JOURNAL OF MATHEMATICAL PSYCHOLOGY **40**, 109–129 (1996)
ARTICLE NO. 0011

A Theory of Ratio Magnitude Estimation*

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Various axiomatic theories of magnitude estimation are presented. The axioms are divided into the following categories: *behavioral*, in which the primitive relationships are in principle observable by the experimenter; *cognitive*, in which the primitive relationships are theoretical in nature and deal with subjective relationships that the subject is supposedly using in making his or her magnitude estimations; and *psychobehavioral*, in which the relationships are theoretical and describe a supposed relationship between the experiment's stimuli and the subject's sensations of those stimuli. The goal of these axiomatizations is to understand from various perspectives what must be observed by the experimenter and assumed about the subject so that the results from an experiment in which the subject is asked to estimate or produce ratios are consistent with the proposition that the subject is, in a scientific sense, "computing ratios" in making his or her magnitude responses.

1. INTRODUCTION

General

Stevens' (1946, 1951) method of magnitude estimation is still widely used and highly controversial. This paper presents axiomatic theories for measuring stimuli in terms of magnitude estimations. These theories are about *idealized* situations and do not involve considerations of error.

Although I believe the axiomatic approaches of this paper will prove useful in clarifying some of the murky and controversial issues that continue to plague the magnitude estimation literature, it is not my intention either to review various parts of this vast literature or to examine it systematically in terms of the new results presented here.

In psychology, Shepard (1978, 1981), Krantz (1972), Marley (1972), and Luce (1990) have provided systematic theories of magnitude estimation and the related technique of cross modality matching. (Krantz's and Luce's theories are axiomatic; Marley's is probabilistic.) The axiomatizations presented here, besides having different primitive concepts and axioms, differ from these and other theories in a number of important respects: In general, the axiomatizations do not assume that the subject in a magnitude estimation experiment has a good understanding of numbers or numerals; nor do they assume (as in many psychophysical applications) that the subject's responses exhibit invariance with respect to transformations of the stimuli; and some of the axiomatizations are based in part on meaningfulness considerations—specifically, that the subject's subjective interpretations of numerals in a magnitude estimation experiment correspond to functions "computed meaningfully" from some underlying "inner psychological measurement structure". (Other axiom systems are also presented that are based only on behavioral considerations and make no assumptions about how the subject produces his or her responses).

Two Principles of the Stevens' Theory of Magnitude Estimation

Let X be a set of stimuli that is presented to a subject. Then *Stevens' method of ratio magnitude estimation* proceeds by having the subject produce a function φ_t from X into $\mathbb{R}^+$ with the following properties: An element t—which we will call the *modulus*—is selected from X. The subject is instructed to consider the number 1 as representing his or her subjective intensity of t, and to keep this consideration in mind in giving his or her numerical estimate of his or her subjective intensity of stimuli x in X. The experimenter uses these verbal estimates of the subject to construct the function φ_t by assigning the number corresponding to the subject's numerical estimate as the value of the function $\varphi_t(x)$.

Although Stevens is a very articulate and, on the surface, an apparently clear writer, his expositions about what underlies and is accomplished by his method of ratio magnitude estimation lacks rigor. I believe the following two assumptions, which I will refer to as *Stevens's Assumptions*, are inherent in his ideas about ratio magnitude estimation:

1. The function φ_t is an element of a ratio scale $\mathscr{S}$ that adequately measures the subject's subjective intensity of stimuli in X.

2. Each element x in X can be used as a modulus and the resulting representation φ_x is in the ratio scale $\mathscr{S}$, i.e., there exists r in $\mathbb{R}^+$ such that $\varphi_x = r\varphi_t$.

* Research for this article was supported by National Science Foundation grant SBR 9520107 to the University of California, Irvine. The author thanks R. Duncan Luce, Tony Marley, and Uta Wille for helpful continents on earlier drafts of this article, and Dieter Heyer and Reinhard Niederée for numerous critical and technical suggestions.

0022-2496/96 $18.00

Let $D = \{\varphi_t \mid t \in X\}$. Then D is the complete data set that is generated by conducting all possible magnitude estimations of stimuli in X with all possible moduli from X. D can be recoded as the set E of 3-tuples of the form $(x, \mathbf{p}, t)$, where

$$(x, \mathbf{p}, t) \in E \qquad \text{iff} \quad \varphi_t(x) = p.$$

In triples $(x, \mathbf{p}, t)$ in E, $\mathbf{p}$ is put in bold typeface because it represents a *numeral* and not a *number*. In this article, numbers are assumed to be *highly abstract scientific objects*, and it is not assumed that subjects understand or use such scientific objects in their calculations or responses, nor is it assumed that subjects have or use a philosophically sound correspondence between (scientific) numbers and numerals. Thus for the purposes of this paper, the occurrence of the numeral $\mathbf{p}$ in the expression "$(x, \mathbf{p}, t)$" should be interpreted as a response item provided by the subject to the experimenter.

Definition 1. Let X and E be as above. Then E is said to have the *multiplicative property* if and only if for all x and t in X and all p in $\mathbb{R}^+$, if $(x, \mathbf{p}, t) \in E$, $(y, \mathbf{q}, x) \in E$, and $(y, \mathbf{r}, t) \in E$, then $r = p \cdot q$.

The multiplicative property obviously imposes powerful constraints on the subject's magnitude estimation behavior. It is implied by Stevens' Assumptions:

Theorem 1. *Let X and E be as above, and suppose Stevens' Assumptions and that $\varphi_z(z) = 1$ for all z in X. Then E has the multiplicative property.*

Proof. Suppose $(x, \mathbf{p}, t) \in E$, $(y, \mathbf{q}, x) \in E$, and $(y, \mathbf{r}, t) \in E$. Then $\varphi_t(x) = p$, $\varphi_x(y) = q$, and $\varphi_t(y) = r$. By Stevens' Assumptions, let u in $\mathbb{R}^+$ be such that

$$\varphi_x = u\varphi_t.$$

Then $1 = \varphi_x(x) = u\varphi_t(x) = u \cdot p$, i.e.,

$$u = \frac{1}{p}.$$

Therefore,

$$q = \varphi_x(y) = u\varphi_t(y) = \frac{1}{p} \cdot \varphi_t(y) = \frac{1}{p} \cdot r,$$

i.e., $r = p \cdot q$.

In almost all the cases in the literature either insufficient or the wrong kind of data have been collected for the valid testing of the multiplicative property. However, because of the strong structural relationship it describes. I suspect that the multiplicative property would fail empirically for most of the kinds of situations where magnitude estimation is employed. If such massive failures are indeed the case, then in light of Theorem 1 something fundamental must be changed in the preceding theory of ratio magnitude estimation.

In my opinion, it is Stevens' method of constructing the representations φ_t that is most suspect: I see no reason why just because a subject says or indicates for a fix modulus t and for various x in X that "x is $\mathbf{p}$ times more intense than t," it then follows that $\varphi_t(x) = p$ is a valid representation of the subject's subjective intensity of x with respect to t. Stevens and other magnitude estimation theorists do not provide any theoretical or even intuitive rationale for this; at most they only note that the method of ratio magnitude estimation produces representations (that they presume to be part of ratio scales) that interrelate in consistent and theoretically interesting ways with other phenomena. It is my conjecture that the consistency results not by reflecting some underlying reality but from a lack of enough relevant data that might reveal structural inconsistencies.

A More General Form of Ratio Magnitude Estimation

Another property of magnitude estimation behavior, which to my knowledge has never been systematically investigated, is the *commutative property*. This latter property, which is weaker than the multiplicative property, is defined formally as follows:

Definition 2. Let X and E be as above. Then E is said to have the *commutative property* if and only if for all p and q in $\mathbb{I}^+$ and all x, y, z, t, and w in X, if $(x, \mathbf{p}, t) \in E$, $(z, \mathbf{q}, x) \in E$, $(y, \mathbf{q}, t) \in E$, and $(w, \mathbf{p}, y) \in E$, then $z = w$.

It is shown latter in this article that there are ratio magnitude estimation situations in which the multiplicative property fails but the commutative property holds and the situation can be measured in such a way that (i) a ratio scale $\mathscr{S}$ on the stimuli results, and (ii) there is a strict order preserving mapping f from the numerals occurring in the magnitude estimation situation into the positive real numbers such that for all stimuli x and t and all numerals $\mathbf{p}$,

$$(x, \mathbf{p}, t) \qquad \text{iff} \quad \psi(x) = f(\mathbf{p})\, \psi(t) \qquad \text{for all} \quad \psi \text{ in } \mathscr{S}. \tag{1}$$

One way of interpreting Eq. (1) is that (1) the numerals represent the subject's subjective measures (and *not numerical measures*) of ratios of subjective intensities, and (2) f is the scientific way of interpreting these subjective measures of ratios as numerical ratios. The form of ratio magnitude estimation embodied in Eq. (1) is clearly a generalization of the kind of magnitude estimation which results in those situations for which Stevens' methods for constructing ratio scales for magnitude estimation data are appropriate: If the multiplicative property holds, then f in Eq. (1) can be chosen so that $f(\mathbf{p}) = p$, i.e., f can be chosen in such a way that it interprets numerals in the same manner as is done in science. However, there is nothing in either the ways of collecting the magnitude estimation data or in the representational theory that is used later in this article to measure such data to suggest that such a coinciding of interpretations is anything more than a coincidence. In order to treat it as something more, further theoretical assumptions need to be made about the subject's mental processing in a ratio magnitude estimation paradigm, and such assumptions necessarily fall outside of behavioral modeling of the data.

Overview and Organization of the Article

To make the technical material easier to understand and flow together, various proofs of theorems and lemmas are placed at the end of the paper in Section 5. Some of the proofs in Section 5 make use of results and concepts presented in previous sections.

Section 2 presents some basic concepts about magnitude estimation and theory of measurement. This material is background to the rest of the paper.

Section 3 presents various axiomatizations of ratio magnitude estimation: It is assumed that the responses of a subject participating in a magnitude estimation experiment are coded as above as triples of the form $(x, \mathbf{p}, t)$. The axiomatizations state what *observable* properties the triple should have and what *theoretical* assumptions should be made about how the numerals $\mathbf{p}$ relate to an inner psychological measurement structure $\mathscr{J}$ on sensations. When the axioms are satisfied, it is shown that $\mathscr{J}$ can be measured in such a way that the measurements of the inner psychological interpretations of numerals $\mathbf{p}$ are functions that are multiplications by positive reals. It is not a consequence of the axiom system that the measurement of the psychological interpretation of the numeral $\mathbf{p}$ is the real number p that is the *scientific* designation of the numeral $\mathbf{p}$. However, an axiom can be added that achieves this, i.e., an axiom can be added so that under measurement each numeral $\mathbf{p}$ will correspond to a function that is multiplication by the number p. Also, axiomatizations are presented for ratio magnitude estimation that is based only on observable responses of the subject.

The axiomatizations discussed above are generalized to situations where totally ordered sets can be taken as "the numerals". An instance of this of particular importance to psychology is where the "numerals" $\mathbf{p}$ correspond to functions on a physical continuum that under *physical* measurement are represented as multiplications by positive reals.

Section 3 also discusses the semiclassical problem about whether or not there is an essential difference between representing numerals as multiplications versus additions (Torgerson, 1961; Birnbaum, 1982). A theorem is presented that shows under very plausible conditions that changes of the instructions to the subject (e.g., "estimate differences" instead of "estimate ratios") do not materially alter the *structure* of the data set. This result provides a theory for an observed empirical fact that equal difference judgments produce the same data as equal ratio judgments.

Section 4 discusses issues considered throughout the paper.

2. PRELIMINARIES

General

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{I}$ the set of integers, and $\mathbb{I}^+$ the set of positive integers, $\geqslant$ denotes the "greater than or equal to" relation on $\mathbb{R}$, and by convention, $\geqslant$ is used also to denote "greater than or equal to" relations on $\mathbb{R}^+$, $\mathbb{I}$, and $\mathbb{I}^+$. The (partial) operation of composing functions is denoted by $*$. For functions T and positive integers k, the notation T^k stands for k functional compositions of T, e.g., $T^2 = T * T$.

For a binary relation $\geqslant_*$, "$x >_* y$" stands for "$x \geqslant_* y$ and not $y \geqslant_* x$" and "$x \not>_* y$" stands for "not $x >_* y$".

By definition, $\geqslant_*$ is said to be a *total ordering* on X if and only if X is a nonempty set, $\geqslant_*$ is a transitive and reflexive relation on X, and for all x and y in X, either $x >_* y$ or $y >_* x$ or $x = y$. By definition, a set X is said to be *denumerable* if and only if there exists a one-to-one function from it onto $\mathbb{I}^+$.

Continua

A continuum is an ordered structure that is isomorphic to $\langle \mathbb{R}^+, \geqslant \rangle$. Continua were qualitatively characterized by Cantor (1895). The following definition and theorem essentially capture Cantor's characterization:

Definition 3. $\langle X, \succcurlyeq \rangle$ is said to be a *continuum* if and only if the following four statements are true:

1. *Total ordering*: $\succcurlyeq$ is a total ordering on X.
2. *Unboundedness*: $\langle X, \succcurlyeq \rangle$ has no greatest or least element.
3. *Density*: For all x and z in X, if $x \succ z$, then there exists y in X such that $x \succ y$ and $y \succ z$.
4. *Dedekind completeness*: Each $\succcurlyeq$-bounded nonempty subset of X has a $\succcurlyeq$-least upper bound.
5. *Denumerable density*: There exists a denumerable subset Y of X such that for each x and z in X, if $x \succ z$, then there exists y in Y such that $x \succcurlyeq y$ and $y \succcurlyeq z$.

Theorem 2. *$\langle X, \succcurlyeq \rangle$ is a continuum if and only if it is isomorphic to $\langle \mathbb{R}^+, \geqslant \rangle$.*

Proof. Cantor (1895). (A proof is also given in Theorem 2.2.2 of Narens, 1985.)

Measurement Scales and Structures

DEFINITION 4. $\mathscr{S}$ is said to be a (*measurement*) *scale* on X if and only if X is a nonempty set and $\mathscr{S}$ is a nonempty set of functions from X into $\mathbb{R}$.

Let $\mathscr{S}$ be a scale. Then $\mathscr{S}$ is said to be *homogeneous* if and only if (i) all elements of $\mathscr{S}$ have the same range R, and (ii) for each x in X and each r in the range R there exists φ in $\mathscr{S}$ such that $\varphi(x) = r$.

$\mathscr{S}$ is said to be a *trivial scale* if and only if it is a scale that consists of a single element.

