbf{C}} \alpha_m$. Then for all δ in* **C**, *there exist a positive integer p and an element γ_p of* **C** *such that*

$$\delta \succsim_{\mathbf{C}} \gamma_p \quad \textit{and} \quad \tau_p(\alpha_m)\gamma_p \sim_{\star} x_p\lambda.$$

Proof. Since $\mathfrak{C} = \langle \mathbf{C}, \succsim_{\mathbf{C}}, \oplus \rangle$ is an extensive structure (Theorem 8.1), it is easy to show that a positive integer k can be found so that

$$k\alpha_m \succsim_{\mathbf{C}} \lambda.$$

Let δ be an arbitrary element of $\mathbf{C}$. Since $\mathfrak{C}$ is an extensive structure and $x_i\alpha_i$ is an unbounded **X–C** sequence, it easily follows from Lemma 8.7 and Definition 8.6 that a positive integer p can be found such that $\alpha_m \succsim_{\mathbf{C}} \alpha_p$ and

$$\delta \succsim_{\mathbf{C}} k\alpha_p. \tag{41}$$

Since by hypothesis $\lambda l \succsim_{\mathbf{C}} \alpha_m$ and by Definition 8.5

$$\tau_p(\alpha_m)\alpha_p \sim_{\star} x_p\alpha_m,$$

it follows from statement 5 of Lemma 8.4 that γ_p can be found such that

$$\tau_p(\alpha_m)\gamma_p \sim_{\star} x_p\lambda. \tag{42}$$

By Definition 8.5,

$$\tau_p(\alpha_m)\alpha_p \sim_{\star} x_p\alpha_m. \tag{43}$$

Thus by applying statement 3 of Lemma 8.4 to Eq. (43) $k-1$ times,

$$\tau_p(\alpha_m)(k\alpha_p) \sim_{\star} x_p(k\alpha_m),$$

which together with $k\alpha_m \succsim_{\mathbf{C}} \lambda$ implies

$$x_p\lambda \precsim_{\star} \tau_p(\alpha_m)(k\alpha_p),$$

which by Eq. (42) yields

$$\tau_p(\alpha_m)\gamma_p \precsim_\star \tau_p(\alpha_m)(k\alpha_p),$$

which yields

$$k\alpha_p \succcurlyeq_{\mathbf{C}} \gamma_p,$$

which by Eq. (41) yields

$$\delta \succcurlyeq_{\mathbf{C}} \gamma_p. \quad \square$$

Lemma 8.12. *Let $x_i\alpha_i$ be an unbounded* **X–C** *sequence. Then for each m in $\mathbb{I}^+$ and each $\lambda \succcurlyeq_{\mathbf{C}} \alpha_m$, there exists a Δ-set $\mathscr{S}$ dependent on $x_i\alpha_i$ such that $\Delta_{m,\lambda}$ is in $\mathscr{S}$.*

Proof. Let m be an arbitrary element of $\mathbb{I}^+$, and λ be an arbitrary element of **C** such that $\lambda \succcurlyeq_{\mathbf{C}} \alpha_m$. Let $\mathscr{S}$ be the largest set of elements of the form $\Delta_{k,\beta}$, where k is in $\mathbb{I}^+$ and $\beta \succcurlyeq_{\mathbf{C}} \alpha_k$, such that $\Delta_{m,\lambda}$ is in Δ and for all $\Delta_{k,\beta}$ in Δ, if $\Delta_{k,\beta}(\delta) = \Delta_{m,\lambda}(\delta)$ for some δ in **C**, then $\Delta_{k,\beta}(\sigma) = \Delta_{m,\lambda}(\sigma)$ for all σ common to the domains of $\Delta_{k,\beta}$ and $\Delta_{m,\lambda}$. $\mathscr{S}$ exists by theorems of set theory. It will be shown that $\mathscr{S}$ satisfies statements 1–4 of Definition 8.8.

1. $\mathscr{S} \neq \emptyset$ since $\Delta_{m,\lambda}$ is in $\mathscr{S}$.

2. Suppose $\Delta_{j,\gamma}$ and $\Delta_{k,\beta}$ are in $\mathscr{S}$. Let α_p be the largest element of $\{\alpha_j, \alpha_k, \alpha_m\}$ and δ be an element of **C** such that $\delta \succ_C \alpha_p$. Then δ is in the domains of $\Delta_{j,\gamma}$, $\Delta_{k,\beta}$, and $\Delta_{m,\lambda}$. By the definition of $\mathscr{S}$,

$$\Delta_{j,\gamma}(\delta) = \Delta_{m,\lambda}(\delta) = \Delta_{k,\beta}(\delta).$$

Thus by Lemma 8.9, $\Delta_{j,\gamma}(\sigma) = \Delta_{k,b}(\sigma)$ for all σ common to the domains of $\Delta_{j,\gamma}$ and $\Delta_{k,\beta}$.

3. Suppose $\Delta_{k,\beta}$ is in $\mathscr{S}$, j is in $\mathbb{I}^+$, $\gamma \succcurlyeq_{\mathbf{C}} \alpha_j$, δ is in **C**, and $\Delta_{j,\gamma}(\delta) = \Delta_{k,\beta}(\delta)$. Let α_p be the largest element of $\{\alpha_j, \alpha_k, \alpha_m\}$ and η be an element of **C** such that $\eta \succ_C \alpha_p$. Then η is in the domains of $\Delta_{j,\gamma}, \Delta_{k,\beta}$, and $\Delta_{m,\lambda}$. By Lemma 8.9, $\Delta_{j,\gamma}(\eta) = \Delta_{k,\beta}(\eta)$. Since $\Delta_{k,\beta}$ is in $\mathscr{S}$, $\Delta_{k,\beta}(\eta) = \Delta_{m,\lambda}(\eta)$. Thus

$$\Delta_{j,\gamma}(\eta) = \Delta_{m,\lambda}(\eta).$$

Therefore by Lemma 8.9, $\Delta_{j,\gamma}(\sigma) = \Delta_{m,\lambda}(\sigma)$ for all σ common to the domains of $\Delta_{j,\gamma}$ and $\Delta_{m,\lambda}$. Thus $\Delta_{j,\gamma}$ is in $\mathscr{S}$.

