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STRATEGIC AND BEHAVIORAL DECOMPOSITION OF GAMES

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Abstract. $k_1 \times k_2 \times \cdots \times k_n$ games are uniquely decomposed into their strategic and behavioral parts. The strategic part contains all information needed to determine a specified strategic outcome (e.g., Nash, Quantal Response Equilibria (QRE)); the behavioral portion is what motivates cooperative approaches such as tit-for-tat, side-payments, etc. The decomposition reduces complexity; e.g., the space of 2×2 games is reduced from $\mathbb{R}^8$ to points in a three-dimensional cube. Applications include characterizing all 2×2 games, finding which Nash (or QRE) structures accompany certain classes of $k_1 \times \cdots \times k_n$ games, and how results concerning two and three player tit-for-tat strategies differ.

1. Introduction

In analyzing cooperative and competitive interactions, games can range from the classic single-shot, to repeated, evolutionary, quantum games, bargaining solutions, etc. Central to all approaches is the "initial game" identifying what is being analyzed. Consequential to these procedures is the game's basic structure, which is developed here.

The surprising complexity of spaces of games has hindered previous efforts to find a fully developed structure. Even the simplest n-person game, where each player has only two strategies, requires $n2^n$ values. As such, each game can be identified with a point in the Euclidean space $\mathbb{R}^{n2^n}$, and each $\mathbb{R}^{n2^n}$ point defines a game. Thus the space of 2×2 games is the eight-dimensional $\mathbb{R}^8$ and the space of $2 \times 2 \times 2$ games is $\mathbb{R}^{24}$.

This intricacy is reflected by a partial characterization of the simplest 2×2 games developed by Robinson and Goforth [11]. By identifying 144 central 2×2 ordinal games, they partitioned $\mathbb{R}^8$ into 144 regions. Hopkins [3] captured the inherent complexity of this space by reducing portions of this analysis to five-dimensional manifolds; another method leads to a torus with genus 37 (think of this as the number of holes). The natural objective of further reducing the dimension and complexity of a space of games is carried out here.

Candogan, Menache, Ozdaglar, and Parrilo [1] developed an approach emphasizing a game's "Nash strategic flow." This interesting decomposition arises from translating a game into a flow on a graph, and using properties of the flow to characterize games, which is of particular interest when studying potential games. Aspects of what they find are captured by our coordinate system approach; elsewhere we plan to use our results to describe and extend a flow component analysis.

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Among the various ways to decompose games (e.g., Jones [5]), a commonly used method separates a game into its cooperative (identical play) and zero sum components such as

$$\begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4\\ \hline 4 & -4 & 2 & 0\\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 6 & 6 & 2 & 2\\ \hline 0 & 0 & 1 & 1\\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 0 & 0 & -2 & 2\\ \hline 4 & -4 & 1 & -1\\ \hline\end{array} \tag{1}$$

Kalai and Kalai [7] use appropriate terms from each component to develop their thought-provoking "coco" (cooperative-competitive) bargaining solution. But decompositions can alter a game's original strategic structure. In Eq. 1, for instance, the original game has one mixed and two pure (TL, BR) (i.e., top-left, bottom-right) Nash strategies, but each component on the right has a single dominant strategy and neither has a mixed strategy. Our approach always preserves the strategic form.

More specifically, any $k_1 \times k_2 \times \ldots \times k_n$ game is separated into the portion that captures all strategic aspects (usually Nash) and the part that affects other behavioral/cooperative actions. This reduces the space of 2×2 games from eight dimensions to where information needed for Nash strategic aspects is given by points in a square; all aspects are captured by points in a three-dimensional cube. Similarly, $2 \times 2 \times 2$ games are reduced from 24 dimensions to where strategic information is in a nine-dimensional cube; all aspects are in a 17-dimensional cube. For n-player, two-strategy games, the analysis is reduced by $2n+1$ dimensions.

$$\mathcal{G}_1 = \begin{array}{|cc|cc|}\hline 0 & 0 & 4 & 2\\ \hline 2 & 4 & 6 & 6\\ \hline\end{array}, \quad \mathcal{G}_2 = \begin{array}{|cc|cc|}\hline 4 & 4 & -2 & 6\\ \hline 6 & -2 & 0 & 0\\ \hline\end{array}, \quad \mathcal{G}_3 = \begin{array}{|cc|cc|}\hline 8 & 0 & 0 & 2\\ \hline 10 & 2 & 2 & 4\\ \hline\end{array}. \tag{2}$$