$\mathscr{S}$ is said to be a *ratio scale* if and only if (i) $\mathscr{S}$ is a scale, (ii) for each φ in $\mathscr{S}$ and each r in $\mathbb{R}^+$, $r\varphi$ is in $\mathscr{S}$, and (iii) for all φ and γ in $\mathscr{S}$ there exists s in $\mathbb{R}^+$ such that $\varphi = s\gamma$.

$\mathscr{S}$ is said to be a *subscale of a ratio scale* if and only if $\mathscr{S}$ is a scale and $\mathscr{S} \subseteq \mathscr{T}$ or some ratio scale $\mathscr{T}$.

$\mathscr{S}$ is said to be a *translation scale* if and only if (i) $\mathscr{S}$ is a scale, (ii) for each φ in $\mathscr{S}$ and each r in $\mathbb{R}$, $r + \varphi$ is in $\mathscr{S}$, and (iii) for all φ and γ in $\mathscr{S}$, there exists s in $\mathbb{R}$ such that $\varphi = s + \gamma$.

DEFINITION 5. Let X be a nonempty set. Elements of X are called 0-*ary relations on* X. n-*ary relations on* X, $n \in \mathbb{I}^+$, are subsets of X^n (the Cartesian product of X with itself n times). In particular, 1-ary relations on X are subsets of X.

Relations based on X are a more general class of relations: Besides 0-ary and n-ary relations on X, they include sets of 0-ary and n-ary relations on X, relations of 0-ary and n-ary relations on X, relations of sets of 0-ary and n-ary relations on X, etc.

Suppose f is a function on X. By definition for a subset Y of X,

$$f(Y) = \{f(x) \mid x \in Y\}.$$

Also by definition, for an n-ary relation R on $X(f(R)$ is the n-ary relation S on the range of f such that for all $x_1, ..., x_n$ in X,

$$R(x_1, ..., x_n) \quad \text{iff} \quad S(f(x_1), ..., f(x_n)).$$

For a relation U based on X, $f(U)$ is defined inductively in the obvious way, e.g., if U is a set of n-ary relations on X, then

$$f(U) = \{f(u) \mid u \in U\},$$

etc.

Note that the relations based on X include elements of X as well as X itself.

For the kinds of general measurement results described in this article, no loss in generality results from assuming that the measurement structures contain relations based on X:

DEFINITION 6. $\mathfrak{X}$ is said to be a *structure* if and only if $\mathfrak{X}$ is of the form

$$\mathfrak{X} = \langle X, R_1, ..., R_j, ...\rangle_{j \in J},$$

where X is a nonempty set and R_j, $j \in J$, are relations based on X.

Let $\mathfrak{X} = \langle X, R_1, ..., R_j, ...\rangle_{j \in J}$ be a structure. Then, by definition, the *primitives* of $\mathfrak{X}$ consist of X and R_j, $j \in J$. Also by definition, X is called the *domain* of $\mathfrak{X}$.

DEFINITION 7. Let $\mathfrak{X} = \langle X, R_1, ..., R_j, ...\rangle_{j \in J}$ and $\mathfrak{Y} = \langle Y, S_1, ..., S_k, ...\rangle_{k \in K}$ be structures. Then the following three definitions hold:

(1) f is said to be an *isomorphism* of $\mathfrak{X}$ onto $\mathfrak{Y}$ if and only if $J = K$, f is a one-to-one function from X onto Y, and for each $j \in J$, $f(R_j) = S_j$.

(2) $\mathfrak{Y}$ is said to be a *numerical representing structure* for $\mathfrak{X}$ if and only if $Y \subseteq \mathbb{R}$ and there exists an isomorphism of $\mathfrak{X}$ onto $\mathfrak{Y}$.

(3) $\mathscr{S}$ is said to be a *scale of isomorphisms* for $\mathfrak{X}$ if and only if there is numerical representing structure $\mathfrak{R}$ such that $\mathscr{S}$ is the nonempty set of isomorphisms of $\mathfrak{X}$ onto $\mathfrak{R}$.

DEFINITION 8. A structure $\mathfrak{X} = \langle X, \succsim_*, R_1, ..., R_j, ...\rangle_{j \in J}$ is said to be *continuous* if and only if $\langle X, \succsim_*\rangle$ is a continuum.

The structures considered in this paper will either be continuous or could have total orderings $\succsim_*$ added as primitives so that the resulting structures are continuous. (The results of the paper, however, will apply with appropriate modifications to more general situations in which the total ordering $\succsim_*$ need not be Dedekind complete.)

DEFINITION 9. An *automorphism* of a structure $\mathfrak{X}$ is, by definition, an isomorphism of $\mathfrak{X}$ onto $\mathfrak{X}$.

DEFINITION 10. Let $\mathfrak{X} = \langle X, \succsim_*, R_1, ..., R_j, ...\rangle_{j \in J}$ be a continuous structure.

$\mathfrak{X}$ is said to be *homogeneous* if and only if for each x and y in X there exists an automorphism α of $\mathfrak{X}$ such that $\alpha(x) = y$.

$\mathfrak{X}$ is said to be 1-*point unique* if and only if for all automorphisms α and β of $\mathfrak{X}$, if $\alpha(x) = \beta(x)$ for some x in X, then $\alpha = \beta$.

THEOREM 3. *Suppose* $\mathfrak{X} = \langle X, \succsim_*, R_1, ..., R_j, ...\rangle_{j \in J}$ *is a continuous structure. Then the following three statements are true*:

1. *There exists a scale of isomorphisms* $\mathscr{S}$ *of* $\mathfrak{X}$ *such that for each* φ *in* $\mathscr{S}$, $\varphi(X) = \mathbb{R}^+$ *and* $\varphi(\succsim_*) = \geqslant$.

2. *Let* $\mathscr{S}$ *be a scale of isomorphisms on* $\mathfrak{X}$. *Then* (*the structure*) $\mathfrak{X}$ *is homogeneous if and only if* (*the scale*) $\mathscr{S}$ *is homogeneous.*

3. *Suppose* $\mathfrak{X}$ *is homogeneous and* 1-*point unique. Then there exists a ratio scale* $\mathscr{S}$ *of isomorphisms of* $\mathfrak{X}$ *such that for each* φ *in* $\mathscr{S}$, $\varphi(X) = \mathbb{R}^+$ *and* $\varphi(\succcurlyeq_*) = \geqslant$.

Proof 1. By Theorem 2, let φ be an isomorphism of $\langle X, \succcurlyeq_* \rangle$ onto $\langle \mathbb{R}^+, \geqslant_* \rangle$. Let $\mathfrak{R} = \langle \mathbb{R}^+, \geqslant, \varphi(R_1), ..., \varphi(R_j), ... \rangle_{j \in J}$. Then it easily follows from Definition 5 that φ is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{R}$. Let $\mathscr{S}$ be the set of isomorphisms of $\mathfrak{X}$ onto $\mathfrak{R}$. Then $\mathscr{S} \neq \varnothing$ since $\varphi \in \mathscr{S}$, and thus $\mathscr{S}$ is a scale of isomorphisms.

Statement 2 easily follows from Definitions 9, 10, and 4.

Statement 3 follows from Theorem 2.12 of Narens (1981) and from Theorem 4.2 of Chapter 2 of Narens (1985).

Magnitude Estimation Situations

CONVENTION 1. Throughout the rest of this article, X will denote a nonempty set and $\succcurlyeq$ will denote a binary relation on X.

In the intended interpretations of the article, elements of X are the possible stimuli to be presented to a subject for judgment and $\succcurlyeq$ is an experimenter determined, intensity ordering on X. (In psychophysical applications where X is a set of physical stimuli, $\succcurlyeq$ will be taken to be the natural physical ordering on X.)

CONVENTION 2. Throughout the article E denotes the magnitude estimation behavior of a subject to stimuli from X. Unless stated to the contrary, all elements of E will have the form of an ordered triple $(x, \mathbf{p}, t)$, where x and t are elements of X and p is a positive integer. (For technical reasons discussed earlier, p is placed in bold typeface when it is part of a triple in E.)

The integral nature of $\mathbf{p}$ in Convention 2 is not essential, and the more general case is discussed in Subsection 3.5.

In the intended interpretations, the experimenter gives the subject a magnitude estimation task, and E is the record of the experimental results. There are many kinds of magnitude estimation tasks where the experimental results can be coded as such a set.

The *axiomatic theories* presented in this paper do not depend on the details of the instructions given to the subject; they depend only on the set E and an explicit theory of the relationship of E to psychological processes. Some researchers, however, may want to give *different interpretations to the axiomatic theory* depending on the instructions given to the subject. For example, the same axiomatic theory may apply to data collected from the following three instructions:

(1) Find a stimulus in X which appears to be p times greater in intensity than the stimulus t.

(2) Pick the number p which best describes the stimulus x as being p times more intense than the stimulus t.

(3) Find the stimulus which in your subjective valuation is $p+$ the valuation of the stimulus t.

For (1) and (2) a researcher might want to represent stimuli numerically so that the numerical interpretation of $(x, \mathbf{p}, t)$ is that the numerical value of x is p times the numerical value of t, whereas in (3) he or she might want to interpret $(x, \mathbf{p}, t)$ so that the numerical value of x equals p plus the numerical value of t.

3. RATIO MAGNITUDE ESTIMATION

In ratio magnitude estimation, the subject is asked to estimate ratios subjectively. This can be done in many ways. For concreteness, throughout most of this section it will be assumed that the subject has been instructed to "find a stimulus in X that appears to be p times greater in intensity than the stimulus t."

In this section, several versions of ratio magnitude estimation are axiomatized. For the purposes of exposition and discussion, it is convenient to divide the axioms into the following three categories: (i) experimental or behavioral assumptions, (ii) inner psychological assumptions, and (iii) psychobehavioral assumptions.

The *experimental and behavioral assumptions* consist of axioms about the stimuli, axioms about the subject's behavior, and axioms about the relationships between the stimuli and the behavior of the subject. The primitives that appear in these axioms are in principle observable to the experimenter.

CONVENTION. To simplify formulations and exposition, the term "behavioral" will be used throughout this article to describe both experimental and behavioral assumptions.

The *inner psychological assumptions* are axioms about the mental activity of the subject, often involving subjective experience. These assumptions are theoretical in nature and are formulated in terms of relationships that are not observable by the experimenter.

The *psychobehavioral assumptions* are axioms that link behavioral objects and relationships with inner psychological ones. These linkages are theoretical in nature and are not observable by the experimenter.

3.1. *Behavioral Axiomatization*

The behavioral axiomatization consists of only behavioral assumptions.

In the following axiom, $\succcurlyeq$ is intended to be a total ordering of the stimulus set selected by the experimenter.

Axiom 1. $\langle X, \succcurlyeq \rangle$ is a continuum.

The assumption that $\langle X, \succcurlyeq \rangle$ is a continuum can be weakened so that the results presented in the paper generalize to more situations. However, to achieve such generalizations more complicated axiomatic systems would be needed, and because the presentations of these generalized systems would add very little to the main issues of the paper, they are not pursued here.

Axiom 2. The following five statements are true:

1. $E \subseteq \{(x, \mathbf{p}, t) \mid x \in X, p \in \mathbb{I}^+, \text{ and } t \in X\}$.
2. For all $(x, \mathbf{p}, t)$ in E, $x \succsim t$.
3. For all t in X, $(t, \mathbf{1}, t)$ is in E.
4. For all x and t in X and all p in $\mathbb{I}^+$, there exist exactly one z in X and exactly one s in X such that

$$(z, \mathbf{p}, t) \in E \qquad \text{and} \qquad (x, \mathbf{p}, s) \in E.$$

5. For all $x, y, t,$ and s in X, if $(x, \mathbf{p}, t) \in E$ and $(y, \mathbf{p}, s) \in E$, then

$$x \succsim y \qquad \text{iff} \quad t \succsim s.$$

Statements 1, 2, and 3 of Axiom 2 are straightforward. (It is possible to reformulate the axiom system so that only numerals greater than **1** are used. If this is done, then Statement 3 can be eliminated by modifying Statement 2 as follows: For all $(x, \mathbf{p}, t)$ in E, $x \succ t$.) Statement 4 describes a highly idealized situation—one in which infinitely many magnitudes ratios **p** and infinitely many stimuli t are presented to subject. Statement 5 says that the magnitudes behave in a strictly increasing manner.

Axiom 3. The following three statements are true:

1. For all $(x, \mathbf{p}, t)$ and $(y, \mathbf{q}, t)$ in E,

$$x \succ y \qquad \text{iff} \quad p > q.$$

2. For all x and t in X, if $x \succ t$, then there exist y in X and p in $\mathbb{I}^+$ such that $y \succ x$ and $(y, \mathbf{p}, t) \in E$.
3. For all x and t in X, if $x \succ t$, then there exist y and z in X and p in $\mathbb{I}^+$ such that

$$(y, \mathbf{p}+\mathbf{1}, t) \in E, \quad (y, \mathbf{p}, z) \in E, \quad \text{and} \quad x \succ z \succ t.$$

Axiom 3 describes natural conditions for a ratio magnitude estimation paradigm. Statement 1 provides the linkage of the usual ordering on numbers (and consequently the usual ordering on numerals) to the experimenter determined ordering on stimuli. Statement 2 plays a role somewhat akin to the Archimedean axiom in measurement theory: It essentially says that no element of X is "infinitely large" in terms of magnitude estimation to another element of X. Statement 3 also plays the role of an Archimedean axiom: It essentially says that no two distinct elements are "infinitesimally close" in terms of magnitude estimation.

Axiom 4 (Commutative Property). For all p and q in $\mathbb{I}^+$ and all x, y, z, t, and w in X, if $(x, \mathbf{p}, t) \in E$, $(z, \mathbf{q}, x) \in E$, $(y, \mathbf{q}, t) \in E$, and $(w, \mathbf{p}, y) \in E$, then $z = w$.

If we let $\mathbf{q} \bullet \mathbf{p}$ stand for first estimating p times a stimulus t and then q times that estimated stimulus, then Axiom 4 says that $\mathbf{q} \bullet \mathbf{p} = \mathbf{p} \bullet \mathbf{q}$, a condition that in algebra is called "commutativity".

Definition 11. Assume the behavioral assumptions Axioms 1 to 4. For each p in $\mathbb{I}^+$, define the binary relation $\bar{p}$ on X as follows: For all x and t in X,

$$x = \bar{p}(t) \qquad \text{iff} \quad (x, \mathbf{p}, t) \in E.$$

$\bar{p}$ may be thought of as the *behavioral interpretation* of p. Since $\bar{p}$ is defined entirely in terms of behavioral concepts, it is a behavioral concept.