4. Suppose $\Delta_{i,\gamma}$ and $\Delta_{k,\beta}$ are in $\mathscr{S}$. Then by the argument presented in the numbered 2 paragraph above, $\Delta_{j,\gamma}(\delta) = \Delta_{k,\beta}(\delta)$ for some δ in **C**. $\square$

Lemma 8.13. *Suppose f is a* **C**-*function. Then f is an automorphism of $\langle \mathbf{C}, \geqslant_C, \oplus \rangle$.*

Proof. Throughout this proof let, by Definitions 8.8 and 8.9, $\mathscr{S}$ be a Δ-set dependent on the sequence $x_i\alpha_i$ such that

$$f = \bigcup S.$$

It will first be shown that f is a function on **C**. Since f is a union of binary relations, f is a binary relation. Suppose (δ, α) and (δ, β) are arbitrary elements of f. By Definitions 8.8 and 8.9 let $\Delta_{j,\gamma}$ and $\Delta_{k,\sigma}$ in $\mathscr{S}$ be such that $\Delta_{j,\gamma}(\delta) = \alpha$ and $\Delta_{k,\sigma}(\delta) = \beta$. By statement 2 of Definition 8.7, $\alpha = \beta$. Since (δ, α) and (δ, β) are arbitrary elements of f, it follows that f is a function.

It will now be shown that f is onto **C**. Let δ be an arbitrary element of **C**. Since by statement 1 of Definition 8.8 $\mathscr{S} \neq \emptyset$, let $\Delta_{m,\lambda}$ be an element of $\mathscr{S}$. By Lemma 8.10,

$$\Delta_{m,\lambda}(\alpha_m) = \lambda.$$

By Lemma 8.11, let p in $\mathbb{I}^+$ and γ in **C** be such that

$$\delta \succcurlyeq_{\mathbf{C}} \gamma \quad \text{and} \quad \tau_p(\alpha_m)\gamma \precsim_\star x_p\lambda.$$

Then by Definition 8.5,

$$\Delta_{p,\gamma}(\alpha_m) = \lambda.$$

Thus by statement 3 of Definition 8.8, $\Delta_{p,\gamma}$ is in $\mathscr{S}$. Since $\delta \succcurlyeq_{\mathbf{C}} \gamma$, it follows from statement 3 of Lemma 8.6 that $\Delta_{p,\gamma}(\theta) = \delta$ for some $\theta \succcurlyeq_{\mathbf{C}} \alpha_p$. Thus f is onto **C**.

It will next be shown that f is defined on all of **C**. Let δ be an arbitrary element of **C**. Since f is onto **C**, it follows from Definitions 8.8 and 8.9 that α_p, α, and γ can be found so that $\Delta_{p,\gamma}(\alpha) = \delta$. Then by Definition 8.5 it follows that $\delta \succcurlyeq_{\mathbf{C}} \alpha_p$ and therefore by Definition 8.5 that $\Delta_{p,\gamma}(\delta)$ exists. Thus by Definitions 8.8 and8.9, $f(\delta)$ is defined.

Statements 1 and 2 of Lemma 8.6 together with Definitions 8.8 and 8.9 show that f preserves $\succcurlyeq_{\mathbf{C}}$ and $\oplus$. Therefore f is an automorphism of $\langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$. $\square$

Lemma 8.14. *Let $x_i\alpha_i$ be an unbounded* **X–C** *sequence, and let φ be an additive representation for $\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$. Then for each i in $\mathbb{I}^+$ there exists a function ψ_i from $\{x | x \in \mathbf{X}$ and $x_i \succcurlyeq_{\mathbf{X}} x\}$ into $\mathbb{R}^+$ such that for all x, y, α, and β, if $x_i \succcurlyeq_{\mathbf{X}} x$, $x_i \succcurlyeq_{\mathbf{X}} y$, $\alpha \succcurlyeq_{\mathbf{C}} \alpha_i$, and $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$, then*

$$x\alpha \precsim_\star y\beta \quad \textit{iff} \quad \frac{\psi_i(x)}{\varphi(\alpha)} \geqslant \frac{\psi_i(y)}{\varphi(\beta)}.$$

Proof. Since φ is an additive representation for $\mathfrak{C}$, it immediately follows that $\mathfrak{A} = \langle \varphi(\mathfrak{C}), \geqslant, + \rangle$ is an extensive structure that is isomorphic to $\mathfrak{C}$. It is well-known that each automorphism $\mathfrak{A}$ is a multiplication by a positive real number.

For each i in $\mathbb{I}^+$ and each x in **X**, if $x_i \succcurlyeq_{\mathbf{X}} x$, then for some δ, $x_i\delta \sim_\star x\alpha_i$ and (by statement 4 of Lemma 8.4) $\delta \succcurlyeq_{\mathbf{C}} \alpha_i$. Therefore, by Definition 8.5, for each i in $\mathbb{I}^+$ and each x in **X** such that $x_i \succcurlyeq_{\mathbf{X}} x$, let

$$\psi_i(x) = \frac{1}{\varphi(\tau_i^{-1}(x))}.$$

Let i, x, y, α, and β, be arbitrary elements such that i is in $\mathbb{I}^+$, $x_i \succcurlyeq_{\mathbf{X}} x$, $x_i \succcurlyeq_{\mathbf{X}} y$, $\alpha \succcurlyeq_{\mathbf{C}} \alpha_i$, and $\beta \succcurlyeq_{\mathbf{C}} \alpha_i$. Then by Definition 8.5,

$$\begin{aligned} x\alpha \precsim_\star y\beta \quad & \text{iff } \tau_i\tau_i^{-1}(x)\alpha \precsim_\star \tau_i\tau_i^{-1}(y)\beta \\ & \text{iff } x_i\Delta_{i,\alpha}[\tau_i^{-1}(x)] \precsim_\star x_i\Delta_{i,\beta}[\tau_i^{-1}(y)] \\ & \text{iff } \Delta_{i,\alpha}[\tau_i^{-1}(x)] \preccurlyeq_{\mathbf{C}} \Delta_{i,\beta}[\tau_i^{-1}(y)] \\ & \text{iff } \varphi(\Delta_{i,\alpha}[\tau_i^{-1}(x)]) \leqslant \varphi(\Delta_{i,\beta}[\tau_i^{-1}(y)]). \end{aligned}$$