The value of knowing a game's strategic information is obvious; to indicate the importance of the behavioral terms, all three Eq. 2 games share the same BR Nash equilibrium, but each is analyzed in a different way. The $\mathcal{G}_1$ game is uncontroversial because the BR Nash strategy is the Pareto superior outcome. Game $\mathcal{G}_2$ is a Prisoner's Dilemma where the BR Nash equilibrium is Pareto inferior to the TL outcome. Game $\mathcal{G}_3$ is more ambiguous; if the entries represent money, then a way to obtain the superior total of the BL outcome would involve side payments to induce the column player to play L. These differences are caused by the behavioral component in our decomposition.

Our method, then, identifies those features that can change how a game is analyzed. To do so, an n player $k_1 \times k_2 \times \cdots \times k_n$ game is uniquely divided into three parts: the first portion identifies strategic interests, the second component is what influences all cooperative behavioral reactions, and the third "kernel" portion merely adds the same value to each of a player's entries. All three Eq. 2 games are Nash equivalent, so all differences in the analysis of, or behaviors elicited by these games are strictly due to their behavioral terms. With the Prisoner's Dilemma ($\mathcal{G}_2$), then, the behavioral portion is what motivates seeking ways to achieve the TL outcome.

The decomposition also identifies large classes of games with specified strategic properties. A desired modeling, for instance, may require a certain Nash structure; e.g., perhaps two competing pure strategies. As developed in Sect. 5.2, *all possible games* with a desired strategic feature can be identified with our decomposition. This permits the design of experiments (currently being carried out with paid subjects) to determine in what ways a game's strategic and behavioral portions influence the selected strategies. Because there exists a continuum of games, where each has an identical strategic component but the behavioral terms can differ as widely as desired, the behavioral components become new variables for analyzing games and designing experiments.

The approach is introduced (Sect. 2) with 2×2 games and generalized to $k_1 \times k_2 \times \cdots \times k_n$ games. The decomposition is mathematically simple; the uniqueness proof (Sect. 7) uses elementary properties of symmetry groups and representation theory. Applications of the decomposition, such as showing how standard tit-for-tat results for two-person Prisoner's Dilemma games can change with more players, are in Sect. 3. In Sect. 4, the decomposition is used to reduce the complexity of the space of games. The positioning of standard games within this structure is described in Sect. 5. Proofs not following statements are in Sect. 7.

2. Decomposition

Start by adopting a solution concept; e.g., Nash equilibria. All games with an identical solution structure are collected into the same set with the following definition (which is later refined).

Definition 1. *(Preliminary) With a given solution concept SC, games $\mathcal{G}_i$ and $\mathcal{G}_j$ satisfy the binary relationship $\sim_{SC}$ if they have an identical SC structure.*

To illustrate Def. 1 with Nash equilibria and its binary relationship $\sim_N$, both

$$(3) \qquad \mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array}, \quad \mathcal{G}_5 = \begin{array}{|cc|cc|}\hline 2 & 11 & 8 & 9 \\ \hline 0 & 0 & 10 & 4 \\ \hline\end{array}.$$

have the same pure strategies of TL, BR, and the mixed strategies of $(\frac{2}{3}, \frac{1}{3})$ for player one (row player), and $(\frac{1}{2}, \frac{1}{2})$ for player two, so $\mathcal{G}_4 \sim_N \mathcal{G}_5$. From the perspective of Nash equilibria, these games are indistinguishable. ($\mathcal{G}_4$ was used in Eq. 1.)