It easily follows from Statement 4 of Axiom 2 that for each p in $\mathbb{I}^+$, $\bar{p}$ is a function on X.

Definition 12. Assume Axioms 1 to 4. Let

$$\mathfrak{B} = \langle X, \succsim, \bar{1}, ..., \bar{p}, ... \rangle_{p \in \mathbb{I}^+}.$$

By definition, $\mathfrak{B}$ is called the *behavioral structure* (*associated with E*).

Note that the primitives of $\mathfrak{B}$ are behavioral concepts.

In terms of the current formulation, the traditional goal of measurement through the magnitude estimation paradigm is achieved by measuring the behavioral structure $\mathfrak{B}$ through a scale $\mathscr{S}$ such that for each p in $\mathbb{I}^+$ and each ϕ in $\mathscr{S}$, $\phi(\bar{p})$ is a function that is multiplication by the integer p. In Section 3.4, we will show that the multiplicative property (Definition 1) is exactly what is needed to add to Axioms 1 to 4 to achieve this goal.

A more limited goal is to measure $\mathfrak{B}$ in such a way that for each p in $\mathbb{I}^+$ and each ϕ in $\mathscr{S}$, $\phi(\bar{p})$ is a function that is multiplication by some positive real c, with not necessarily $c = p$. The axiom system, Axioms 1 to 4, already accomplishes this:

Definition 13. ϕ is said to be a *multiplicative representing function* for $\mathfrak{B}$ if and only if ϕ is a function from X into $\mathbb{R}^+$ such that for each $p \in \mathbb{I}^+$, $\phi(\bar{p})$ is a function that is multiplication by a positive real.

A scale $\mathscr{S}$ on X is said to be a *multiplicative scale* for $\mathfrak{B}$ if and only if each element of $\mathscr{S}$ is a multiplicative representing function for $\mathfrak{B}$.

Theorem 4. *Assume Axioms* 1 *to* 4. *Then there exists a numerical representing structure* $\mathfrak{R}$ *such that the scale of isomorphisms* $\mathscr{S}$ *of* $\mathfrak{B}$ *onto* $\mathfrak{R}$ *is* (i) *a multiplicative scale for* $\mathfrak{B}$ *and* (ii) *a ratio scale.*

Proof. Theorem 15.

THEOREM 5. *Assume Axioms* 1 *to* 4. *Suppose* $\mathcal{S}$ *is a ratio scale of isomorphisms of* $\mathfrak{B}$. *Then* $\mathcal{S}$ *is a multiplicative scale for* $\mathfrak{B}$.

Proof. Theorem 16.

3.2. *Cognitive Axiomatization*

Some researchers are content to deal with behavioral issues only. For these, there is no need to go beyond behavioral primitives and behavioral assumptions.

Others are interested in the interplay between cognition and behavior. For this case one needs to include additional psychobehavioral and psychological primitives and assumptions. With the addition of these primitives one has the ability to formulate clearly conditions for measurements of the behavioral structure $\mathfrak{B}$ to translate into measurements of a cognitive structure based on sensations.

For magnitude estimation, the obvious cognitive question is "How is the subject producing his or her responses in the magnitude estimation paradigm?" In this section, a *minimal theory* (Axioms 5 to 8 below) is presented for answering this. ("Minimal" is meant to convey here the author's belief that any plausible cognitive theory at the same level of idealization as the minimal theory designed to answer the question will imply the minimal theory.)

The minimal theory is based on the idea that the responses of the subject correspond to inner psychological functions that are computed by the subject from an "inner psychological measurement structure". The exact form of the "computation" and the specific primitives that make up the "inner psychological measurement structure" are not given; only their most general features are specified. (This is what gives the "minimalness" to the theory.)

Magnitude estimation is usually not the primary goal of empirical studies: It is generally used as an instrument to investigate a substantive domain of interest. In such situations, the choice between behavioral and cognitive scales will depend on the particular objectives of the research.

Although the minimal theory together with Axioms 1 to 4 force a strong relationship between the behavioral and cognitive, they do not force these scales to have identical measurement properties: By Theorem 4, X is measured behaviorally by a ratio scale of isomorphisms of $\mathfrak{B}$, whereas by Theorem 6 below, the inner psychological measurement structure upon which the magnitude estimations depend is measured by a scale that is a subscale of a ratio scale. Theorem 6 shows that a necessary and sufficient condition for this inner psychological measurement structure to be measurable by a ratio scale is that it be homogeneous. With this additional assumption of homogeneity, the behavioral and cognitive scales that result from magnitude estimation are for practical purposes identical (Theorem 7 below).

Psychobehavioral Assumptions

Axiom 5. Ψ is a function from X into the set of the subject's sensations.

Axiom 5 makes the philosophical distinction between objects in X and sensations of objects in X. Objects in X are considered behavioral but not inner psychological, whereas sensations of objects in X are considered inner psychological but not behavioral.

(Using the word "sensation" to describe mental impressions of stimuli in X may not be appropriate in several kinds of magnitude estimation situations, e.g., when X consists of objects or circumstances of social value and the subject magnitude estimates the social value of elements of X. For such applications other appropriate concepts can be substituted for "sensation," and the results of such substitutions will have no effect on the theory and results of the paper.)

In the following axiom $\succsim_\psi$ is the inner psychological intensity ordering on the set of sensations $\Psi(X)$ referred to in Axiom 7.

Axiom 6. For all x and y in X,

$$x \succ y \qquad \text{iff} \quad \Psi(x) \succ_\psi \Psi(y).$$

Inner Psychological Assumptions

Axiom 7. The subject has an inner psychological structure

$$\mathcal{J} = \langle \Psi(X), \succsim_\psi, R_1, ..., R_j, ... \rangle_{j \in J}$$

for "measuring" the intensity of sensations in $\Psi(X)$.

It follows from Axioms 1 and 6 that $\succsim_\psi$ is a total ordering and that Ψ is a one-to-one function.

In the inner psychological structure $\mathcal{J}$, the primitives $\Psi(X)$, $\succsim_\psi$, and R_j, $j \in J$, are considered to be inner psychological.

The primitive $\succsim_\psi$ is intended to be the subject's ordering of his or her subjective intensities of sensations in $\Psi(X)$.

Except for the domain $\Psi(X)$ and the primitive $\succsim_\psi$ (which is linked to the behavioral ordering $\succsim$ through the psychobehavioral Axiom 6), other *individual* primitives of $\mathcal{J}$ are not explicitly mentioned in axioms of this section. However, the *structure* $\mathcal{J}$ of primitives plays an important role in various assumptions of the section, e.g., in Axiom 8 below which assumes that certain functions are definable in terms of the primitives of $\mathcal{J}$, or in various hypotheses of theorems which assume that the structure $\mathcal{J}$ is homogeneous.

DEFINITION 14. For each p in $\mathbb{I}^+$, define $\hat{p}$ from $\Psi(X)$ into $\Psi(X)$ as follows: For each x and t in X,

$$\hat{p}[\Psi(t)] = \Psi(x) \qquad \text{iff} \quad (x, \mathbf{p}, t) \in E.$$

It easily follows from previous axioms that $\hat{p}$ is a function.

The expression "$\hat{p}[\Psi(t)] = \Psi(x)$" may be considered as the inner psychological correlate of the expression $(x, \mathbf{p}, t) \in E$. With this in mind, it is natural to consider the function $\hat{p}$ as the inner psychological interpretation of the numeral **p**.

The next axiom says that the function $\hat{p}$ is defined (constructed, calculated) in terms of the primitives of $\mathcal{J}$. As now discussed, by various theories (Narens, 1988; Narens & Mausfeld, 1992; Narens, in press) this is equivalent to saying that $\hat{p}$ is *meaningful* with respect to the structure $\mathcal{J}$.

In the traditional approaches to the theory of measurement (e.g., Krantz *et al.*, 1972; Suppes *et al.*, 1990, Luce *et al.*, 1990; Narens, 1985), meaningfulness has been identified with invariance. In the present context, the traditional approaches would consider the appropriate invariance concept for the meaningfulness of inner psychological relationships dependent on $\mathcal{J}$ to be invariance under the set of automorphisms of $\mathcal{J}$. Narens (1988, manuscript) shows that relationships that are invariant under the automorphisms of a structure coincide with those that are definable in terms of the primitives of the structure through an extremely powerful logical language—a language of equivalent or greater power than axiomatic set theory. Narens (1988, in press) argues that the kinds of extremely powerful logical languages used for this purpose are for many applications too powerful, in the sense that they are too permissible in the kinds of relations that they allow as "meaningful," and he suggests that in scientific applications less powerful languages be employed. In terms of invariance under automorphisms, a consequence of this suggestion is that *invariance under automorphisms should be considered as a necessary condition for meaningfulness but not necessarily as a sufficient condition for meaningfulness.*

Intuitively, this is how meaningfulness enters in the present context: Subjective magnitude is captured by the inner psychological structure $\mathcal{J}$. We do not know much about $\mathcal{J}$ except that its domain consists of sensations of stimuli, and we believe that $\mathcal{J}$ has a primitive ordering of sensations corresponding to "subjective intensity". We assume that the subject's magnitude estimations involve the structure $\mathcal{J}$—that is, the subject somehow performs a calculation or evaluation involving $\mathcal{J}$ to produce his or her responses to trials in the magnitude estimation experiment. We assume that the subject does this in a way that gives a constant meaning to each numeral **p**; i.e., *it is assumed that the interpretation that the subject gives to the numeral* **p** *is calculated or defined in terms of the primitives of* $\mathcal{J}$. Of course, something needs to be said about the subjective methods of calculation or definition of the numeral **p**. They are inner psychological, and it is natural, therefore, to suspect that they would have special properties reflecting that they are products of mental activity. Nevertheless, without knowing the details of these properties, it is reasonable to believe they can be captured formally in terms of the extremely powerful logical languages (which among other things contain the equivalents of all known mathematics), and therefore by the above mentioned results of Narens (1988, in press) that *these inner psychological methods of calculation or definition are invariant under the automorphisms of* $\mathcal{J}$. These intuitive considerations are summarized in the following axiom:

Axiom 8. Assume Axioms 5 and 7. For each p in $\mathbb{I}^+$, $\hat{p}$ is meaningful with respect to $\mathcal{J}$.

In Axioms 1 to 8, most of the mathematical structure about magnitude estimation is contained in the behavioral axioms 1 to 4, often as testable hypotheses. The mathematical content in the inner psychological and psychobehavioral axioms is very minimal and, for reasons stated previously, is necessary for any reasonable theory of ratio magnitude estimation that includes mental phenomena.

Definition 15. φ is said to be a *multiplicative representing function* for $\mathcal{J}$ if and only if φ is a function from $\Psi(X)$ into $\mathbb{R}^+$ such that for each $p \in \mathbb{I}^+$, $\varphi(\hat{p})$ is a function that is multiplication by a positive real.

A scale $\mathcal{S}$ on $\Psi(X)$ is said to be a *multiplicative scale* for $\mathcal{J}$ if and only if each element of $\mathcal{S}$ is a multiplicative representing function for $\mathcal{J}$.

Axioms 1 to 8 yield the following theorem:

Theorem 6. *Assume Axioms* 1 *to* 8. *Then there exists a numerical structure* $\mathfrak{N}$ *with domain* $\mathbb{R}^+$ *such that the following three statements are true*:

1. *The set* $\mathcal{S}$ *of isomorphisms of the inner psychological structure* $\mathcal{J}$ *onto* $\mathfrak{N}$ *is a subscale of a ratio scale.*

2. $\mathcal{S}$ (*as defined in Statement* 1) *is a multiplicative scale for* $\mathcal{J}$.

3. *If* $\mathcal{J}$ *is homogeneous, then the following two statements are true*:

(i) $\mathcal{S}$ (*as defined in Statement* 1) *is a ratio scale.*

(ii) *Let t be an arbitrary element of X, and by* (i) *let* φ *be the unique element of* $\mathcal{S}$ *such that* $\varphi(\Psi(t)) = 1$, *and by Statement* 2, *for each p in* $\mathbb{I}^+$, *let* c_p *be the positive real such that multiplication by* c_p *is* $\varphi(\hat{p})$. *Then for all p in* $\mathbb{I}^+$ *and x in X,*

$$(x, \mathbf{p}, t) \in E \quad \textit{iff} \quad \varphi(\Psi(x)) = c_p.$$

Proof. Theorem 13.

Assume Axioms 1 to 8. By Theorem 4, the *data* in E can be appropriately measured behaviorally by a ratio scale of isomorphisms $\mathcal{S}$ of $\mathfrak{B}$. Properly speaking, $\mathcal{S}$ only measures the subject's behavior in an experiment. It can, however, be used to define a closely related *derived sensory scale* $\mathscr{S}'$ that measures intensity of sensations of stimuli of X by letting

$$\mathscr{S}' = \{\varphi' \mid \varphi \in \mathscr{S} \text{ and for each } x \text{ in } X, \varphi'(\Psi(x)) = \varphi(x)\}.$$

Nevertheless, $\mathscr{S}'$ may be inappropriate for measuring subjective intensity. One reason is that the qualitative structure for subjective intensity—the structure $\mathscr{J}$ in Axiom 7—may not be homogeneous, and $\mathscr{S}'$, like $\mathscr{S}$, is a ratio scale and therefore is homogeneous. However, under the assumption that the inner psychological structure $\mathscr{J}$ is homogeneous, the following strong relationship obtains between behaviorally based scales and inner psychologically based ones:

THEOREM 7. *Assume Axioms* 1 *to* 8 *and that* $\mathscr{J}$ *is homogeneous. Then* (i) *for each scale* $\mathscr{S}$ *of isomorphisms of* $\mathfrak{B}$, *its derived sensory scale* $\mathscr{S}'$ (*discussed above*) *is a scale of isomorphisms of* $\mathscr{J}$, *and* (ii) *for each scale* $\mathscr{T}$ *of isomorphisms of* $\mathscr{J}$, *there exists a scale* $\mathscr{U}$ *of isomorphisms of* $\mathfrak{B}$ *such that* $\mathscr{U}' = \mathscr{T}$, *where* $\mathscr{U}'$ *is the derived sensory scale for* $\mathscr{U}$.

Proof. Theorem 19.