Since by Lemmas 8.12 and 8.13 $\Delta_{i,\alpha}$ is part of an automorphism of $\mathfrak{C}$ and φ is an isomorphism of $\mathfrak{C}$ onto $\mathfrak{A}$,

$$\varphi(\Delta_{i,\alpha}[\tau_i^{-1}(x)]) = \varphi[\Delta_{i,\alpha}](\varphi(\tau_i^{-1}(x)),$$

where $\varphi[\Delta_{i,\alpha}]$ is part of an automorphism of $\mathfrak{A}$. Since by Lemma 8.10 $\Delta_{i,\alpha}(\alpha_i) = \alpha$, it then follows that $\varphi[\Delta_{i,\alpha}]$ must be multiplication by $\varphi(\alpha)/\varphi(\alpha_i)$. Therefore, by the above sequence of logical equivalences,

$$\begin{aligned} x\alpha \precsim_\star y\beta \quad & \text{iff } \varphi(\Delta_{i,\alpha}[\tau_i^{-1}(x)]) \leqslant \varphi(\Delta_{i,\beta}[\tau_i^{-1}(y)]) \\ & \text{iff } \varphi[\Delta_{i,\alpha}](\varphi(\tau_i^{-1}(x)) \leqslant \varphi[\Delta_{i,\beta}](\varphi(\tau_i^{-1}(y)) \\ & \text{iff } \frac{\varphi(\alpha)}{\varphi(\alpha_i)}\varphi(\tau_i^{-1}(x)) \leqslant \frac{\varphi(\beta)}{\varphi(\alpha_i)}\varphi(\tau_i^{-1}(y)) \\ & \text{iff } \frac{\varphi(\alpha)}{\varphi(\alpha_i)}\frac{1}{\psi_i(x)} \leqslant \frac{\varphi(\beta)}{\varphi(\alpha_i)}\frac{1}{\psi_i(y)} \\ & \text{iff } \frac{\psi_i(x)}{\varphi(\alpha)} \geqslant \frac{\psi_i(y)}{\varphi(\beta)}. \qquad \square \end{aligned}$$

Lemma 8.15. *Let $x_i\alpha_i$, φ_i, and ψ_i be as in Lemma 8.14. For each positive integer i let z_i be x_1 if $x_i \succcurlyeq_{\mathbf{X}} x_1$, and z_i be x_i if $x_1 \succcurlyeq_{\mathbf{X}} x_i$. (Thus in particular, $z_1 \succcurlyeq_{\mathbf{X}} z_i$ for all positive integers i.) Let*

$$\psi = \bigcup_{i=1}^{\infty} \frac{\psi_1(z_i)}{\psi_i(z_i)}\psi_i.$$

Then ψ is a function from $\mathbf{X}$ into $\mathbb{R}^+$.

Proof. Since ψ is a union of functions, ψ is a set of ordered pairs.

Let k be an arbitrary positive integer. Let

$$r = \frac{\psi_1(z_k)}{\psi_k(z_k)}\psi_k(z_k) = \psi_1(z_k).$$

Then the ordered pair (z_k, r) is in ψ. Suppose z_k is in the domain of ψ_j. Then (z_k, r') is in ψ, where

$$\frac{\psi_1(z_j)}{\psi_j(z_j)}\psi_j(z_k) = r'. \tag{44}$$

By statement 1 of Lemma 8.4, let δ and γ be such that

$$z_j\delta \sim_\star z_k\gamma.$$

Since $z_1 \succcurlyeq_{\mathbf{X}} z_j$ and $z_1 \succcurlyeq_{\mathbf{X}} z_k$, it follows by Lemma 8.14 that

$$\frac{\psi_j(z_j)}{\psi_j(z_k)} = \frac{\varphi(\delta)}{\varphi(\gamma)} = \frac{\psi_1(z_j)}{\psi_1(z_k)}.$$

Thus

$$\psi_j(z_k) = \frac{\psi_1(z_k)}{\psi_1(z_j)}\psi_j(z_j),$$

which by Eq. (44) yields

$$r' = \frac{\psi_1(z_k)}{\psi_j(z_j)}\psi_j(z_j) = \psi_1(z_k) = r. \tag{45}$$

To show ψ is a function, suppose (x, s) and (x, s') are elements of ψ. By the definition of ψ, let j and k be such that

$$\frac{\psi_1(z_j)}{\psi_j(z_j)}\psi_j(x) = s$$

and

$$\frac{\psi_1(z_k)}{\psi_k(z_k)}\psi_k(x) = s'.$$

Without loss of generality, suppose that $z_j \succcurlyeq_{\mathbf{X}} z_k$. Then z_k is in the domain of ψ_j. By statement 1 of Lemma 8.4, let μ and v be such that

$$x\mu \sim_\star z_k v.$$

Then by Lemma 8.14,

$$\frac{\psi_j(x)}{\psi_j(z_k)} = \frac{\varphi(\mu)}{\varphi(v)} = \frac{\psi_k(x)}{\psi_k(z_k)},$$

and thus

$$\psi_j(x) = \frac{\psi_j(z_k)}{\psi_k(z_k)}\psi_k(x). \tag{46}$$

From

$$s = \frac{\psi_1(z_j)}{\psi_j(z_j)}\psi_j(x),$$

it follows by Eq. (46) that

$$s = \frac{\psi_1(z_j)}{\psi_j(z_j)}\frac{\psi_j(z_k)}{\psi_k(z_k)}\psi_k(x) = \left[\frac{\psi_1(z_j)}{\psi_j(z_j)}\psi_j(z_k)\right]\left[\frac{1}{\psi_k(z_k)}\right]\psi_k(x),$$

which by Eqs. (44) and (45) yields

$$s = \frac{\psi_1(z_k)}{\psi_k(z_k)}\psi_k(x) = s'.$$

To show that ψ is defined on all of $\mathbf{X}$, let x be an arbitrary element of $\mathbf{X}$. Since $x_i\alpha_i$ is and unbounded $\mathbf{X}$–$\mathbf{C}$ sequence, let p be such that $x_p \succcurlyeq_{\mathbf{C}} x$. Then x is in the domain of ψ_p and thus is in the domain of ψ. $\square$