We also consider Quantal Response Equilibria (QRE) developed by McKelvey and Palfrey [9]. While the motivation and details for QRE are carefully described in [9], QRE can be treated as modeling limitations (e.g., errors) of players in computing best responses: shortcomings that cause deviations from Nash equilibrium play. The "logit" QRE systems analyzed here use a particular way (see Sect. 3.4) to model the players' limitations; the level of ability is captured by a parameter $\lambda \in (0, \infty)$ where larger λ values indicate more adept players. One must anticipate that λ plays a central role in the $\sim_{SC}$ relationship, which is reflected by our $\sim_{QRE,\lambda}$ notation. A natural question (answered below) is whether a $\lambda > 0$ value exists (i.e., a limited level of ability to handle best response computations) where $\mathcal{G}_4 \sim_{QRE,\lambda} \mathcal{G}_5$, or, more dramatically, where $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$. If a $\lambda > 0$ does not exist, it would demonstrate that QRE distinguishes between players' abilities to respond to different kinds of games. But should such a λ exist, it would mean that, from the perspective of QRE and the specified λ value, the uncontroversial $\mathcal{G}_1$ would be indistinguishable from the Prisoner's Dilemma.

For all considered solution concepts, $\sim_{SC}$ is an equivalence relationship. (Namely, $\mathcal{G}_i \sim_{SC} \mathcal{G}_i$. If $\mathcal{G}_i \sim_{SC} \mathcal{G}_j$, then $\mathcal{G}_j \sim_{SC} \mathcal{G}_i$. If $\mathcal{G}_i \sim_{SC} \mathcal{G}_j$ and $\mathcal{G}_j \sim_{SC} \mathcal{G}_k$, then $\mathcal{G}_i \sim_{SC} \mathcal{G}_k$.) Thus $\sim_{SC}$ partitions the space of games into equivalence classes. Each $\sim_N$ equivalence class, for instance, consists of all games with the same Nash equilibria strategies; this means that $\mathcal{G}_4$ and $\mathcal{G}_5$ are but two of an infinite number of entries in their $\sim_N$ class.

To analyze the $\sim_{SC}$ equivalence classes, the goals are:

(1) Characterize each equivalence class with an essential game theoretic aspect that defines all games in the class. By characterizing an equivalence class, this trait *cannot* be satisfied by a game from any other class.
(2) Determine how games within a class differ.
(3) Determine how different solution concepts partition the space of games in different ways. Differences between $\sim_{QRE,\lambda}$ equivalences sets with different λ values, for instance, would identify what structural variations of games explain different QRE predictions. All dissimilarities and similarities among $\sim_N$ and different $\sim_{QRE,\lambda}$ equivalence classes are fully identified.

2.1. **The second issue and a decomposition.** As shown next, $\mathcal{G}_1 \sim_N \mathcal{G}_2 \sim_N \mathcal{G}_3$ (with Def. 1 and a later refined definition). That these diverse games belong to the same Nash equivalence class underscores the importance of the second question, which is to understand how games within a class differ. Answers involve the behavioral term.

As developed starting in Sect. 2.2, each game $\mathcal{G}$ has a *unique decomposition* into the $\mathcal{G}^N$ part that determines the Nash strategies, the $\mathcal{G}^B$ part that captures other behavioral aspects, and the $\mathcal{G}^K$ part that just scales the entries. The unique division of the Eq. 2 games, given in the $\mathcal{G} = \mathcal{G}^N + \mathcal{G}^B + \mathcal{G}^K$ order, follows:

$$\mathcal{G}_1 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline -2 & -2 & 2 & -2 \\ \hline -2 & 2 & 2 & 2 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline\end{array}, \tag{4}$$

$$\mathcal{G}_2 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 \\ \hline\end{array}. \tag{5}$$

$$\mathcal{G}_3 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 4 & -1 & -4 & -1 \\ \hline 4 & 1 & -4 & 1 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 5 & 2 & 5 & 2 \\ \hline 5 & 2 & 5 & 2 \\ \hline\end{array}. \tag{6}$$

These three games share the same $\mathcal{G}^N$ Nash strategic form, which completely determines each game's Nash behavior; the games are "best response equivalent." That $\mathcal{G}^N$ fully extracts all Nash strategic elements of a game comes from the form of the $\mathcal{G}^K$ "kernel" component (which adds the same value to each of a player's entries) and the "behavioral" component $\mathcal{G}^B$ that admits no differences in rows for the row player, nor in the columns for the column player. Thus, $\mathcal{G}^B$ outcomes are determined by coordination of players' actions, such as behavioral aspects ranging from altruism, cooperation, tit-for-tat, to "You scratch my back, I'll scratch yours," etc.