The principal difference between using Axioms 1 to 4 and measuring the behavioral structure $\mathfrak{B}$ and using Axioms 1 to 8 and measuring the inner psychological structure $\mathscr{J}$ is the choice of primitives: In $\mathfrak{B}$ the primitives are

$$X, \succcurlyeq, \bar{1}, ..., \bar{p}, ...,$$

whereas in $\mathscr{J}$ the primitives are

$$\Psi(X), \succcurlyeq_\psi, R_1, ..., R_j, ...$$

Thus the inner psychological structure $\mathfrak{B}'$ that is coordinate to $\mathfrak{B}$ has the form

$$\mathfrak{B}' = \langle \Psi(X), \succcurlyeq_\psi, \hat{1}, ..., \hat{p}, ... \rangle_{p \in \mathbb{I}^+}.$$

It easily follows from Axioms 7 and 6 that $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic. The primitives of $\mathfrak{B}'$ need not be the inner psychological primitives for subjective sensitivity (i.e., are not the primitives of $\mathscr{J}$); they are only definable from the latter (Axiom 8). Because of this, $\mathfrak{B}'$ (and therefore $\mathfrak{B}$) may have a different scale type than $\mathscr{J}$, and therefore may be inappropriate for measuring subjective intensity.

3.3. *Translation Scales*

DEFINITION 16. φ is said to be an *additive representing function* for $\mathscr{J}$ if and only if φ is a function from $\Psi(X)$ into $\mathbb{R}$ such that for each $p \in \mathbb{I}^+$, $\varphi(\hat{p})$ is a function that is an addition by a nonnegative real.

A scale $\mathscr{S}$ on $\Psi(X)$ is said to be an *additive scale* for $\mathscr{J}$ if and only if each element of $\mathscr{S}$ is an additive representing function for $\mathscr{J}$.

By transforming the structure $\mathfrak{R}$ in Theorem 6 via the function $r \to \log(r)$, the following theorem is obtained:

THEOREM 8. *Assume Axioms* 1 *to* 8. *Then the following two statements are true*:

1. *There exists an additive scale of isomorphisms of* $\mathscr{J}$.
2. *If* $\mathscr{J}$ *is homogeneous, then there exists a translation scale of isomorphisms for* $\mathscr{J}$ *that is an additive scale for* $\mathscr{J}$.

Measuring objects of a domain by a scale of isomorphisms of a structure based on the domain is a variant of the *representational theory of measurement*. The representational theory is at the present time the dominant theory of measurement in the literature, and it is the basis of the massive treatise, *Foundations of Measurement*, Vols. I–III, by (in various orders of authors) Krantz, Luce, Suppes, and Tversky.

The representational theory of measurement does not choose among isomorphic numerical representing structures with respect to their appropriateness for measuring a given situation. In particular, if Axioms 1–8 hold, the representational theory is not able to decide whether a multiplicative scale of isomorphisms of $\mathscr{J}$ is more appropriate in a particular situation than an additive scale of isomorphisms of $\mathscr{J}$. (Torgerson, 1961, reached a similar conclusion. His arguments, when applied to the representational theory, essentially consists of the above observation.)

It is natural to ask what happens when the subject engages in *different kinds* of magnitude estimation tasks on the *same* set of stimuli, e.g., one task in which he or she is instructed to estimate ratios and another in which he or she is instructed to estimate differences. Suppose Axioms 1 to 8 hold for both tasks. Then by Theorem 4 the subjects' data can be measured *separately* by multiplicative scales of the behavioral structures associated with the data sets from the tasks. In addition, if the *same homogeneous* inner psychological structure is used by the subject to "compute" his or her responses in both tasks, then it is a consequence of Theorem 9 below that a scale for X exists that is *simultaneously* a multiplicative scale of the behavioral structures associated with the two sets of data collected in the two tasks.

Although the representational theory of measurement cannot justify the selection of one scale of isomorphisms of a structure over another, it still is able in some circumstances to make *relative* distinctions between various scales of isomorphisms of *different* structures with the *same* domain. For the above situation with two magnitude estimation tasks, consider the case where a scale on X is simultaneously a multiplicative scale of isomorphisms of the behavioral structure associated with the first task and an additive scale of isomorphisms of the behavioral structure associated the second task. (It is trivial to construct examples of this case.)

However, by the above discussion, in the context of Axioms 1–8 such relative distinctions can be made only when either the subject uses a *different* inner psychological measurement structure for each task or the subject uses a *nonhomogeneous* inner psychological measurement structure for both tasks.

The following theorem is the technical result used in reaching the above conclusions:

THEOREM 9. *Suppose for Task* 1 (i) *the subject has been instructed to* "*Find a stimulus in X which appears to be p times greater in intensity than the stimulus t*," (ii) *E is his or her responses to this task*, (iii) *Axioms* 1–8 *hold and the inner psychological structure* $\mathscr{J}$ *is homogeneous, and* (iv) $\mathscr{S}$ *is a multiplicative scale of isomorphisms of* $\mathscr{J}$. (*The existence of* $\mathscr{S}$ *follows from* (iii) *and Theorem* 6.)

Also suppose in Task 2 *that different instructions are given to the subject*, *e.g*, "*Find the stimulus which in your subjective valuation is* $q+$ *the valuation of the stimulus t*,"*and as a result of these instructions the subject produces the partial data set H where elements of H have the form* $(x, \mathbf{q}, t)$, *where q is a fixed positive integer*, *t ranges over the elements of X*, *and* (1) *for each t in X there exists exactly one x in X such that* $(x, \mathbf{q}, t) \in H$, (2) *for all* $(x, \mathbf{q}, t)$ *in H*, $x \succcurlyeq t$, *and* (3) *for all x*, *y*, *t*, *and v in X*, *if* $(x, \mathbf{q}, t)$ *and* $(y, \mathbf{q}, v)$ *are in H and* $t > v$, *then* $x \succ y$. *Let* $\hat{q}$ *be the following function on* $\Psi(X)$: *For all x and t in X*,

$$\hat{q}(\Psi(t)) = \Psi(x) \qquad \textit{iff} \quad (x, \mathbf{q}, t) \in H.$$

Assume $\hat{q}$ *is meaningful with respect to* $\mathscr{J}$. *Then there exists a positive real r such that for all* φ *in* $\mathscr{S}$, $\varphi(\hat{q})$ *is the function that is multiplication by r*.

Proof. Theorem 14.

Theorem 9 provides a theoretical basis for the empirical findings discussed in Torgerson (1961) about difference and ratio judgments on the same stimuli set:

> The situation turns out to be much the same in the quantitative judgment domain. Again, we have both distance methods, where the subject is instructed to judge subjective differences between stimuli, and ratio methods, where the subject is instructed to judge subjective ratios. Equisection and equal appearing intervals are examples of distance methods. Fractionation and magnitude estimation are examples of ratio methods.
>
> In both classes of methods, the subject is supposed to tell us directly what the differences and ratios are. We thus have the possibility of settling things once and for all. Judgments of differences take care of the requirements of the addition commutative group. Judgments of ratios take care of the multiplication commutative group. All we need to show is that the two scales combine in the manner required by the number system. This amounts to showing that scales based on direct judgments of subjective differences are linearly related to those based on subjective ratios.
>
> Unfortunately, they are not. While both procedures are subject to internal consistency checks, and both often fit their own data, the two scales are not linearly related. But when we plot the logarithm of the ratio scale against the difference scale spaced off in arithmetic units, we usually do get something very close to a straight line. Thus, according to the subject's own judgments, stimuli separated by equal subjective intervals are also separated by approximately equal subjective ratios.
>
> This result suggests that the subject perceives or appreciates but a single quantitative relation between a pair of stimuli. This relation to begin with is neither a quantitative ratio or difference, of course —ratios and differences belong only to the formal number system. It appears that the subject simply interprets this single relation in whatever way the experimenter requires. When the experimenter tells him to equate differences or to rate on an equal interval scale, he interprets the relation as a distance. When he is told to assign numbers according to subjective ratios, he interprets the same relation as a ratio. Experiments on context and anchoring show that he is also able to compromise between the two. (pp. 202–203)

3.4. *Numeral Scales*

Assume Axioms 1 to 8. By Theorem 6, let $\mathscr{S}$ be a multiplicative scale for $\mathscr{J}$. Then it follows by Theorem 6 that there exists a function f from $\mathbb{I}^+$ onto $\mathbb{R}^+$ such that for all x and t in X and all p in $\mathbb{I}^+$ and all φ in $\mathscr{S}$,

$$(x, \mathbf{p}, t) \in E \qquad \text{iff} \quad \varphi(x) = f(p) \cdot \varphi(t).$$

The literature has almost universally restricted its attention to the case where f is the identity function on $\mathbb{I}^+$. This is obviously a very special and important case.

DEFINITION 17. φ is said to be a *numeral* multiplicative representing function for $\mathscr{J}$ (respectively, $\mathfrak{B}$) if and only if it is a multiplicative representing function for $\mathscr{J}$ (respectively, $\mathfrak{B}$) and for each $p \in \mathbb{I}^+$, $\varphi(\hat{p})$ (respectively, $\varphi(\bar{p})$) is the function that is multiplication by p.

A scale $\mathscr{S}$ on $\Psi(X)$ is said to be a *numeral* multiplicative scale for $\mathscr{J}$ (respectively, $\mathfrak{B}$), if and only if it is a multiplicative scale for $\mathscr{J}$ (respectively, $\mathfrak{B}$) such that each of its elements is a numeral multiplicative representing function for $\mathscr{J}$ (respectively, $\mathfrak{B}$).

The following theorem is an immediate consequence of Definition 17:

THEOREM 10. *Suppose* φ *is a numeral multiplicative representing function for* $\mathscr{J}$ (*respectively*, $\mathfrak{B}$) *and* $r \in \mathbb{R}^+$. *Then* $r\varphi$ *is a numeral multiplicative representing function for* $\mathscr{J}$ (*respectively*, $\mathfrak{B}$).

Proof. Left to the reader.

The following *behavioral* axiom is important for establishing the existence of numeral multiplicative scales:

Axiom 9 (Multiplicative Property). For all p, q, and r in $\mathbb{I}^+$ and all t, x, y, and z in X, if $(x, \mathbf{p}, t) \in E$, $(z, \mathbf{q}, x) \in E$, and $r = qp$, then $(z, \mathbf{r}, t) \in E$.

Note that Axiom 9 implies Axiom 4.

THEOREM 11. *Assume Axioms* 1 *to* 8. *Then the following two statements are logically equivalent*:

1. *Axiom* 9.
2. *There exists a representing structure*

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, S_1, ..., S_j, ... \rangle_{j \in J}$$

such that the set $\mathcal{S}$ *of isomorphisms of the inner psychological structure* $\mathcal{J}$ *onto* $\mathfrak{N}$ *is a numeral multiplicative scale for* $\mathcal{J}$ (*Definition* 17).

Proof. Theorem 17.

Assume Axioms 1 to 9. Then it is easy to show that the scale $\mathcal{S}$ in Statement 2 of Theorem 11 is a ratio scale if and only if $\mathcal{J}$ is homogeneous.

The following is the "behavioral version" of Theorem 11:

THEOREM 12. *Assume Axioms* 1 *to* 4. *Then the following two statements are logically equivalent*:

1. *Axiom* 9.
2. *There exists a representing structure*

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, T_1, ..., T_i, ... \rangle_{i \in \mathbb{I}^+}$$

such that the set $\mathcal{S}$ *of isomorphisms of the behavioral structure* $\mathfrak{B}$ *onto* $\mathfrak{N}$ *is a ratio scale and is a numeral multiplicative scale for* $\mathfrak{B}$ (*Definition* 17).

Proof. Theorem 18.

If Axioms 1 to 9 are assumed, then Theorem 11 and Theorem 12 *seem* to suggest that numeral multiplicative scales of isomorphisms are more appropriate for the measurements of $\mathcal{J}$ and $\mathfrak{B}$ than other kinds of multiplicative scales of isomorphisms. However, as previously discussed, the representational theory of measurement does not distinguish between the appropriateness of different isomorphic numerical representing structures for the same qualitative structure, and therefore is not able to distinguish between the appropriateness of scales of isomorphisms associated with these structures.

Stevens (1946, 1951) presented a theory of measurement that when applied to magnitude estimation yielded unique numeral scales. Unfortunately, his theory is not fully specified, particularly as to what constitutes a proper construction of a scale. The following argument, which I believe is inherent in Stevens perspective about measurement, attempts to provide intuitive reasons for accepting the "special appropriateness" of numeral multiplicative scales:

Assume Axioms 1 to 9. Let $\mathbf{q} \bullet \mathbf{p}$ stand for first estimating p times a stimulus t and then q times that estimated stimulus. Then Axiom 9 says that

$$\mathbf{q} \bullet \mathbf{p} = \mathbf{r} \quad \text{iff} \quad q \cdot p = r,$$

i.e., that the subject does "correct arithmetic in his or her magnitude estimation". In multiplicative scales of isomorphisms of $\mathfrak{B}$, the operation $\bullet$ is interpreted in terms of the operation of multiplication of positive real numbers, $\cdot$, and the numeral multiplicative scale of isomorphisms of $\mathfrak{B}$ is the only multiplicative scale of isomorphisms of $\mathfrak{B}$ that preserves the important *behaviorally observed* fact that the subject does correct arithmetic in his or her magnitude estimation. Therefore, the numeral multiplicative scale of isomorphism of $\mathfrak{B}$ should be used for the behavioral measurement of $\mathfrak{B}$.

With respect to the representational theory, the above intuitive argument is somewhat obscure:

In the inner psychology of the subject, numerals $\mathbf{p}$ correspond to functions $\hat{p}$ that are meaningful with respect to $\mathcal{J}$, and the operation $\bullet$ corresponds to the operation of composing such functions, $*$. However, in doing "correct arithmetic" on numerals much more structure is required. For example, in understanding why $\mathbf{2} \bullet \mathbf{3} = \mathbf{6}$, more than the formal properties of multiplication are used: In particular, the *definitions* of the numbers 2, 3, and 6 are also used. One usual definition of the number 1 is purely in terms of multiplication: It is the unique number x such that $x \cdot y = y$ for all numbers y. The usual way of defining 2 is in terms of the number 1 and addition, i.e., $2 = 1 + 1$,. In general, the usual definitions of individual positive integers > 1 in ordinary arithmetic depend on the successor function $\rho(p) = p + 1$. Thus it is natural to expect that subjects who use "correct arithmetic" in magnitude estimation *could use their understanding of individual numerals and therefore of the numeral analog of the successor function.*

In the subject's inner psychology, functions $\hat{p}$ correspond to numerals $\mathbf{p}$, and the operation of function composition $*$ on $\{\hat{p} \mid p \in \mathbb{I}^+\}$ corresponds to operation of multiplication of numerals, $\bullet$. Since for $p \in \mathbb{I}^+$ $\hat{p}$ is meaningful with respect to the inner psychological structure $\mathcal{J}$, the operation $*$ is also meaningful, since it is reasonable to assume that the formula

$$\forall x[\hat{p} * \hat{q}(x) = \hat{p}(\hat{q}(x))]$$

will lead to a proper (inner psychological) definition of $*$ in terms of the primitives of $\mathcal{J}$. (In the above formula, "$\forall$"

stands for the logical quantifier "for all".) Since the data set E is consistent with the subject having the means to do "correct arithmetic" on numerals, we will also assume that his or her calculations used in this kind of "correct arithmetic" is the result of calculations based on $\mathscr{J}$. In accord with the above discussion, this will interpreted to mean (i) at least there is an inner psychological correlate to the arithmetic the successor function $\rho(p) = p + 1$ that the subject could use in defining individual numerals > 1, and (ii) ρ is meaningful with respect to $\mathscr{J}$. It is the lack of a theory of how such an inner psychological correlate interacts with the data E that makes obscure the assumption that the subject is using "correct arithmetic".