Lemma 8.16. *Let φ be an additive representation for $\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$. Then there exists a function ψ from $\mathbf{X}$ into $\mathbb{R}^+$ such that for all $x\alpha$ and $y\beta$ in the domain of $\precsim_\star$,*

$$x\alpha \precsim_\star y\beta \quad \text{iff } \frac{\psi(x)}{\varphi(\alpha)} \geqslant \frac{\psi(y)}{\varphi(\beta)}.$$

Proof. Let $x_i\alpha_i$ be an unbounded **X–C** sequence and ψ and ψ_i as in Lemmas 8.15 and 8.14, and let $x\alpha$ and $y\beta$ be arbitrary elements of the domain of $\succsim_\star$. Since $x_i\alpha_i$ is an unbounded **X–C** sequence, let j be such that

$$x_j \succcurlyeq_{\mathbf{C}} x \quad \text{and} \quad x_j \succcurlyeq_{\mathbf{C}} y.$$

By use of Lemma 8.4 it is easy to show that we can find δ and γ in **C** such that

$$\delta \succcurlyeq_{\mathbf{C}} \alpha_j, \quad \gamma \succcurlyeq_{\mathbf{C}} \alpha_j, \quad \text{and} \quad x\delta \sim_\star y\gamma.$$

It will first be shown that

$$x\alpha \succsim_\star y\beta \quad \text{iff } x(\alpha \oplus \delta) \succsim_\star y(\beta \oplus \gamma). \tag{47}$$

First suppose $x\alpha \succsim_\star y\beta$. Since $\mathfrak{C}$ is an extensive structure, $\alpha \oplus \delta \succcurlyeq_{\mathbf{C}} \alpha$ and $\beta \oplus \gamma \succcurlyeq_{\mathbf{C}} \beta$. Thus $x(\alpha \oplus \delta)$ and $y(\beta \oplus \gamma)$ are in the domain of $\succsim_\star$. Therefore, since $x\delta \sim_\star y\gamma$, it follows that from statement 3 of Lemma 8.4 that $x(\alpha \oplus \delta) \succsim_\star y(\beta \oplus \gamma)$. Now suppose not $x\alpha \succsim_\star y\beta$. Then $y\beta \succ_\star x\alpha$. Thus, since $x\delta \sim_\star y\gamma$, by statement 3 of Lemma 8.4, $y(\beta \oplus \gamma) \succ_\star x(\alpha \oplus \delta)$, and thus not $x(\alpha \oplus \delta) \succsim_\star y(\beta \oplus \gamma)$.

Lemma 8.14 applied to $x\delta \sim_\star y\gamma$ yields

$$\frac{\psi_j(x)}{\varphi(\delta)} = \frac{\psi_j(y)}{\varphi(\gamma)},$$

and thus

$$\psi_j(x)\varphi(\gamma) = \psi_j(y)\varphi(\delta). \tag{48}$$

Therefore by Eq. (47), Lemma 8.14, Definition 8.6, Eq. (48), and Lemma 8.15,

$$\begin{aligned}
x\alpha \succsim_\star y\beta \quad & \text{iff } x(\alpha \oplus \delta) \succsim_\star y(\beta \oplus \gamma) \\
& \text{iff } \frac{\psi_j(x)}{\varphi(\alpha \oplus \delta)} \geqslant \frac{\psi_j(y)}{\varphi(\beta \oplus \gamma)} \\
& \text{iff } \frac{\psi_j(x)}{\varphi(\alpha) + \varphi(\delta)} \geqslant \frac{\psi_j(y)}{\varphi(\beta) + \varphi(\gamma)} \\
& \text{iff } \psi_j(x)\varphi(\beta) + \psi_j(x)\varphi(\gamma) \\
& \geqslant \psi_j(y)\varphi(\alpha) + \psi_j(y)\varphi(\delta) \\
& \text{iff } \psi_j(x)\varphi(\beta) \geqslant \psi_j(y)\varphi(\alpha) \\
& \text{iff } \frac{\psi_j(x)}{\varphi(\alpha)} \geqslant \frac{\psi_j(y)}{\varphi(\beta)} \\
& \text{iff } \frac{\psi(x)}{\varphi(\alpha)} \geqslant \frac{\psi(y)}{\varphi(\beta)}.
\end{aligned}$$

8.2. Proofs for Section 2

Lemma 8.17. *Let $\langle u, v \rangle$ be a basic belief representation for $\succsim$ and a and b be in X. Then*

$$a \sim_X b \quad \textit{iff } u(a)v(a) = u(b)v(b).$$

Proof.

$$\begin{aligned}
a \sim_X b \quad & \text{iff } (a|a,b) \sim (b|a,b) \\
& \text{iff } \frac{u(a)v(a)}{u(a)+u(b)} = \frac{u(b)v(b)}{u(a)+u(b)} \\
& \text{iff } u(a)v(a) = u(b)v(b). \qquad \square
\end{aligned}$$

Definition 8.10. For all a and b in X, $a \approx b$ if and only if there exists e in X such that $e \neq a$, $e \neq b$ and

$$(e|e,a) \sim (e|e,b).$$

Theorem 8.2 (Theorem 2.1). *Assume the basic belief axioms (Definition 2.4) are true. Then the following two statements hold*:

1. (Representation theorem) *There exists a basic belief representation for $\succsim$ (Definition 2.3).*
2. (Uniqueness theorem) *Let*

$$\mathscr{U} = \{u | \langle u, v \rangle \ u \text{ is a basic belief representation for } \succsim\},$$

and}

$$\mathscr{V} = \{v | \langle u, v \rangle \ v \text{ is a basic belief representation for } \succsim\}.$$

Then $\mathscr{U}$ and $\mathscr{V}$ are ratio scales.

Proof. By Theorem 8.1 let φ be and additive representation for $\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$.

For each $A \in \mathscr{C}$, let α_A be the element of **C** such that $A \in \alpha_A$.