A key $\mathcal{G}^B$ feature is its Pareto superior entry, e.g., the dominant $\mathcal{G}_1^N$ and $\mathcal{G}_1^B$ entries are both located at BR. But for $\mathcal{G}_2$, the BR Nash location for $\mathcal{G}_2^N$ is the $\mathcal{G}_2^B$ Pareto inferior position; the $\mathcal{G}_2^B$ Pareto superior TL position is the $\mathcal{G}_2^N$ Nash inferior location (*both* players wish to move). This conflict between $\mathcal{G}_2$ components—strategic and cooperative—generates the Prisoner's Dilemma. For $\mathcal{G}_3$, the BL position of the $\mathcal{G}_3^B$ Pareto superior term sways interest from the BR Nash point to the BL choice.

2.2. **The Nash decomposition.** Using $\mathcal{G}_4$ to introduce the decomposition, if the column player's L-R strategy is given by probabilities $(q, 1-q)$, $q \in [0,1]$, the row player's preferences between T and B are determined by their expected values; e.g., the row player's expected outcome by playing T is $EV(T) = 6q + 0(1-q)$ and B is $EV(B) = 4q + 2(1-q)$. The difference (in a "best response" analysis) is

$$EV(T) - EV(B) = [6q + 0(1-q)] - [4q + 2(1-q)] = [6-4]q + [0-2](1-q). \tag{7}$$

With a positive Eq. 7 value, the player should play T; with a negative value, the player should play B.

The critical terms are the bracket values in the final Eq. 7 equality. Rather than 6 and 4, their difference of 2 determines the q coefficient; rather than 0 and 2, their -2 difference determines the $(1-q)$ coefficient. Thus the strategic analysis remains unchanged by replacing each term in each pair by how it differs from the pair's average; e.g., replacing 6 with $6 - \frac{6+4}{2} = 1$ and 4 with $4 - \frac{6+4}{2} = -1$ (in Eq. 8) does not change the q coefficient in Eq. 7. Similarly, replacing 0 with $0 - \frac{0+2}{2} = -1$ and 2 with $2 - \frac{0+2}{2} = 1$ does not affect the $(1-q)$ coefficient. An important but not obvious fact is that terms *must be replaced* by how they deviate from the pair's average to separate a game's strategic and behavioral aspects: Other choices need not extract all strategic information (Sect. 7) because (as with Eq. 1) a remaining component could include Nash information.

The same argument holds when comparing the second player's expected values for L and R in response to the row player's mixed strategies of p and $(1-p)$ for T and B. This defines the game $\mathcal{G}_4^N$, which has the same Nash structure as $\mathcal{G}_4$.

$$\mathcal{G}_4^N = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -2 & 1 & 2 \\ \hline\end{array} \tag{8}$$

Because $\mathcal{G}_4^N$ is defined by differences from averages, these averages define the $\mathcal{G}_4 - \mathcal{G}_4^N$ entries; e.g., the first term in the upper-left corner of $\mathcal{G}_4 - \mathcal{G}_4^N$ is $6 - 1 = 6 - [6 - \frac{6+4}{2}] = \frac{6+4}{2}$, or the average of the 6 and 4 entries. This leads to the decomposition

$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array} = \mathcal{G}_4^N + \begin{array}{|cc|cc|}\hline 5 & 5 & 1 & 5 \\ \hline 5 & -2 & 1 & -2 \\ \hline\end{array}. \tag{9}$$

The second term is further decomposed by extracting deviations from the average of each player's entries. Let κ_j be the j^{th} player's average of $\mathcal{G}_4$ entries; e.g., $\kappa_1 = (6+4+0+2)/4 = 3$, and $\kappa_2 = 1.5$. Player one's column entries in the last Eq. 9 bimatrix are 5 and 1 and $\kappa_1 = 3$, so replace 5 with $5 - 3 = 2$ and 1 with $1 - 3 = -2$. Doing the same for player two with $\kappa_2 = 1.5$ leads to the following $\mathcal{G}_4$ decomposition (that differs from Eq. 1):

$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -2 & 1 & 2 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 2 & 3.5 & -2 & 3.5 \\ \hline 2 & -3.5 & -2 & -3.5 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 3 & 1.5 & 3 & 1.5 \\ \hline 3 & 1.5 & 3 & 1.5 \\ \hline\end{array}. \tag{10}$$

The same approach holds for any $k_1 \times \cdots \times k_n$ game: Replace values at each stage by how they differ from appropriate "averages." For the first stage, a pure strategy selected for each of the other players defines an array of payoffs for the j^{th} player: To define this player's $\mathcal{G}^N$ entries, replace each term in this array by how it differs from the array's average.