3.5. *Generalized Ratio Magnitude Estimation*

The theory of ratio magnitude estimation encompassed by Axioms 1–8 does not make strong assumptions about the structure of numerals—it only uses assumptions that are formulable in terms of the ordering of the numerals. Because of this, results that follow from these axioms easily generalize to other kinds of magnitude estimation situations:

Let $\mathbb{N}$ be a nonempty set, and (with a mild abuse of notation) let $\geqslant$ be a binary relation on $\mathbb{N}$. Elements $\mathbf{p}$, $p \in \mathbb{N}$, are called *generalized numerals*. (Note the use of bold typeface when referring to generalized numerals and regular typeface when referring to elements of $\mathbb{N}$.)

The following are the two most important cases of generalized numerals:

(1) $\mathbb{I}^+ \subseteq \mathbb{N} \subseteq \{r \mid r \in \mathbb{R}^+ \text{ and } r \geqslant 1\}$, and

(2) $\langle C, \succcurlyeq \rangle$ is a physical continuum, $e \in C$, and $\mathbb{N} = \{a \mid a \in C \text{ and } a \succcurlyeq e\}$.

Case (1) generalizes ratio magnitude estimation to situations where the subject can use nonintegral numerals, and Case (2) applies to situations where the subject's behavior is characterized in terms of generalized numerals based on physical stimuli, e.g., in an experiment where $(x, \mathbf{p}, t)$ stands for the pressure p that results when the subject squeezes a ball to display how much he or she believes that crime x is more serious than crime t.

The axioms for generalized magnitude estimation are the same as Axioms 1–8 with the following exceptions:

1. $\mathbb{N}$ is substituted throughout for $\mathbb{I}^+$.

2. $\geqslant$ is assumed to be a total ordering on $\mathbb{N}$ with a least element e and no greatest element.

3. Statement 3 of Axiom 2, which states,

For all t in X, $(t, \mathbf{1}, t)$ is in E,

is replaced by,

For all t in X, $(t, \mathbf{e}, t)$ is in E.

4. Statement 3 of Axiom 3, which states,

For all x and t in X, if $x \succ t$, then there exist y and z in X and p in $\mathbb{I}^+$ such that

$$(y, \mathbf{p+1}, t) \in E, \quad (y, \mathbf{p}, z) \in E, \quad \text{and} \quad x \succ z \succ t,$$

is replaced by,

For all x and t in X, if $x \succ t$, then there exist y and z in X and p and q in $\mathbb{N}$ such that

$$q > p, \quad (y, \mathbf{q}, t) \in E, \quad (y, \mathbf{p}, z) \in E, \quad \text{and} \quad x \succ z \succ t.$$

It is an easy (but somewhat tedious) matter to verify that all the above consequences of the behavioral axiomatization consisting of Axioms 1–4 and the cognitive axiomatization consisting of Axioms 1–8, when appropriately reformulated using the above conventions, are consequences of the corresponding axiom systems made up from the above axioms for generalized magnitude estimation.

4. DISCUSSION

Axioms 1–4 are necessary for any idealized theory of magnitude estimation in which numerals are to be interpreted as multiplications. By Theorem 4, they are also sufficient for obtaining ratio multiplicative scales for representing the data in E. This by itself is not enough to conclude that "the subject has a ratio scale for subjective intensity" or that "the subject uses a ratio scale to formulate his or her responses"; to draw such conclusions additional assumptions are needed about how the subject's responses are related to his or her experiences. A minimal form of the relationship is described in Axioms 5–8. Axioms 1–8, however, are still not enough to conclude that the "subject has a ratio scale of subjective intensity": it is only enough to conclude that the subject's scale is a subscale of a ratio scale (Theorem 6). To insure that this subscale is a ratio scale, an additional assumption must be imposed—namely, that of homogeneity of the inner psychological structure $\mathscr{J}$ used by the subject to compute the inner psychological correlates of the numerals (Statement 3 of Theorem 6).

In the axiomatization for ratio magnitude estimation, the behavioral Axioms 1 to 4 carry the mathematical structure of Axioms 1 to 8; the only *mathematical role* of Axioms 5–8 is to provide the theoretical links between the observed behavior of the subject and his or her mental processing. (I believe that the latter axioms accomplish this role in as minimal a way as one would want.)

In Axioms 1–4, Axioms 1, 2, and 3, which state elementary properties of magnitude estimation that one would expect to hold in idealized magnitude estimation settings, are very limited mathematically. It is Axiom 4, which states

> $\mathbf{q} \bullet \mathbf{p} = \mathbf{p} \bullet \mathbf{q}$, where "$\mathbf{q} \bullet \mathbf{p}$" stand for a first estimating p times a stimulus t and then q times that estimated stimulus

that contains the important algebraic structure for multiplicative representations, and therefore it is the axiom that in empirical settings should be carefully evaluated.

Assume Axioms 1–8. By Theorem 4 let $\mathscr{S}$ be a multiplicative scale of isomorphisms of $\mathfrak{B}$. Suppose a researcher uses $\mathscr{S}$ to measure *subjective intensity*. Then implicitly, the researcher is assuming that a function Ψ from stimuli onto sensations exists so that the derived sensory scale

$$\mathscr{S}' = \{\varphi' \mid \mathscr{S} \text{ and for each } x \text{ in } X, \varphi'(\Psi(x)) = \varphi(x)\}$$

measures subjective intensity of sensations. By Theorem 7, if the inner psychological structure $\mathscr{J}$ is homogeneous, then $\mathscr{S}'$ could also be achieved through a scale of isomorphisms of $\mathscr{J}$. Thus, under the *assumption of the homogeneity of* $\mathscr{J}$, $\mathscr{S}'$ is by the representational theory of measurement a proper scale for measuring subjective intensity. However, if $\mathscr{J}$ is not homogeneous, then according to the representational theory of measurement, $\mathscr{S}'$ would not be a proper scale for measuring subjective intensity.

Although the representational theory is currently the dominant theory of measurement in the literature, there are other theories that would always permit $\mathscr{S}'$ to be a proper scale of measurement for intensity of sensations. One of these, which is based on meaningfulness considerations instead on representational ideas, is described in Narens (manuscript). (The discussion of these alternative approaches to measurement is outside the intended scope of the paper.)

The representational theory, which is based on isomorphisms, cannot distinguish between multiplicative and additive scales for $\mathscr{J}$ (Theorems 6 and 8), even if the subject were asked to participate in different experiments on the same stimuli and were instructed to use ratio calculations in one of the experiments and difference calculations in the other (Theorem 9). However, the kind of measurement theory used by Stevens (and many researchers in the behavioral sciences) can in principle distinguish between multiplicative and additive representations, since these theories make the additional assumption (which are rarely made explicit or even acknowledged by their adherents) that the data E reflect an inner psychological "arithmetic" that is structurally like ordinary arithmetic. In the simplest case this assumption would consist of at least the behavioral Axiom 9 and an additional inner psychological axiom that there is an inner psychological mapping of the inner psychological operation of function composition $*$ on $\{\hat{p} \mid p \in \mathbb{I}^+\}$ onto the inner psychological arithmetic multiplication operation.

If in addition to Axioms 1–8 the behavioral Axiom 9,

$$\mathbf{q} \bullet \mathbf{p} = \mathbf{r} \quad \text{iff} \quad q \cdot p = r,$$

where $\mathbf{q} \bullet \mathbf{p}$ stands for first estimating p times a stimulus t and then q times that estimated stimulus, is assumed, then the representational theory of measurement also yields a numeral multiplicative scale. However, unlike Stevens, the representational theory places no special significance on numeral scales: In the representational theory, any scale of isomorphisms for $\mathscr{J}$ is just as good as any other scale of isomorphisms for $\mathscr{J}$. Thus in the representational theory, even the restriction of measurement to multiplicative scales for $\mathscr{J}$ is not enough to specify a numeral scale for $\mathscr{J}$, since the scale formed by taking powers of the representations a numeral scale for $\mathscr{J}$ is easily verified to be a multiplicative scale for $\mathscr{J}$. Therefore the best one can achieve within the representational theory with respect to giving numeral scales "special significance" is to state additional axioms involving at least one of the primitives of the inner psychological structure $\mathscr{J}$ that is different from X and $\succsim_\psi$ so that the numeral multiplicative representations for $\mathscr{J}$ are the only representations for $\mathscr{J}$ that are multiplicative and represent at least one of the newly axiomatized primitives in a prescribed way, e.g., represent a primitive operation $\oplus$ as $+$.

5. PROOFS

LEMMA 1. *Assume Axioms* 1 *to* 8. *Then for all p and q in* $\mathbb{I}^+$, $\hat{p} * \hat{q} = \hat{q} * \hat{p}$.

Proof. Let p and q be arbitrary elements of $\mathbb{I}^+$ and t be an arbitrary element of X. Let

$$z = \hat{p} * \hat{q}[\Psi(t)] = \hat{p}[\hat{q}[\Psi(t)]],$$

and

$$w = \hat{q} * \hat{p}[\Psi(t)] = \hat{q}[\hat{p}[\Psi(t)]].$$

Then by Definition 14, $(\Psi^{-1}[\hat{p}[\Psi(t)]], \mathbf{p}, t)$, $(\Psi^{-1}[(\hat{q}[\Psi(t)]], \mathbf{q}, t)$, $(z, \mathbf{p}, \Psi^{-1}[\hat{q}[\Psi(t)]])$, and $(w, \mathbf{q}, \Psi^{-1}[\hat{p}[\Psi(t)]])$ are in E. Thus by Axiom 4, $z = w$, i.e., $\hat{p} * \hat{q}[\Psi(t)] = \hat{q} * \hat{p}[\Psi(t)]$. Since t is an arbitrary element of X, $\hat{p} * \hat{q} = \hat{q} * \hat{p}$.

LEMMA 2. *Assume Axioms* 1–8. *Then the following two statements are true for each p in* $\mathbb{I}^+$:

1. $\hat{p}$ *is onto* $\Psi(X)$ *and for each x and y in X*,

$$\Psi(x) \succ_\psi \Psi(y) \quad \textit{iff} \quad \hat{p}[\Psi(x)] \succ_\psi \hat{p}[\Psi(y)].$$

2. *$\hat{p}^{-1}$ is a function from $\Psi(X)$ onto $\Psi(X)$ such that for each x and y in X,*

$$\Psi(x) \succ_{\psi} \Psi(y) \qquad \textit{iff} \quad \hat{p}^{-1}[\Psi(x)] \succ_{\psi} \hat{p}^{-1}[\Psi(y)].$$

Proof. Statement 1 follows from Definition 14 and Statements 4 and 5 of Axiom 2.

Statement 2 follows from Statement 1 and Statement 4 of Axiom 2.

LEMMA 3. *Assume Axioms* 1 *to* 8. *Then the following two statements are true for all p and q in* $\mathbb{I}^+$:

1. $\hat{p} * \hat{q}^{-1} = \hat{q}^{-1} * \hat{p}$.
2. $\hat{p}^{-1} * \hat{q}^{-1} = \hat{q}^{-1} * \hat{p}^{-1}$.

Proof. 1. By Lemma 1, $\hat{p} * \hat{q} = \hat{q} * \hat{p}$. Thus $\hat{p} = \hat{q} * \hat{p} * \hat{q}^{-1}$. Therefore, $\hat{q}^{-1} * \hat{p} = \hat{p} * \hat{q}^{-1}$.

2. By Lemma 1,

$$\hat{p} * \hat{q} = \hat{q} * \hat{p}.$$

Taking inverses then yields

$$\hat{q}^{-1} * \hat{p}^{-1} = \hat{p}^{-1} * \hat{q}^{-1}.$$

DEFINITION 18. Let H be the smallest set of functions such that

(i) $\{\hat{p} \mid p \in \mathbb{I}^+\} \subseteq H$, and
(ii) if α and β are in H, then $\alpha * \beta$ is in H.

LEMMA 4. *Assume Axioms* 1–8. *Then for each α and β in H, $\alpha * \beta = \beta * \alpha$.*

Proof. The lemma follows by a simple induction argument from Lemma 1.

DEFINITION 19. Let $H^{-1} = \{\alpha^{-1} \mid \alpha \in H\}$.

LEMMA 5. *Assume Axioms* 1–8. *Then the following three statements are true for all α and β in H^{-1},*

1. $\alpha * \beta \in H^{-1}$
2. $\alpha * \beta = \beta * \alpha$.
3. *For each x and y in X, there exists γ in H^{-1} such that $\Psi(x) \succ_{\psi} \gamma(\Psi(y))$.*

Proof. Statement 1 follows from Definition 18 and noting that $\alpha^{-1} * \beta^{-1} = (\beta * \alpha)^{-1}$.

Statement 2 follows from Lemma 4 and noting that

$$\alpha * \beta = (\beta^{-1} * \alpha^{-1})^{-1} = (\alpha^{-1} * \beta^{-1})^{-1} = \beta * \alpha.$$

To show Statement 3, let x and y be arbitrary elements of X. If $x \succ y$ then it easily follows from Statement 3 of Axiom 2 and Axiom 6 that for $\gamma = \hat{1} = \hat{1}^{-1}$, $\Psi(x) \succ_{\psi} \gamma(\Psi(y))$. Thus suppose $y \succcurlyeq x$. By Statement 2 of Axiom 3, let z in X and p in $\mathbb{I}^+$ be such that $z \succ y$ and $(z, \mathbf{p}, x) \in E$. Then by Statement 1 of Axiom 3, Axiom 6. and Lemma 2, it follows that for $\gamma = \hat{q}^{-1}$, where $\mathbf{q} = \mathbf{p} + \mathbf{1}$, that $\Psi(x) \succ_{\psi} \gamma(\Psi(y))$.