For each c in X, let (by Axiom 10 and Definition 8.10) c' be such that $c \approx c'$ and $c \neq c'$, and let u be the function on X defined by

$$u(c) = \frac{\varphi(\alpha_{\{c,c'\}})}{2}.$$

(To show u is well-defined, let c'' be such that $c \approx c''$ and $c \neq c''$. It is only necessary to show $\alpha_{\{c,c'\}} = \alpha_{\{c,c''\}}$. It follows from Axiom 4 that $\{c,c'\} \sim_{\mathscr{C}} \{c,c''\}$, and thus that $\alpha_{\{c,c'\}} = \alpha_{\{c,c''\}}$.)

1. Let $A = \{a, a_1, \ldots, a_n\}$ be an arbitrary element of $\mathscr{C}$ such that $a, a_1, \ldots, a_n$ are distinct. Then by Axiom 10 and Definition 8.10, let $A' = \{a', a'_1, \ldots, a'_n\}$ be such that $a', a'_1, \ldots, a'_n$ are distinct, $A \sim_{\mathscr{C}} A'$, $A \cap A' = \emptyset$, $a \approx a'$, and for $i = 1, \ldots, n$, $a_i \approx a'_i$. Because $A \sim_{\mathscr{C}} A'$, it follows that $\alpha_A = \alpha_{A'}$, and thus that

$$\varphi(\alpha_A) = \varphi(\alpha_{A'}).$$

Therefore, because φ is an additive representation for $\mathfrak{C}$,

$$\begin{aligned}
2\varphi(\alpha_A) &= \varphi(\alpha_A \oplus \alpha_{A'}) \\
&= \varphi(\alpha_{A \cup A'}) \\
&= \varphi[\alpha_{\{a,a'\}} \oplus \alpha_{\{a_1,a'_1\}} \oplus \cdots \oplus \alpha_{\{a_n,a'_n\}}]
\end{aligned}$$

$$= \varphi(\alpha_{\{a,a'\}}) + \sum_{1=1}^{n} \varphi(\alpha_{\{a_i,a_i'\}})$$
$$= 2u(a) + \sum_{i=1}^{n} 2u(a_i),$$

and thus

$$\varphi(\alpha_A) = \sum_{e \in A} u(e). \tag{49}$$

Because φ is an additive representation for $\mathfrak{C}$ and Eq. (49) holds for all A in $\mathscr{C}$, it follows that for all B and C in $\mathscr{C}$,

$$B \gtrsim_{\mathscr{C}} C \quad \text{iff} \quad \alpha_B \succcurlyeq_{\mathbf{C}} \alpha_C$$
$$\text{iff} \quad \varphi(\alpha_B) \geqslant \varphi(\alpha_C)$$
$$\text{iff} \quad \sum_{b \in B} u(b) \geqslant \sum_{c \in C} u(c).$$

Let ψ be as in Lemma 8.16. Define the function v on X as follows: for each e in X, let x_e be the element of $\mathbf{X}$ such that $e \in x_e$ and let

$$v(e) = \frac{\psi(x_e)}{u(e)}.$$

By Lemma 8.16 and Eq. (49) and the above definition of v,

$$(a|A) \gtrsim (b|B) \quad \text{iff} \quad x_a \alpha_A \gtrsim_{\bigstar} x_b \alpha_B$$
$$\text{iff} \quad \frac{\psi(x_a)}{\varphi(\alpha_A)} \geqslant \frac{\psi(x_b)}{\varphi(\alpha_B)}$$
$$\text{iff} \quad \frac{v(a)u(a)}{\sum_{e \in A} u(e)} \geqslant \frac{v(b)u(b)}{\sum_{e \in B} u(e)}$$
$$\text{iff} \quad v(a)\frac{u(a)}{\sum_{e \in A} u(e)} \geqslant v(b)\frac{u(b)}{\sum_{e \in B} u(e)}.$$

Let A and B be arbitrary elements of $\mathscr{C}$. By Definition 8.1 and Eq. (49),

$$A \gtrsim_{\mathscr{C}} B \text{ iff } \alpha_A \succcurlyeq_{\mathbf{C}} \alpha_B \text{ iff } \varphi(\alpha_A) \geqslant \varphi(\alpha_B)$$
$$\text{iff } \sum_{e \in A} u(e) \geqslant \sum_{e \in B} u(e).$$

2. Let $\langle u, v \rangle$ be a basic belief representation for $\gtrsim$. It is an immediate verification that for each s in $\mathbb{R}^+$, $\langle su, v \rangle$ is a basic belief representation for $\gtrsim$.

Suppose $\langle t, v \rangle$ is also a basic belief representation for $\gtrsim$. Then it follows from Eq. (49) that

$$u(A) = \sum_{e \in A} u(e) \quad \text{and} \quad t(A) = \sum_{e \in A} t(e),$$

for A in $\mathscr{C}$. It is easy to verify that $\hat{u}$ and $\hat{t}$ defined on $\mathbf{C}$ by

$$\hat{u}(\alpha_A) = u(A) \quad \text{and} \quad \hat{t}(\alpha_A) = t(A)$$

are additive representations for $\mathfrak{C} = \langle \mathbf{C}, \succcurlyeq_{\mathbf{C}}, \oplus \rangle$. Since by Lemma 8.7 the additive representations for $\mathfrak{C}$ form a ratio scale, let r be a positive real such that $r\hat{u} = \hat{t}$. Then $ru = t$.

The above establishes that

$$\mathscr{U} = \{u | \langle u, v \rangle \ u \text{ is a basic belief representation for } \gtrsim\}$$

is a ratio scale.

Because $\langle u, v \rangle$, a basic belief representation for $\gtrsim$, it is an immediate verification that for each s in $\mathbb{R}^+$, $\langle u, sv \rangle$ is a basic belief representation for $\gtrsim$.