Entries in $\mathcal{G} - \mathcal{G}^N$ are the averages of these arrays. Let κ_j be the average of the j^{th} player's $\mathcal{G} - \mathcal{G}^N$ entries (which is the average of the j^{th} player's $\mathcal{G}$ entries) to define $\mathcal{G}^K$. To define $\mathcal{G}^B$, replace the j^{th} player's $\mathcal{G} - \mathcal{G}^N$ entries by how they differ from κ_j.

2.3. **The general case.** By using "differences from averages," an equivalent approach is to start with the various averages. To illustrate, consider the 4×3 game

$$\mathcal{G}_6 = \begin{array}{|rr|rr|rr|}\hline 0 & -1 & 1 & -4 & 5 & -4 \\ \hline -2 & 2 & 2 & 4 & 5 & 3 \\ \hline -5 & -4 & -1 & -5 & 8 & -3 \\ \hline -5 & 1 & -2 & -1 & 6 & 0 \\ \hline \end{array} \tag{11}$$

The positive payoffs for both players are the $(5, 3)$ and $(2, 4)$ cells.

Each player's $\mathcal{G}_6^K$ entry is the average of the player's $\mathcal{G}_6$ entries. The sum of the row player's entries is 12, with an average of 1. The column player's average is -1, so

$$\mathcal{G}_6^K = \begin{array}{|rr|rr|rr|}\hline 1 & -1 & 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 & 1 & -1 \\ \hline \end{array}. \tag{12}$$

To compute the behavioral $\mathcal{G}_6^B$, replace the $\mathcal{G}_6 - \mathcal{G}_6^K$ arrays with their averages. Namely, a pure strategy selected for the other player identifies a $\mathcal{G}_6 - \mathcal{G}_6^K$ array for the j^{th} player; replace each term in this array with its average. With $\mathcal{G}_6$ and the column player's far left choice, the row player's $\mathcal{G}_6 - \mathcal{G}_6^K$ four entries, from top down, are $0-1$, $-2-1$, $-5-1$, $-5-1$ with an average value of -4. Doing so for all columns and rows leads to

$$\mathcal{G}_6^B = \begin{array}{|rr|rr|rr|}\hline -4 & -2 & -1 & -2 & 5 & -2 \\ \hline -4 & 4 & -1 & 4 & 5 & 4 \\ \hline -4 & -3 & -1 & -3 & 5 & -3 \\ \hline -4 & 1 & -1 & 1 & 5 & 1 \\ \hline \end{array}. \tag{13}$$

By construction, all rows are identical for the row player; all columns are identical for the column player.

The strategic term, $\mathcal{G}_6^N$ is what remains; e.g., $\mathcal{G}_6^N = \mathcal{G}_6 - [\mathcal{G}_6^B + \mathcal{G}_6^K]$, or

$$\mathcal{G}_6^N = \begin{array}{|rr|rr|rr|}\hline 3 & 2 & 1 & -1 & -1 & -1 \\ \hline 1 & -1 & 2 & 1 & -1 & 0 \\ \hline -2 & 0 & -1 & -1 & 2 & 1 \\ \hline -2 & 1 & -2 & -1 & 0 & 0 \\ \hline \end{array}. \tag{14}$$