DEFINITION 20. Let K be the smallest subset of functions such that

(i) $H \subseteq K$;
(ii) $H^{-1} \subseteq K$; and
(iii) if α and β are in K, then $\alpha * \beta$ is in K.

LEMMA 6. *Assume Axioms* 1–8. *Then $\langle K, * \rangle$ is a commutative group, i.e.,* (i) *$*$ is associative and $K \neq \emptyset$,* (ii) *for all α and β in K, $\alpha * \beta$ is in K,* (iii) *for all α in K, α^{-1} is in K, and* (iv) *for all α and β in K, $\alpha * \beta = \beta * \alpha$.*

Proof. (i) Since $*$ is functional composition, it is associative, and it follows from the definitions of E, H, and K that $K \neq \emptyset$.

(ii) follows from Condition (iii) of Definition 20.

(iii) By using Lemma 1, Lemma 3, and induction, it follows that for all δ and θ in H, $\delta * \theta^{-1} = \theta^{-1} * \delta$. By this and the associativity and commutativity of $*$ on H, it then easily follows from Lemma 5 and Definition 20 that K consists of elements of the form,

$$\gamma = \alpha_1 * \cdots * \alpha_m * (\beta_1 * \cdots * \beta_n)^{-1},$$

where $\alpha_1, ..., \alpha_m$ and $\beta_1, ..., \beta_n$ are elements of H. But for such γ, γ^{-1} has the same form, i.e.,

$$\gamma^{-1} = (\beta_1 * \cdots * \beta_n) * (\alpha_1 * \cdots * \alpha_m)^{-1}.$$

(iv) follows from Lemma 5 and an argument similar to (iii) above.

LEMMA 7. *Assume Axioms* 1 *to* 8. *Then for all x and y in X and all α in K,*

$$\Psi(x) \succcurlyeq_{\psi} \Psi(y) \qquad \textit{iff} \quad \alpha(\Psi(x)) \succcurlyeq_{\psi} \alpha(\Psi(y)).$$

Proof. The lemma follows from Lemma 2 and the definitions of H and K.

LEMMA 8. *Assume Axioms* 1–8. *Suppose u, x, and z are in X and $\Psi(u) \succ_{\psi} \Psi(x) \succcurlyeq_{\psi} \Psi(z)$. Then for some δ in K,*

$$\Psi(u) \succ_{\psi} \delta[\Psi(z)] \succ_{\psi} \Psi(x).$$

Proof. Suppose not. A contradiction will be shown. Let

$$Z = \{w \mid w \in X \text{ and } x \succcurlyeq w \text{ and for all } \beta \in K$$
$$\text{either } \beta(\Psi(w)) \succcurlyeq_\psi \Psi(u) \text{ or } \Psi(x) \succcurlyeq_\psi \beta(\Psi(w))\}.$$

Then by hypothesis, $z \in Z$. By the definition of Z, x is a $\succcurlyeq$-upper bound of Z. Therefore by Axiom 1, let v be the least upper $\succcurlyeq$-bound of Z. Then

$$x \succcurlyeq v. \tag{2}$$

By Statement 3 of Axiom 3 and Lemma 2, p in $\mathbb{I}^+$ can be found so that $q = p + 1$, $\delta = \hat{p}^{-1} * \hat{q}$, and

$$\Psi(u) \succ_\psi \delta(\Psi(x)) \succ_\psi \Psi(x). \tag{3}$$

Consider $\delta(\Psi(v))$. Since $q = p + 1 > p$, it follows by Statement 1 of Axiom 3 and Lemma 2 that $\hat{q}(\Psi(v)) \succ_\psi \hat{p}(\Psi(v))$ and therefore by the definition of δ and Lemma 2 that

$$\delta(\Psi(v)) \succ_\psi \Psi(v). \tag{4}$$

Since by Equation 2 $x \succcurlyeq v$, it follows from Lemma 7 and Eq. (3) that $\Psi(u) \succ \delta(\Psi(v))$. Then, since $\succcurlyeq$ is a total ordering, either (i) $\Psi(x) \succcurlyeq \delta(\Psi(v))$ or (ii) $\Psi(u) \succ \delta(\Psi(v)) \succ \Psi(x)$. Part (i) and Eq. (4) contradicts v being the least upper bound of Z. Therefore (ii) holds. Since by Axioms 1 and 6, $\langle \Psi(X), \succcurlyeq_\psi \rangle$ is a continuum, let t in X be such that

$$\Psi(u) \succ \delta(\Psi(v)) \succ \Psi(t) \succ \Psi(x).$$

Then by Lemma 7, $\Psi(v) \succ \delta^{-1}(\Psi(t))$. Thus, since v is the least upper bound of Z, let s in Z be such that

$$\Psi(v) \succcurlyeq \Psi(s) \succ \delta^{-1}(\Psi(t)).$$

Then $x \succcurlyeq v \succcurlyeq s$ and

$$\Psi(u) \succ \delta(\Psi(v)) \succcurlyeq \delta(\Psi(s)) \succ \Psi(t) \succ \Psi(x),$$

which contradicts the assumption that s is in Z.

LEMMA 9. *Assume Axioms* 1–8. *Then for all* α *and* β *in* K, $\alpha(\Psi(x)) \succ_\psi \beta(\Psi(x))$ *for some* x *in* X *if and only if for all* y *in* X, $\alpha(\Psi(y)) \succ_\psi \beta(\Psi(y))$.

Proof. Suppose x in X is such that

$$\alpha(\Psi(x)) \succ_\psi \beta(\Psi(x)).$$

Then by Lemma 7,

$$(\beta^{-1} * \alpha)(\Psi(x)) \succ_\psi \Psi(x). \tag{5}$$

Suppose y in X is such that $\alpha(\Psi(y)) \not\succ_\psi \beta(\Psi(y))$. A contradiction will be shown. Since $\succ_\psi$ is a total ordering,

$$\beta(\Psi(y)) \succcurlyeq_\psi \alpha(\Psi(y)). \tag{6}$$

By Statement 3 of Lemma 5, let γ in K be such that

$$\Psi(x) \succcurlyeq_\psi \gamma(\Psi(y)).$$

By Eq. (5) and Lemma 8, let $\delta \in K$ be such that

$$\beta^{-1} * \alpha(\Psi(x)) \succ_\psi \delta[\gamma(\Psi(y))] \succ_\psi \Psi(x). \tag{7}$$

Then

$$\alpha(\Psi(x)) \succ_\psi (\beta * \delta * \gamma)(\Psi(y)),$$

which by commutativity of $\langle K, * \rangle$ yields

$$\alpha(\Psi(x)) \succ_\psi (\delta * \gamma * \beta)(\Psi(y)),$$

which by Eq. (6) yields

$$\alpha(\Psi(x)) \succ_\psi (\delta * \gamma * \alpha)(\Psi(y)),$$

which by commutativity of $\langle K, * \rangle$ yields

$$\alpha(\Psi(x)) \succ_\psi (\alpha * \delta * \gamma)(\Psi(y)),$$

which yields

$$\Psi(x) \succ_\psi (\delta * \gamma)(\Psi(y)),$$

which contradicts Eq. (7).

Suppose for all y in X, $\alpha(\Psi(y)) \succ_\psi \beta(\Psi(y))$. Then for some x in X, $\alpha(\psi(x)) \succ_\psi \beta(\Psi(x))$.

LEMMA 10. *Assume Axioms* 1–8. *Then for all* α *in* K, *if* $\alpha(\Psi(x)) \succ_\psi \Psi(x)$ *for some* x *in* X, *then* $\alpha(\Psi(y)) \succ_\psi \Psi(y)$ *for all* y *in* X.

Proof. Let α be an element of K and x be an element of X such that $\alpha(\Psi(x)) \succ_\psi \Psi(x)$. Then

$$\alpha(\Psi(x)) \succ_\psi \iota(\Psi(x)),$$

where ι is the identity element of K. Then by Lemma 9, for each y in X,

$$\alpha(\Psi(y)) \succ_\psi \iota(\Psi(y)) = \Psi(y).$$

DEFINITION 21. For each nonempty subset U of K, α is said to be an *upper bound* of U if and only if α is a function from $\Psi(X)$ into $\Psi(X)$ such that for all β in U and all x in X,

$$\alpha(\Psi(x)) \succcurlyeq_\psi \beta(\Psi(x)).$$

By definition for each nonempty subset U of K, $\gamma = \text{l.u.b. } U$ if and only if γ is an upper bound of U and for all upper bounds α of U, $\alpha(\Psi(x)) \succcurlyeq_\psi \gamma(\Psi(x))$ for all x in X. If U is a nonempty subset of K and $\gamma = \text{l.u.b. } U$, then γ is said to be the *least upper bound* of U. *Lower bound* of U and *greatest lower bound* of U (g.l.b. U) have the obvious analogous definitions.

Since $\langle X, \succcurlyeq \rangle$ is a continuum, it easily follows that least upper bounds for upper bounded, nonempty subsets U of K always exist and are unique.

Definition 22. By definition, let

$$L = \{\text{l.u.b. } U \mid U \neq \varnothing \text{ and } U \subseteq K \text{ and } U \text{ has an upper bound}\}.$$

Lemma 11 *Assume Axioms* 1–8. *Then the following four statements are true*:

1. *$\langle L, * \rangle$ is a commutative group.*
2. *$\langle L, * \rangle$ is homogeneous, i.e., for all $\Psi(x)$ and $\Psi(y)$ in $\Psi(X)$ there exists α in L such that $\alpha(\Psi(x)) = \Psi(y)$.*
3. *For all α in L and all x and y in X*

$$\Psi(x) \succcurlyeq_\psi \Psi(y) \quad \textit{iff} \quad \alpha(\Psi(x)) \succcurlyeq_\psi \alpha(\Psi(y)).$$

Proof. 1. Suppose α and β are arbitrary elements of L. By Definition 22, let U and V be nonempty subsets of K such that

$$\alpha = \text{l.u.b. } U \quad \text{and} \quad \beta = \text{l.u.b. } V.$$

We will first show that $\alpha * \beta$ is an element of L. Let

$$W = \{\gamma' \in K \mid \exists\alpha'\exists\beta'[\alpha' \in U \text{ and } \beta' \in V \text{ and for all } x \text{ in } X, \alpha' * \beta'(\Psi(x)) \succcurlyeq_\psi \gamma'(\Psi(x))]\}.$$

(In the above equation, the symbol "∃" stands for the existential quantifier "for some".) Then W is nonempty, since $\alpha' * \beta'$ is in W for all α' in U and all β' in V. W is bounded above by $\alpha * \beta$. Let $\gamma = \text{l.u.b. } W$. It easily follows from the definitions of α, β, and γ that $\alpha * \beta(\Psi(x)) \succcurlyeq_\psi \gamma(\Psi(x))$ for all x in X. Thus to show $\alpha * \beta = \gamma$, it is sufficient to show that the assumption that for some x in X, $\alpha * \beta(\Psi(x)) \succ_\psi \gamma(\Psi(x))$ leads to a contradiction:

Suppose x in X is such that $\alpha * \beta(\Psi(x)) \succ_\psi \gamma(\Psi(x))$. It follows from Lemma 8 that κ in K can be found so that

$$\alpha * \beta(\Psi(x)) \succ_\psi \kappa * \gamma(\Psi(x)) \succ_\psi \gamma(\Psi(x)).$$

Then

$$\alpha * \beta(\Psi(x)) \succ_\psi (\kappa^{-1} * \alpha) * \beta(\Psi(x)) \succ_\psi \gamma(\Psi(x)).$$

In particular,

$$\alpha[\beta(\Psi(x))] \succ_\psi \kappa^{-1} * \alpha[\beta(\Psi(x))].$$

Thus, since $\alpha = \text{l.u.b. } U$, let α' in U be such that

$$\alpha[\beta(\Psi(x))] \succcurlyeq_\psi \alpha'[\beta(\Psi(x))] \succcurlyeq_\psi \kappa^{-1} * \alpha[\beta(\Psi(x))] \succ_\psi \gamma(\Psi(x)). \quad (8)$$

By applying Lemma 8 to Eq. (8), δ in K can be found so that

$$\alpha'[\beta(\Psi(x))] \succ_\psi \delta * \gamma(\Psi(x)) \succ_\psi \gamma(\Psi(x)),$$

which yields

$$\alpha'[\beta(\Psi(x))] \succ_\psi \delta^{-1} * \alpha'[\beta(\Psi(x))] \succ_\psi \gamma(\Psi(x)).$$

Since by the Lemma 6, * is commutative on K, it follows that

$$\alpha'[\delta^{-1} * \beta(\Psi(x))] \succ_\psi \gamma(\Psi(x)). \quad (9)$$

Since, by choice of δ and Lemma 10 $\delta(\Psi(z)) \succ_\psi \Psi(z)$ for all $z \in X$, it follows that

$$\beta(\Psi(x)) \succ_\psi \delta^{-1} * \beta(\Psi(x))).$$

Since $\beta = \text{l.u.b. } V$, let β' in V be such that

$$\beta(\Psi(x)) \succcurlyeq_\psi \beta'(\Psi(x)) \succcurlyeq_\psi \delta^{-1} * \beta(\Psi(x))).$$

Then by Eq. (9) and Lemma 7

$$\alpha' * \beta'(\Psi(x)) \succ_\psi \gamma(\Psi(x)),$$

which contradicts that γ is the l.u.b. of W.

To show that inverses of elements of L are in L, let

$$\alpha_1 = \text{g.l.b.}\{\eta^{-1} \mid \eta \in U\}.$$

Then it easily follows that α_1 is in L and $\alpha_1 = \alpha^{-1}$.

The identity function ι on $\Psi(X)$ is in L since it is in K.

Since it is immediate that $*$ is associative, the above shows that $\langle L, * \rangle$ is a group. Since by Lemma 6 $*$ is commutative in K,

$$\begin{aligned}\alpha * \beta &= \text{l.u.b.}\{\eta * v \mid \eta \in U \quad \text{and} \quad v \in V\}\\ &= \text{l.u.b.}\{v * \eta \mid \eta \in U \quad \text{and} \quad v \in V\}\\ &= \beta * \alpha,\end{aligned}$$

and thus $*$ is a commutative operation on L.