Let a be an element of X, $\langle t, v' \rangle$ be an arbitrary basic belief representation for $\gtrsim$, and $r \in \mathbb{R}^+$ be such that

$$v'(a) = rv(a). \tag{50}$$

Let b be an arbitrary element of X. To complete the proof of statement 2, it is sufficient to show that

$$v'(b) = rv(b).$$

Because

$$\mathscr{U} = \{u | \langle u, v \rangle \ u \text{ is a basic belief reprsentation for } \gtrsim\}$$

is a ratio scale, $\langle ru, v \rangle$ is a basic belief representation for $\gtrsim$. For each C in $\mathscr{C}$, let

$$u(C) = \sum_{c \in C} u(c).$$

Case 1: $a \gtrsim_X b$. Let B in $\mathscr{C}$ be such that $b \in B$. Then by Axiom 12 let c in X and C in $\mathscr{C}$ be such that

$$c \sim_X a \quad \text{and} \quad (c|C) \sim (b|B). \tag{51}$$

Then by Eq. (51) and Lemma 8.17,

$$ru(a)v(a) = ru(c)v(c) \quad \text{and} \quad ru(a)v'(a) = ru(c)v'(c). \tag{52}$$

By Eq. (51),

$$\frac{ru(c)v(c)}{ru(C)} = \frac{ru(b)v(b)}{ru(B)} \tag{53}$$

and

$$\frac{ru(c)v'(c)}{ru(C)} = \frac{ru(b)v'(b)}{ru(B)}. \tag{54}$$

Eqs. (51)–(53) yield

$$\frac{u(a)v(a)}{u(C)} = \frac{u(b)v(b)}{u(B)}, \tag{55}$$

and Eqs. (51), (52), and (54) yield

$$\frac{u(a)v'(a)}{u(C)} = \frac{u(b)v'(b)}{u(B)}. \tag{56}$$

Then by Eqs. (50), (55), and (56),

$$v'(b) = rv(b).$$

Case 2: $b \gtrsim_X a$. Let A in $\mathscr{C}$ be such that $a \in A$. Then by Axiom 12 let c in X and C in $\mathscr{C}$ be such that

$$c \sim_X b \quad \text{and} \quad (c|C) \sim (a|A). \tag{57}$$

Then by Eq. (57) and Lemma 8.17,

$$ru(b)v(b) = ru(c)v(c) \quad \text{and} \quad ru(b)v'(b) = ru(c)v'(c). \tag{58}$$

By Eq. (57),

$$\frac{ru(c)v(c)}{ru(C)} = \frac{ru(a)v(a)}{ru(A)} \tag{59}$$

and

$$\frac{ru(c)v'(c)}{ru(C)} = \frac{ru(a)v'(a)}{ru(A)}. \tag{60}$$

Eqs. (57)–(59) yield

$$\frac{u(b)v(b)}{u(C)} = \frac{u(a)v(a)}{u(A)}, \tag{61}$$

and Eqs. (57), (58), and (60) yield

$$\frac{u(b)v'(b)}{u(C)} = \frac{u(a)v'(a)}{u(A)}. \tag{62}$$

Then by Eqs. (50), (61), and (62),

$v'(b) = rv(b)$. □

Theorem 8.3 (Theorem 2.2). *Assume the basic belief axioms with binary symmetry* (*Definition* 2.5). *Then the following two statements hold*:

1. (Representation theorem) *There exists a basic choice representation for* $\succsim$ (*Definition* 2.6).
2. (Uniqueness theorem) *The set of basic choice representations for* $\succsim$ *forms a ratio scale.*

Proof. 1. Because the basic belief axioms with binary symmetry imply the basic belief axioms, by Theorem 8.2 let $\langle u, v\rangle$ be a basic belief representation for $\succsim$. Then by Theorem 8.2, (i) u is a function from X into $\mathbb{R}^+$, (ii) for all A and B in $\mathscr{C}$,

$$A \succsim_{\mathscr{C}} B \quad \text{iff} \quad \sum_{a\in A} u(a) \geqslant \sum_{b\in B} u(b),$$

and (iii) for all finite conditional events $(a|A)$ and $(b|B)$ of X,

$$(a|A) \succsim (b|B) \quad \text{iff} \quad v(a)\frac{u(a)}{\sum_{e\in A} u(e)} \geqslant v(b)\frac{u(b)}{\sum_{e\in B} u(e)}.$$

Thus to show statement 1, it is sufficient to show that for all a and b in X,

$$v(a) = v(b) \quad \text{and} \quad [a \succsim_X b \text{ iff } u(a) \geqslant u(b)].$$

Let a and b be arbitrary elements of X and A and B be arbitrary elements of $\mathscr{C}$. Suppose $a \in A$ and $b \in B$. By Axiom 9, let a' and b' be such that $a \neq a'$, $b \neq b'$, $a \sim_X a'$, and $b \sim_X b'$. Then by Definition 2.2,

$$(a|a,a') \sim (a'|a,a') \quad \text{and} \quad (b|b,b') \sim (b'|b,b'). \tag{63}$$

Applying Theorem 8.2 to Eq. (63) yields,

$$\frac{v(a)u(a)}{u(a)+u(a')} = \frac{v(a')u(a')}{u(a)+u(a')} \quad \text{and}$$
$$\frac{v(b)u(b)}{u(b)+u(b')} = \frac{v(b')u(b')}{u(b)+u(b')}, \tag{64}$$

and applying Axiom 14 and Theorem 8.2 to Eq. (63) yields >

$$v(a)\frac{u(a)}{u(a)+u(a')} = v(b)\frac{u(b)}{u(b)+u(b')} \tag{65}$$

and

$$\frac{v(a)u(a)}{u(a)+u(b)} = \frac{v(a')u(a')}{u(a')+u(b')}. \tag{66}$$

Another application of Axiom 14 and Theorem 8.2 to Eq. (63) yields

$$\frac{v(a)u(a)}{u(a)+u(b')} = \frac{v(a')u(a')}{u(a')+u(b)}. \tag{67}$$

Eq. (64) implies $v(a)u(a) = v(a')u(a')$. Thus Eq. (66) implies

$$u(a) + u(b) = u(a') + u(b'), \tag{68}$$

and Eq. (67) implies

$$u(a) + u(b') = u(a') + u(b). \tag{69}$$

Adding Eqs. (68) and (69) and reducing then yields,

$u(a) = u(a')$,

which by Eq. (69) yields,

$u(b) = u(b')$,

and thus by Eq. (65),

$$v(a) = v(b). \tag{70}$$

Thus to complete the proof of statement 1 it needs to only be shown that

$$a \succsim_X b \quad \text{iff} \quad u(a) \geqslant u(b).$$

By Definition 2.2, Theorem 8.2, and Eq. (70),

$$\begin{aligned} a \succsim_X b \quad &\text{iff} \quad (a|a,b) \succsim (b|a,b) \\ &\text{iff} \quad v(a)\frac{u(a)}{u(a)+u(b)} \geqslant v(b)\frac{u(b)}{u(a)+u(b)} \\ &\text{iff} \quad u(a) \geqslant u(b). \end{aligned}$$

2. By Theorem 8.2,

$$\mathscr{U} = \{u | \langle u, v\rangle \text{ is a basic belief representation for } \succsim\},$$

forms a ratio scale. Thus statement 2 is true. □

8.3. Proofs for Section 3

Lemma 8.18. *Assume the belief axioms. By Theorem* 2.1, *let* $\langle u, v\rangle$ *be a basic belief representation for* $\succsim$. *Define* $\bar{u}$

and $\bar{v}$ on $\mathscr{F}$ as follows: Let A be an arbitrary element of $\mathscr{F}$.