A similar computation shows that $\mathcal{G}_6 \sim_N \mathcal{G}_7$ where $\mathcal{G}_7$ is the seemingly dissimilar

$$\mathcal{G}_7 = \begin{array}{|rr|rr|rr|}\hline 0 & -1 & 8 & -4 & -5 & -4 \\ \hline -2 & -3 & 9 & -1 & -5 & -2 \\ \hline -5 & -1 & 6 & -2 & -2 & 0 \\ \hline -5 & 7 & 5 & 5 & -4 & 6 \\ \hline \end{array} = \mathcal{G}_6^N + \begin{array}{|rr|rr|rr|}\hline -3 & -3 & 7 & -3 & -4 & -3 \\ \hline -3 & -2 & 7 & -2 & -4 & -2 \\ \hline -3 & -1 & 7 & -1 & -4 & -1 \\ \hline -3 & 6 & 7 & 6 & -4 & 6 \\ \hline \end{array} \tag{15}$$

Although $\mathcal{G}_6 \sim_N \mathcal{G}_7$, the only jointly desired $\mathcal{G}_7$ cell of $(5, 5)$ has a different location from the two $\mathcal{G}_6$ cells with positive payoffs. By being Nash equivalent, all differences between $\mathcal{G}_6$ and $\mathcal{G}_7$ are strictly determined by differences in their behavioral components.

2.4. **A $2 \times 2 \times 2$ game.** To illustrate with a multiplayer game, consider the following $2 \times 2 \times 2$ game where the row, column and front-back (F, Ba) payoffs are

$$\mathcal{G}_8 : \text{ Front } = \begin{array}{|rrr|rrr|}\hline 7 & 7 & 7 & 5 & 11 & 5 \\ \hline 11 & 5 & 4 & 9 & 9 & 0 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|rrr|rrr|}\hline 4 & 4 & 11 & 0 & 8 & 9 \\ \hline 8 & 0 & 8 & 4 & 4 & 4 \\ \hline \end{array} \tag{16}$$

Each player's average entry is 6, so $\mathcal{G}_8^K$ has $(6, 6, 6)$ in each of its eight cells. After subtracting six from each $\mathcal{G}_8$ entry, the row player's average entry value for the LF strategies is $\frac{1}{2}[(7-6)+(11-6)] = 3$, which defines each cell's first entry in the LF column of $\mathcal{G}_8^B$. In this manner, the behavioral component is

$$\mathcal{G}_8^B : \text{ Front } = \begin{array}{|rrr|rrr|}\hline 3 & 3 & 3 & 1 & 3 & 1 \\ \hline 3 & 1 & 0 & 1 & 1 & -4 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|rrr|rrr|}\hline 0 & 0 & 3 & -4 & 0 & 1 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline \end{array} \tag{17}$$

What remains is the Nash strategic component $\mathcal{G}_8^N = \mathcal{G}_8 - (\mathcal{G}_8^B + \mathcal{G}_8^K)$

$$\mathcal{G}_8^N : \text{ Front } = \begin{array}{|rrr|rrr|}\hline -2 & -2 & -2 & -2 & 2 & -2 \\ \hline 2 & -2 & -2 & 2 & 2 & -2 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|rrr|rrr|}\hline -2 & -2 & 2 & -2 & 2 & 2 \\ \hline 2 & -2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \tag{18}$$

We now refine Def. 1 from its "best-response equivalence" description to a form that emphasizes the precise structure of the Nash information.

Definition 2. *Two $k_1 \times \cdots \times k_n$ games are said to be Nash equivalent. "$\sim_N$," if and only if they have the same $\mathcal{G}^N$ component.*

The following identical play (Eq. 19) and zero sum (Eq. 20) games have the same Nash best response (i.e., if player two plays L, one plays T; if two plays R, one plays B; if two plays $q = .5$, one is indifferent; whatever one plays, two plays R) so they satisfy Def. 1. But the decomposition shows that they are not Nash informationally equivalent (Def. 2).

$$\mathcal{G}_{IP} = \begin{array}{|rr|rr|}\hline -2 & -2 & 2 & 2 \\ \hline -4 & -4 & 4 & 4 \\ \hline \end{array} = \begin{array}{|rr|rr|}\hline 1 & -2 & -1 & 2 \\ \hline -1 & -4 & 1 & 4 \\ \hline \end{array} + \begin{array}{|rr|rr|}\hline -3 & 0 & 3 & 0 \\ \hline -3 & 0 & 3 & 0 \\ \hline \end{array} \tag{19}$$

$$\mathcal{G}_{ZS} = \begin{array}{|rr|rr|}\hline 3 & -3 & -3 & 3 \\ \hline 1 & -1 & -1 & 1 \\ \hline \end{array} = \begin{array}{|rr|rr|}\hline 1 & -3 & -1 & 3 \\ \hline -1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|rr|rr|}\hline 2 & 0 & -2 & 0 \\ \hline 2 & 0 & -2 & 0 \\ \hline \end{array} \tag{20}$$

As proved in Sect. 4.3, only in special cases can zero-sum and identical play games be in the same $\mathcal{G}^N$ equivalence class.