2. Let $\Psi(x)$ and $\Psi(y)$ be arbitrary elements of $\Psi(X)$. It will be shown that $\alpha(\Psi(x)) = \Psi(y)$ for some α in L. If $\Psi(x) = \Psi(y)$, then $\iota(\Psi(x)) = \Psi(y)$ where ι is the identity element of L. So suppose $\Psi(x) \neq \Psi(y)$. Without loss of generality, suppose $\Psi(y) \succ_\psi \Psi(x)$. Let

$$U = \{\beta \mid \beta \in K \quad \text{and} \quad \Psi(y) \succsim_\psi \beta(\Psi(x))\}.$$

Then it easily follows that U is a nonempty bounded subset of K. Let $\alpha = \text{l.u.b. } U$. If $\alpha(\Psi(x)) = \Psi(y)$, then Statement 2 has been shown. Suppose $\alpha(\Psi(x)) \neq \Psi(y)$. A contradiction will be shown. Since $\alpha(\Psi(x)) \neq \Psi(y)$, it follows from the definitions of U and α,

$$\Psi(y) \succ_\psi \alpha(\Psi(x)) \succsim_\psi \Psi(x).$$

By Lemma 8, let δ in K be such that

$$\Psi(y) \succ_\psi \delta(\Psi(x)) \succ_\psi \alpha(\Psi(x)).$$

Then δ is in U and $\delta(\Psi(x)) \succ_\psi \alpha(\Psi(x))$, which contradicts $\alpha = \text{l.u.b. } U$.

3. Let α be an arbitrary element of L and x and y be arbitrary elements of X. It follows from the definition of L that $\Psi(x) \succsim_\psi \Psi(y)$ implies $\alpha(\Psi(x)) \succsim_\psi \alpha(\Psi(y))$. Assume

$$\alpha(\Psi(x)) \succsim_\psi \alpha(\Psi(y)).$$

Then, since $\alpha^{-1} \in L$, it immediately follows that $\Psi(x) \succsim_\psi \Psi(y)$.

LEMMA 12. *Assume Axioms* 1–8. *Then there exists a numerical structure*

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, S_1, ..., S_j, ... \rangle_{j \in J}$$

such that the following three statements are true:

1. *The set $\mathscr{S}$ of isomorphism of the inner psychological structure $\mathscr{J}$ (as defined in Axiom* 6) *onto $\mathfrak{N}$ is nonempty.*

2. *For all φ in $\mathscr{S}$ (as defined in Statement* 1) *and all p in $\mathbb{I}^+$, $\varphi(\hat{p})$ is a function that is a multiplication by a positive constant.*

3. *If $\mathscr{J}$ is homogeneous, then the following two statements are true*:

(i) *$\mathscr{S}$ (as defined in Statement* 1) *is a ratio scale.*

(ii) *Let t be an arbitrary element of X, and by* (i) *let φ be the unique element of $\mathscr{S}$ such that $\varphi(\Psi(t)) = 1$, and by Statement* 2, *for each p in $\mathbb{I}^+$, let c_p be the positive real such that multiplication by $c_p = \varphi(\hat{p})$. Then for all p in $\mathbb{I}^+$ and x in X,*

$$(x, \mathbf{p}, t) \in E \qquad \textit{iff} \quad \varphi(\Psi(x)) = c_p.$$

Proof. Since by Lemma 11 $\langle L, * \rangle$ is a homogeneous, commutative group on the continuum $\langle \Psi(X), \succsim_\psi \rangle$, it follows by the proof of Theorem 2.12 of Narens (1981) (see also Theorem 4.2 of Chapter 2 of Narens, 1985) that a ratio scale $\mathscr{T}$ from $\Psi(X)$ onto $\mathbb{R}^+$ and a function φ can be found such that $\varphi \in \mathscr{T}$, $\varphi(\succsim_\psi) = \geqslant$, and for each τ in $\mathscr{T}$, $\tau(L)$ is the set of multiplications by positive reals. Let φ be an element of $\mathscr{T}$. Let

$$\mathfrak{N} = \varphi(\mathscr{J}) = \langle \varphi(\Psi(X)), \varphi(\succsim_\psi), \varphi(R_1), ..., \varphi(R_j), ... \rangle_{j \in J},$$

and let $\mathscr{S}$ be the set of isomorphisms of $\mathscr{J}$ onto $\mathfrak{N}$.

1. $\mathscr{S} \neq \varnothing$, since $\varphi \in \mathscr{S}$.

In order to show Statements 2 and 3, it will first be shown that for each automorphism α of $\mathscr{J}$ that $\varphi(\alpha)$ is a function that is multiplication by a positive real:

Let α be an automorphism of $\mathscr{J}$. Since for each p in $\mathbb{I}^+$, $\hat{p}$ is meaningful with respect to $\mathscr{J}$, it follows from previous remarks about meaningfulness that $\hat{p}$ is invariant under the automorphisms of $\mathscr{J}$. Using this fact, it is easy to verify that the elements of H, K, and L are invariant under the automorphisms of $\mathscr{J}$. Therefore, in particular, for each β in L and each x in X,

$$\alpha[\beta(\Psi(x))] = \beta[\alpha(\Psi(x))]. \tag{10}$$

Let f be the function $\varphi(\alpha)$ from $\mathbb{R}^+$ onto $\mathbb{R}^+$, and for each β in L, let r_β be the positive real such that $\varphi(\beta)$ is multiplication by r_β, and for each x in X, let s_x be the positive real such that $\varphi(\Psi(x)) = s_x$. Then from Eq. (10), one obtains

$$f(r_\beta \cdot s_x) = r_\beta \cdot f(s_x). \tag{11}$$

Since α is also an automorphism $\langle \Psi(X), \succ_\psi \rangle$, it follows that f is a strictly increasing function. Also,

$$\mathbb{R}^+ = \{r_\beta \mid \beta \in L\} = \{s_x \mid x \in X\}.$$

It is well-known that all strictly increasing functions g on $\mathbb{R}^+$ that satisfy the functional equation

$$g(r \cdot s) = r \cdot g(s)$$

for all r and s in $\mathbb{R}^+$ have the form

$$g(u) = c \cdot u,$$

where c is a positive constant. Thus it follows from Eq. (11) that $\varphi(\alpha)$ is a multiplication by a positive constant.

2. Suppose p is an arbitrary element of $\mathbb{I}^+$ and η is an arbitrary element $\mathscr{S}$. Since $\hat{p} \in L$, let s be the positive real such that $\varphi(\hat{p})$ is multiplication by s. Then since $\mathscr{S}$ is a scale of isomorphisms of $\mathscr{J}$ onto $\mathfrak{R}$, $\alpha = \varphi^{-1} * \eta$ is an automorphism of $\mathscr{J}$. Since $\hat{p}$ is meaningful, it is invariant under α, i.e.,

$$\alpha * \hat{p} = \hat{p} * \alpha.$$

By an above argument given in Statement 1, let r in $\mathbb{R}^+$ be such that $\varphi(\alpha)$ is multiplication by r. Then for each x in X,

$$\begin{aligned} \eta[\hat{p}(\Psi(x))] &= \varphi[\varphi^{-1} * \eta * \hat{p}(\Psi(x))] \\ &= \varphi[\alpha * \hat{p}(\Psi(x))] \\ &= \varphi[\hat{p}(\alpha(\Psi(x)))] \\ &= \varphi(\hat{p})[\varphi(\alpha(\Psi(x)))] \\ &= s \cdot (r \cdot \varphi(\Psi(x))), \end{aligned}$$

i.e., $\eta(\hat{p})$ is the function that is multiplication by $s \cdot r$.

3. (i) Suppose $\mathscr{J}$ is homogeneous. Let r be an arbitrary element of $\mathbb{R}^+$. Since $\varphi(L)$ is the set of multiplications by positive reals, let β in L be such that $\varphi(\beta)$ is multiplication by r. Let x be an element of X. By the homogeneity of $\mathscr{J}$, let α be an automorphism of $\mathscr{J}$ such that $\alpha(\Psi(x)) = \beta(\Psi(x))$. Since $\varphi(L)$ is the set of multiplications by positive reals and since by an above argument given in Statement 1 $\varphi(\alpha)$ is a function that is multiplication by a positive real, it follows that $\varphi(\alpha)$ is multiplication by r, because

$$\varphi[\alpha(\Psi(x))] = \varphi[\beta(\Psi(x))] = r \cdot \varphi(\Psi(x)),$$

i.e., $\varphi(\alpha)$ is multiplication by r. It is easy to verify for each automorphism γ of $\mathscr{J}$ that $\varphi * \gamma$ is an isomorphism of $\mathscr{J}$ onto $\mathfrak{R}$. Therefore, $\eta = \varphi * \alpha$ is in $\mathscr{S}$, i.e., for all y in X,

$$\eta(\Psi(y)) = r \cdot \varphi(\Psi(y)).$$

Therefore, since r is an arbitrary element of $\mathbb{R}^+$, it has been shown that $s \cdot \varphi \in \mathscr{S}$ for each s in $\mathbb{R}^+$. To show Clause (i) that $\mathscr{S}$ is a ratio scale, one then just needs to note that from above $\varphi(\delta)$ for an automorphism δ of $\mathscr{J}$ is a multiplication by a positive real and that for all η in $\mathscr{S}$ and all x in X,

$$\eta(\Psi(x)) = t \cdot \varphi(\Psi(x)),$$

where multiplication by t is $\varphi(\theta)$ where θ is the automorphism $\varphi^{-1} * \eta$ of $\mathscr{J}$, i.e.,

$$\begin{aligned} \eta(\Psi(x)) &= \varphi[\varphi^{-1} * \eta(\Psi(x))] \\ &= \varphi[\theta(\Psi(x))] \\ &= \varphi(\theta) * \varphi(\Psi(x)) \\ &= t \cdot \varphi(\Psi(x)). \end{aligned}$$

Since $\mathscr{T}$ and $\mathscr{S}$ are ratio scales and $\varphi \in \mathscr{S} \cap \mathscr{T}$, it follows that $\mathscr{S} = \mathscr{T}$. Clause (ii) then follows, since $\varphi * \hat{p}$ is an element of $\varphi(L) = \mathscr{T}$ for each p in $\mathbb{I}^+$.

LEMMA 13. *Let $\mathfrak{R}$ and $\mathscr{S}$ be as in Lemma* 12. *Then there exists a set K of elements of $\mathbb{R}^+$ such that for each φ in $\mathscr{S}$,*

(i) $1 \in K$, *and for each r and s in K, $r^{-1} \in K$ and $rs \in K$; and*

(ii) $\mathscr{S} = \{r \cdot \varphi \mid r \in K\}$.

Proof. The theorem is an easy consequence of the following facts: (1) $\mathscr{S}$ is a scale of isomorphisms of $\mathscr{J}$; (2) for all η and γ in $\mathscr{S}$, $\eta^{-1} * \gamma$ is an automorphism of $\mathscr{J}$; (3) the automorphisms of $\mathscr{J}$ form a group; and (4) by the proof of Lemma 12 $\eta(\alpha)$ is a function that is multiplication by a positive real for each η in $\mathscr{S}$ and each automorphism α of $\mathscr{J}$.

THEOREM 13 (Theorem 6). *Assume Axioms* 1–8. *Then there exists a numerical structure $\mathfrak{R}$ with domain $\mathbb{R}^+$ such that the following three statements are true*:

1. *The set $\mathscr{S}$ of isomorphisms of the inner psychological structure $\mathscr{J}$ onto $\mathfrak{R}$ is a subscale of a ratio scale.*

2. *$\mathscr{S}$ (as defined in Statement* 1) *is a multiplicative scale* (*Definition* 13) *for $\mathscr{J}$.*

3. *If $\mathscr{J}$ is homogeneous, then the following two statements are true*:

(i) *$\mathscr{S}$ (as defined in Statement* 1) *is a ratio scale.*

(ii) *Let t be an arbitrary element of X, and by* (i) *let φ be the unique element of $\mathscr{S}$ such that $\varphi((\Psi(t))) = 1$, and by Statement* 2, *for each p in $\mathbb{I}^+$, let c_p be the positive real such that multiplication by c_p is $\varphi(\hat{p})$. Then for all p in $\mathbb{I}^+$ and x in X,*

$$(x, \mathbf{p}, t) \in E \quad \textit{iff} \quad \varphi(\Psi(x)) = c_p.$$

Proof. Follows immediately from Lemmas 12 and 13.

THEOREM 14 (Theorem 9). *Suppose for Task* 1 (i) *the subject has been instructed to "Find a stimulus in X which appears to be p times greater in intensity than the stimulus t,"* (ii) *E is his or her responses to this task,* (iii) *Axioms* 1–8 *hold and the inner psychological structure $\mathscr{J}$ is homogeneous, and* (iv) *$\mathscr{S}$ is a multiplicative scale of isomorphisms of $\mathscr{J}$. Suppose in Task* 2 *different instructions are given to the subject, e.g.,* *"Find the stimulus which in your subjective valuation is $p +$ the valuation of the stimulus t," and as a result of these instructions the subject produces the partial data set H where elements of H have the form $(x, \mathbf{q}, t)$, where q is a fixed positive integer, t ranges over the elements of X, and (1) for each t in X there exists exactly one x in X such that $(x, \mathbf{q}, t) \in H$, (2) for all $(x, \mathbf{q}, t)$ in H, $x \succcurlyeq t$, and (3) for all x, y, t, and v in X, if $(x, \mathbf{q}, t)$ and $(y, \mathbf{q}, v)$ and $t > v$, then $x > y$. Let $\hat{q}$ be the following function on $\Psi(X)$: For all x and t in X,*

$$\hat{q}(\Psi(t)) = \Psi(x) \qquad \text{iff} \quad (x, \mathbf{q}, t) \in H.$$

Assume $\hat{q}$ is meaningful with respect to $\mathscr{J}$. Then there exists a positive real r such that for all φ in $\mathscr{S}$, $\varphi(\hat{q}) =$ is the function that is multiplication by r.

Proof. Let $\mathscr{P}$ be the set of automorphisms of $\mathscr{J}$. Since by assumption $\hat{q}$ is meaningful with respect to $\mathscr{J}$, it is invariant under the elements of $\mathscr{P}$, i.e., for all β in $\mathscr{P}$ and all x in X,

$$\hat{q}[\beta(\Psi(x))] = \beta[\hat{q}(\Psi(x))]. \qquad (12).$$

Let φ be an arbitrary element of $\mathscr{S}$. Then, since $\mathscr{J}$ is homogeneous, it follows by the proof of Lemma 12 that $\varphi(\mathscr{P})$ is the set of multiplications by positive reals. Let f be the function $\varphi(\hat{q})$ from $\mathbb{R}^+$ onto $\mathbb{R}^+$, and for each β in $\mathscr{P}$, let r_β be the positive real such that $\varphi(\beta)$ is multiplication by r_β, and for each x in X, let s_x be the positive real such that $\varphi(\Psi(x)) = s_x$. Then from Eq. (12), one obtains

$$f(r_\beta \cdot s_x) = r_\beta \cdot f(s_x). \qquad (13)$$

It easily follows from assumptions (1) and (3) above that $\hat{q}$ is a strictly increasing function onto $\mathbb{R}^+$. It is well known that all strictly increasing functions g on $\mathbb{R}^+$ that satisfy the functional equation

$$g(r \cdot s) = r \cdot g(s)$$

for all r and s in $\mathbb{R}^+$ have the form

$$g(u) = c \cdot u,$$

where c is a positive constant. Thus it follows from Eq. (13) that $\varphi(\hat{q})$ is a multiplication by a positive constant.