(i) *Suppose $A=\emptyset$. Define $\bar{u}(A)=0$ and $\bar{v}(A)=1$.*
(ii) *Suppose $A\neq\emptyset$. Let B be an element of $\mathscr{F}$ such that $A\subset B$. By Axiom 17, let e, E, and f be such that $B\sim_{\mathscr{C}}E$,*

$$(A|B)\sim_{\mathscr{E}}(e|E),$$

$$f\neq e,\ f\notin A, \text{ and}$$

$$(f|e,f)\sim_{\mathscr{E}}(f|A\cup\{f\}).$$

Then define $\bar{u}(A)=u(e)$ and $\bar{v}(A)=v(e)$.

Then the following three statements are true:

1. *$\bar{u}(\emptyset)=0$ and $\bar{v}(\emptyset)=1$.*
2. *If for some a, $A=\{a\}$, then $\bar{u}(\{a\})=u(a)$ and $\bar{v}(\{a\})=v(a)$.*
3. *$\bar{u}$ and $\bar{v}$ are well-defined on $\mathscr{F}$: That is, if $A\neq\emptyset$ and B', e', E' and f' are such that B' is in $\mathscr{F}$, $A\subset B'$, $B'\sim_{\mathscr{C}}E'$,*

$$(A|B')\sim_{\mathscr{E}}(e'|E'),$$

$$f'\neq e',\ f'\notin A, \text{ and}$$

$$(f'|e',f')\sim_{\mathscr{E}}(f'|A\cup\{f'\}),$$

then $\bar{u}(A)=u(e')$ and $\bar{v}(A)=v(e')$.

Proof. To simplify notation, for each nonempty C in $\mathscr{C}$, let

$$u(C)=\sum_{c\in C}u(c).$$

1. Statement 1 immediately follows from condition (i).
2. Suppose $A=\{a\}$. Then By condition (ii),

$$(f|e,f)\sim_E(f|a,f),$$

and thus

$$v(f)\frac{u(f)}{u(e)+u(f)}=v(f)\frac{u(f)}{u(a)+u(f)},$$

yielding

$$\bar{u}(\{a\})=u(e)=u(a). \tag{71}$$

Also by condition (ii),

$$B\sim_{\mathscr{C}}E \quad\text{and}\quad (\{a\}|B)\sim_{\mathscr{E}}(e|E),$$

and thus

$$u(B)=u(E) \quad\text{and}\quad \frac{u(\{a\})}{u(B)}v(\{a\})=\frac{u(e)}{u(E)}v(e),$$

which together with Eq. (71) yields,

$$\bar{v}(\{a\})=v(e)=v(a).$$

3. Assume the hypotheses of statement 3. Suppose $A\neq\emptyset$. By condition (ii),

$$(f|e,f)\sim_E(f|A\cup\{f\}),$$

and thus

$$v(f)\frac{u(f)}{u(e)+u(f)}=v(f)\frac{u(f)}{u(A)+u(f)},$$

yielding

$$u(e)=u(A). \tag{72}$$

Similarly, the hypothesis

$$(f'|e',f')\sim_{\mathscr{E}}(f'|A\cup\{f'\})$$

yields

$$u(e')=u(A). \tag{73}$$

Thus by Eqs. (72) and (73),

$$\bar{u}(A)=u(e)=u(e'). \tag{74}$$

Because by hypothesis,

$$(A|B)\sim_{\mathscr{E}}(e|E),\ (A|B')\sim_{\mathscr{E}}(e'|E'),\ B\sim_{\mathscr{C}}E, \text{ and } B'\sim_{\mathscr{C}}E',$$

it follows by Axiom 18 that

$$(e|e,e')\sim_{\mathscr{E}}(e'|e,e').$$

Thus by Theorem 2.1,

$$\frac{u(e)v(e)}{u(e)+u(e')}=\frac{u(e')v(e')}{u(e)+u(e')},$$

which by Eq. (74) yields

$$\bar{v}(e)=v(e)=v(e'). \qquad\square$$

Lemma 8.19. *Assume the hypotheses and notation of Lemma 8.18. Let A be a nonempty finite event of X, and let, by definition,*

$$u(A)=\sum_{a\in A}u(a).$$

Then

$$\bar{u}(A)=\sum_{a\in A}\bar{u}(\{a\})=\sum_{a\in A}u(a)=u(A).$$

Proof. Since $\langle u,v\rangle$ is a basic belief representation for $\succsim$, it follows by statement 2 of Lemma 8.18 that for each x in X,

$$\bar{u}(\{x\})=u(x).$$

Thus,

$$\sum_{a\in A}\bar{u}(\{a\})=\sum_{a\in A}u(a)=u(A),$$

and by Eqs. (73) and (74),

$$\bar{u}(A)=u(A). \qquad\square$$

Theorem 8.4 (Theorem 3.1). *Assume the belief axioms (Definition 3.2). Then the following two statements are true:*

1. *There exists a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v.*

2. *Let $\mathbb{B}$ be a belief representation for $\succsim_{\mathscr{E}}$ with context function u and definiteness function v. Then the following two statements are true*:
   (i) *For all positive reals r and s there exists a belief representation for $\succsim_{\mathscr{E}}$ with context function ru and definiteness function sv.*
   (ii) *Let $\mathbb{B}_1$ be a belief representation for $\succsim_{\mathscr{E}}$ with context function u_1 and definiteness function v_1 Then for some positive real numbers r and s,*

$$u_1 = ru \quad \text{and} \quad v_1 = sv.$$

Proof. 1. Let $u,\ v,\ \bar{u}$, and $\bar{v}$ be as in Lemma 8.18. For each nonempty finite event of X, let, by definition,

$$u(C) = \sum_{c\in C} u(c).$$

And for each finite event $(A|B)$ of X, let

$$\mathbb{B}(A|B) = \bar{v}(A)\frac{\bar{u}(A)}{\bar{u}(B)}. \tag{75}$$

Then it will be shown that $\mathbb{B}$ is a belief representation for $\succsim_E$ with context function $\bar{u}$ and definiteness function $\bar{v}$ (Definition 3.3). It is immediate that conditions (i), (ii), and (iv) of Definition 3.3 hold for $\mathbb{B}$, $\bar{u}$, and $\bar{v}$. Condition (iii) follows from the definition of $\bar{u}$ and Lemma 8.19. To show condition (v), let $(A|B)$ and $(C|D)$ be arbitrary conditional events.

Case 1: *Either $A=\emptyset$ or $C=\emptyset$.* Suppose $(A|B)\succsim_{\mathscr{E}}(C|D)$. (i) If $A=\emptyset$, then by Axioms 15 and 16, $C=\emptyset$. Thus by Eq. (75) and statement 1 of Lemma 8.18,

$$\mathbb{B}(A|B) = 0 \geqslant 0 = \mathbb{B}(C|D).$$

(ii) If $C=\emptyset$, then by Eq. (75) and statement 1 of Lemma 8.18,

$$\mathbb{B}(A|B) \geqslant 0 = \mathbb{B}(C|D).$$

Suppose $\mathbb{B}(A|B)\geqslant\mathbb{B}(C|D)$. If $A=\emptyset$, then $\mathbb{B}(C|D)=0$, which by Eq. (75) and the definition of $\bar{u}$ in Lemma 8.18 yields $C=\emptyset$, which by Axioms 15 and 16 yields $(A|B)\succsim_{\mathscr{E}}(C|D)$. If $C=\emptyset$, then $(A|B)\succsim_{\mathscr{E}}(C|D)$ by Axiom 12.

Case 2: *$A\neq\emptyset$, $C\neq\emptyset$, and both B and D have at least two elements*: By Axiom 17, let $e,\ E,\ f$ and $g,\ G,\ h$ be such that the following two conditions hold:

(1) $B\sim_{\mathscr{C}}E,\ D\sim_{\mathscr{C}}G,\ e\notin A,\ g\notin C$, and
$$(A|B)\sim_{\mathscr{E}}(e|E) \quad \text{and} \quad (C|D)\sim_{\mathscr{E}}(g|G).$$

(2) $f\neq e,\ f\notin A,\ h\neq g,\ h\notin C$, and
$$(f|e,f)\sim_{\mathscr{E}}(f|A\cup\{f\}) \quad \text{and} \quad (h|g,h)\sim_{\mathscr{E}}(h|C\cup\{h\}).$$

Then by Lemma 8.18,

$$\bar{u}(A) = u(e) \quad \text{and} \quad \bar{v}(A) = v(e).$$

Because $B\sim_{\mathscr{C}}E$ and $\langle u,v\rangle$ is a basic belief representation for $\succsim$, $u(B)=u(E)$. By Lemma 8.19,

$$\bar{u}(B) = u(B).$$

Thus,

$$\mathbb{B}(A|B) = \bar{v}(A)\frac{\bar{u}(A)}{\bar{u}(B)} = \bar{v}(A)\frac{u(A)}{u(B)} = v(e)\frac{u(e)}{u(E)}.$$

Similarly,

$$\mathbb{B}(C|D) = \bar{v}(C)\frac{\bar{u}(C)}{\bar{u}(D)} = v(g)\frac{u(g)}{u(G)}.$$

Thus, because $\langle u,v\rangle$ is a basic belief representation for $\succsim$ and $\succsim_{\mathscr{E}}$ is an extension of $\succsim$,

$$\begin{aligned}(A|B)\succsim_{\mathscr{E}}(C|D) \quad &\text{iff } (e|E)\succsim(g|G)\\ &\text{iff } v(e)\frac{u(e)}{u(E)}\geqslant v(g)\frac{u(g)}{u(G)}\\ &\text{iff } \mathbb{B}(A|B)\geqslant\mathbb{B}(C|D).\end{aligned}$$

Case 3: *$A=B$ or $C=D$.* Because of statement 1 of Axiom 19, we may assume that B and D are in $\mathscr{C}$. Then by statement 2 of Axiom 12, let B' and D' be such that $B\sim_{\mathscr{C}}B',\ B\cap B'\neq\emptyset,\ D\sim_{\mathscr{C}}D'$, and $D\cap D'\neq\emptyset$, and

$$(A|B)\succsim_{\mathscr{E}}(C|D) \quad \text{iff } (A|B\cup B')\succsim_{\mathscr{E}}(C|D\cup D'). \tag{76}$$

Thus by Case 2 (which has already been shown),

$$(A|B\cup B')\succsim_{\mathscr{E}}(C|D\cup D') \quad \text{iff } \mathbb{B}(A|B\cup B')\geqslant\mathbb{B}(C|D\cup D'). \tag{77}$$

But by conditions (iii) and (iv) of Definition 3.3,

$$\mathbb{B}(A|B\cup B') = \tfrac{1}{2}\mathbb{B}(A|B) \quad \text{and} \quad \mathbb{B}(C|D\cup D') = \tfrac{1}{2}\mathbb{B}(C|D). \tag{78}$$

Thus by Eqs. (76)–(78).

$$(A|B)\succsim_{\mathscr{E}}(C|D) \quad \text{iff } \mathbb{B}(A|B)\geqslant\mathbb{B}(C|D).$$

2. Part (i), statement 2 of the theorem follows by direct verification. Because $\bar{u}(\emptyset)=0$ and for $A\neq\emptyset$, $\bar{u}(A)$ and $\bar{v}(A)$ are defined to be, respectively, $u(e)$ and $v(e)$ for an appropriately chosen element e of X, part (ii) follows from statement 2 of Theorem 2.1, statement 2 of Lemma 8.18, and the definitions of $\bar{u}$ and $\bar{v}$. □

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