THEOREM 15 (Theorem 4). *Assume Axioms* 1–4. *Then there exists a numerical representing structure $\mathfrak{N}$ such that the scale of isomorphisms $\mathscr{S}$ of $\mathfrak{B}$ onto $\mathfrak{N}$ is* (i) *a multiplicative scale for $\mathfrak{B}$ and* (ii) *is a ratio scale.*

Proof. Let Ψ be the identity function on X, $\succcurlyeq_\psi = \succcurlyeq$, $\mathbb{I}^+ = J$, and the inner psychological structure $\mathscr{J}$ be

$$\mathscr{J} = \langle \Psi(X), \succcurlyeq_\psi, \bar{1}, ..., \bar{j}, ... \rangle_{j \in \mathbb{I}^+}.$$

Then it is easy to verify that Axioms 5–8 hold. Thus by Theorem 13 let $\mathscr{S}$ be a multiplicative scale of isomorphisms of $\mathscr{J}$ onto the structure $\mathfrak{N}$, where

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, M_1, ..., M_j, ... \rangle_{j \in \mathbb{I}^+}.$$

Then $\mathscr{S}$ is a multiplicative scale of isomorphisms of $\mathfrak{B}$ onto $\mathfrak{N}$. By Lemma 13, for all φ and γ in $\mathscr{S}$ there exists s in $\mathbb{R}^+$ such that $\gamma = s \cdot \varphi$. Thus to show that $\mathscr{S}$ is a ratio scale, it is sufficient to show that $r \cdot \varphi \in \mathscr{S}$ for each r in $\mathbb{R}^+$ and each φ in $\mathscr{S}$.

Let r be an element of $\mathbb{R}^+$ and φ be an element of $\mathscr{S}$. It follows from Lemma 4 that for each j in $\mathbb{I}^+$ that $\bar{j}$ is an automorphism of $\mathscr{J}$. Therefore by the proof of Lemma 12 it follows that $\varphi(\bar{j})$ is multiplication by a positive real, s_j. Therefore M_j is multiplication by s_j. Since for all j in $\mathbb{I}^+$ and all x and t in X,

$$x = \bar{j}(t) \quad \text{iff} \quad \varphi(x) = s_j \cdot \varphi(t)$$
$$\text{iff} \quad r \cdot \varphi(x) = r \cdot s_j \cdot \varphi(t) = s_j \cdot r \cdot \varphi(t),$$

and for all u and v in X,

$$u \succcurlyeq v \quad \text{iff} \quad \varphi(u) \geqslant \varphi(v) \quad \text{iff} \quad r \cdot \varphi(u) \geqslant r \cdot \varphi(v),$$

it follows that $r \cdot \varphi$ is an isomorphism of $\mathscr{J}$ onto $\mathfrak{N}$ and therefore is an element of $\mathscr{S}$.

THEOREM 16 (Theorem 5). *Assume Axioms* 1–4. *Suppose $\mathscr{S}$ is a ratio scale of isomorphisms of $\mathfrak{B}$. Then $\mathscr{S}$ is a multiplicative scale of isomorphisms of $\mathfrak{B}$.*

Proof. By Theorem 15, let $\mathscr{T}$ be a multiplicative scale of isomorphisms of $\mathfrak{B}$ that is a ratio scale. Since both $\mathscr{S}$ and $\mathscr{T}$ are ratio scales, it then follows from Theorem 2.7 of Narens (1981) that r in $\mathbb{R}^+$ can be found so that $\mathscr{S} = \{\varphi^r \mid \varphi \in \mathscr{T}\}$. Because $\mathscr{T}$ is a multiplicative scale, it easily follows that φ^r is a multiplicative representing function for each φ in $\mathscr{T}$. Therefore $\mathscr{S}$ is multiplicative scale of isomorphisms of $\mathfrak{B}$.

THEOREM 17 (Theorem 11). *Assume Axioms* 1–8. *Then the following two statements are logically equivalent*:

1. *Axiom* 9.
2. *There exists a representing structure*

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, S_1, ..., S_j, ... \rangle_{j \in J}$$

such that the set $\mathscr{S}$ of isomorphisms of the inner psychological structure $\mathscr{J}$ onto $\mathfrak{N}$ is a numeral multiplicative scale for $\mathscr{J}$ (*Definition* 17).

Proof. It immediately follows from Definition 17 that Statement 2 implies Statement 1. Assume Statement 1, i.e., assume Axiom 9.

By Theorem 13, let $\mathscr{T}$ be a multiplicative scale of isomorphisms of $\mathscr{J}$ onto the structure

$$\mathfrak{I} = \langle \mathbb{R}^+, \geqslant, S_1, ..., S_j, \ldots \rangle_{j \in J}.$$

For each r in $\mathbb{R}^+$, let $\mathfrak{I}_r$ be the structure

$$\mathfrak{I}_r = \langle \mathbb{R}^+, \geqslant, S_{1,r}, ..., S_{j,r}, \ldots \rangle_{j \in J},$$

where for each $j \in J$, $S_{j,r}$ is the $m(j)$-ary relation on $\mathbb{R}^+$ defined by: For all $a_1, ..., a_{m(j)}$ in $\mathbb{R}^+$,

$$S_{j,r}(a_1^r, ..., a_{m(j)}^r) \quad \text{iff} \quad S_j(a_1, ..., a_{m(j)}).$$

Then for each r in $\mathbb{R}^+$, the scale $\mathscr{T}_r = \{\varphi^r \mid \varphi \in \mathscr{T}\}$ is a multiplicative scale of isomorphisms of $\mathscr{J}$ onto $\mathfrak{I}_r$.

Let r in $\mathbb{R}^+$ be such that for each φ in $\mathfrak{I}_r$ $\varphi(\hat{2})$ is the function that is multiplication by 2, and let

$$\mathscr{S} = \mathscr{T}_r \qquad \text{and} \qquad \mathfrak{N} = \mathfrak{I}_r.$$

Then $\mathscr{S}$ is a multiplicative scale of isomorphisms of $\mathscr{J}$ onto $\mathfrak{N}$ and $\varphi(\hat{2})$ is the function that is multiplication by 2 for each φ in $\mathscr{S}$. It will be shown by contradiction that $\mathscr{S}$ is a numeral multiplicative scale for $\mathscr{J}$.

Suppose $\mathscr{S}$ is not a numeral multiplicative scale for $\mathscr{J}$. Let k be a positive integer such that $\varphi(\hat{k}) \neq$ multiplication by k. Then it easily follows that $k \neq 1$ and $k \neq 2$. Thus $k > 2$. Let φ be an element of $\mathscr{S}$ and let c be the positive real number such that $\varphi(\hat{k})$ is the function that is multiplication by c. There are two cases to consider:

Case 1. $k > c$. Then $\log k > \log c$, and thus by elementary properties of real numbers, positive integers m and n can be found so that

$$m \log k > n \log 2 > m \log c,$$

i.e.,

$$k^m > 2^n > c^m. \tag{14}$$

Let μ be the function on the positive reals such that for each positive real t, $\mu(t)$ is the function that is multiplication by t. Then by Axioms 1–8 and Axiom 9,

$$\mu(2^n) = \varphi(\hat{2})^n = \varphi(\widehat{2^n})$$

and

$$\mu(c^m) = \varphi(\hat{k})^m = \varphi(\widehat{k^m}).$$

Then, because by Equation (14) $2^n > c^m$, $\mu(2^2) > \mu(c^m)$, and thus

$$\varphi(\widehat{2^n}) > \varphi(\widehat{k^m}). \tag{15}$$

Because Axioms 1–8 require $\varphi(\hat{p}) > \varphi(\hat{q})$ whenever $p > q$, it follows from Eq. (14) that

$$\varphi(\widehat{k^m}) > \varphi(\widehat{2^n}),$$

contradicting Eq (15).

Case 2. $k < c$. Similar to Case 1.

Theorem 18 (Theorem 12). *Assume Axioms* 1–4. *Then the following two statements are logically equivalent*:

1. *Axiom* 9.
2. *There exists a representing structure*

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, T_1, ..., T_i, \ldots \rangle_{i \in \mathbb{I}^+}$$

such that the set $\mathscr{S}$ of isomorphisms of the behavioral structure $\mathfrak{B}$ onto $\mathfrak{N}$ is a ratio scale and is a numeral multiplicative scale for $\mathfrak{B}$ (Definition 17).

Proof. The proof of the existence of a numeral multiplicative scale $\mathscr{S}$ of isomorphisms of $\mathfrak{B}$ is similar to Theorem 17. By Theorem 15 $\mathfrak{B}$ has a ratio scale of isomorphisms, and therefore $\mathfrak{N}$ can be chosen in such a way that $\mathscr{S}$ is a multiplicative ratio scale of isomorphisms.

Theorem 19 (Theorem 7). *Assume Axioms* 1 *to* 8 *and that $\mathscr{J}$ is homogeneous. For each function h from X into $\mathbb{R}^+$, let h' be the function on $\Psi(X)$ that is defined by*: *For all t in X,*

$$h'(\Psi(t)) = h(t).$$

Then the following two statements are true:

1. *For each scale $\mathscr{S}$ of isomorphisms of $\mathfrak{B}$, the set of functions*

$$\mathscr{S}' = \{\varphi' \mid \varphi \in \mathscr{S}\}$$

is a scale of isomorphisms of $\mathscr{J}$.

2. *For each scale $\mathscr{T}$ of isomorphisms of $\mathscr{J}$ there exists a scale $\mathscr{S}$ of isomorphisms of $\mathfrak{B}$ such that*

$$\mathscr{T} = \{\varphi' \mid \varphi \in \mathscr{S}\}.$$

Proof. 1. Let $\mathscr{S}$ be a scale of isomorphisms from $\mathfrak{B}$ onto

$$\mathfrak{N} = \langle \mathbb{R}^+, \geqslant, 1', ..., k', \ldots \rangle_{k \in \mathbb{I}^+}.$$

Since for k in $\mathbb{I}^+$, $\hat{k}$ is meaningful with respect to $\mathscr{J}$, it follows from previous remarks about meaningfulness that $\hat{k}$ is invariant under the automorphisms of $\mathscr{J}$. Therefore, the structure

$$\mathscr{J}' = \langle \Psi(X), \succcurlyeq_\psi, R_1, ..., R_j, ..., \hat{1}, ..., \hat{k}, \ldots \rangle_{j \in J, k \in \mathbb{I}^+}$$

has the same set of automorphisms as $\mathscr{J}$. Let φ be an element of $\mathscr{S}$ and let

$$\mathfrak{N}' = \langle \mathbb{R}^+, \geqslant, \varphi'(R_1), ..., \varphi'(R_j), ..., 1', ..., k', ... \rangle_{j \in J, k \in \mathbb{I}^+}.$$

Then φ' is an isomorphism of $\mathscr{J}'$ onto $\mathfrak{N}'$. Let $\mathscr{T}$ be the scale of isomorphisms from $\mathscr{J}'$ onto $\mathfrak{N}$. Since $\mathscr{J}$ and $\mathscr{J}'$ have the same set of automorphisms, it easily follows that $\mathscr{T}$ is a scale of isomorphisms of $\mathscr{J}$ onto

$$\langle \mathbb{R}^+, \geqslant, \varphi'(R_1), ..., \varphi'(R_j), ..., \rangle_{j \in J}.$$

It will be shown that $\mathscr{T} = \{\xi' \mid \xi \in \mathscr{S}\}$.

Let $\mathscr{H}$ be the set of automorphisms of $\mathfrak{N}$ and $\mathscr{H}'$ be the set of automorphisms of $\mathfrak{N}'$. Then $\mathscr{H}' \subseteq \mathscr{H}$. Since by assumption $\mathscr{J}$ is homogeneous and since $\mathscr{J}$ and $\mathscr{J}'$ have the same set of automorphisms, $\mathscr{J}'$ is homogeneous. Therefore by isomorphism $\mathfrak{N}'$ is homogeneous. $\mathfrak{N}$ is 1-point unique, since it is isomorphic to $\mathfrak{B}$, which by Theorem 15 has a ratio scale of isomorphisms. Let θ be an arbitrary element of $\mathscr{H}$. It will be shown that $\theta \in \mathscr{H}'$, thus establishing $\mathscr{H} = \mathscr{H}'$. Let a be an element of $\mathbb{R}^+$. Since $\mathfrak{N}'$ is homogeneous, let η in $\mathscr{H}'$ be such that $\theta(a) = \eta(a)$. Then, since $\mathfrak{N}$ is 1-point unique, $\theta = \eta$. Thus

$$\mathscr{H} = \mathscr{H}'.$$

Let γ be an arbitrary element of $\mathscr{S}$. Then it easily follows from $\mathscr{S}$ being a scale of isomorphisms that an automorphism α in $\mathscr{H}$ can be found such that

$$\gamma = \alpha * \varphi.$$

However, since for each t in X

$$\begin{aligned} \gamma'(\Psi(t)) &= \gamma(t) \\ &= \alpha * \varphi(t) \\ &= \alpha * \varphi'(\Psi(t)) \end{aligned}$$

and since $\alpha \in \mathscr{H} = \mathscr{H}'$ and $\varphi' \in \mathscr{T}$, it follows that $\gamma' \in \mathscr{T}$.

Let δ be an arbitrary element of $\mathscr{T}$. Then it easily follows from $\mathscr{T}$ being a scale of isomorphisms onto $\mathfrak{N}'$ that an automorphism β in $\mathscr{H}'$ can be found such that

$$\delta = \beta * \varphi'.$$

Since $\mathscr{H} = \mathscr{H}'$, β is also an automorphism of $\mathfrak{N}$. Therefore, $\beta * \varphi$ is in $\mathscr{S}$. Since for all t in X,

$$\delta(t) = \beta * \varphi'(\Psi(t)) = \beta * \varphi(t) = (\beta * \varphi)'(\Psi(t)),$$

$\delta = (\beta * \varphi)'$.

2. By an analogous argument, Statement 2 follows.

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Received: April 27, 1992