2.5. **Notation and game structure.** Notation for a $k_1 \times \cdots \times k_n$ game $\mathcal{G}$ is given for the j^{th} agent, $j = 1, \dots, n$, when the pure strategies for the other $(n-1)$ agents is given by $\mathbf{s}$:

- κ_j is the agent's common $\mathcal{G}^K$ value.
- $\beta_{j;\mathbf{s}}$ is the agent's common $\mathcal{G}^B$ entry for the given $\mathbf{s}$.
- $\eta_{i,j;\mathbf{s}}$, $i = 1, \dots, k_j$, is the j^{th} agent's i^{th} strategy in $\mathcal{G}^N$ for the given $\mathbf{s}$.
- $g_{i,j;\mathbf{s}}$. $i = 1, \dots, k_j$, is the $\mathcal{G}$ entry.

With this notation, the structure of the space of games can be described.

Theorem 1. *For any $k_1 \times \cdots \times k_n$ game:*

(1) *For each i, j and $\mathbf{s}$,*

$$g_{i,j;\mathbf{s}} = \eta_{i,j;\mathbf{s}} + \beta_{j;\mathbf{s}} + \kappa_j. \tag{21}$$

For each j and $\mathbf{s}$,

$$\sum_{i=1}^{k_j} \eta_{i,j;\mathbf{s}} = 0. \tag{22}$$

For each j,

$$\sum_{\mathbf{s}} \beta_{j;\mathbf{s}} = 0, \tag{23}$$

where the summation is over all choices of $\mathbf{s}$.

(2) *Matrices $\mathcal{G}^B$, $\mathcal{G}^K$, and $\mathcal{G}^B + \mathcal{G}^K$ have no Nash information about the game $\mathcal{G}$. All Nash information is in $\mathcal{G}^N$, and $\mathcal{G} \sim_N \mathcal{G}^N$.*

(3) *A necessary condition for $\mathcal{G}$ to have a pure Nash equilibrium is if the associated $\mathcal{G}^N$ cell has all positive entries. A necessary and sufficient condition is if for each j, this entry is the largest $\eta_{i,j;\mathbf{s}}$ for the given $\mathbf{s}$.*

(4) *If each player has a non-zero $\mathcal{G}^B$ entry, then for $n = 2$, matrix G^B has a Pareto superior cell with all positive entries, and a Pareto inferior cell with all negative entries. This need not be the case for $n \geq 3$.*

Part 3 of Thm. 1 describes a way to find all pure Nash equilibria: Find $\mathcal{G}^N$, which is a series of elementary arithmetic steps ($k_1 k_2 \dots k_n(n + \frac{1}{k_1} + \dots \frac{1}{k_n})$ of them); then identify those cells with all positive entries. Illustrating with the $\mathcal{G}_6^N = \mathcal{G}_7^N$ structure, $\mathcal{G}_6$ and $\mathcal{G}_7$ have three pure Nash equilibria along the diagonal: (TL), (Second from top, Middle), (Second from bottom, Right). Game $\mathcal{G}_8$ has only the (BRBa) equilibrium.

Similarly, to determine the pure Nash equilibria of

$$\mathcal{G}: \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 6 & 7 & 3 & 2 & 5 & 5 \\ \hline 4 & 3 & 9 & 4 & 5 & 2 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 6 & 7 & 5 & -8 & 9 & 7 \\ \hline 8 & -7 & 7 & -6 & -5 & 4 \\ \hline \end{array}$$

compute $\mathcal{G}^N$. (Only $\mathcal{G}^N$ is needed, which requires $2^3(3 + \frac{3}{2}) = 36$ operations to determine how appropriate terms differ from their average.) It follows immediately from

$$\mathcal{G}^N: \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 1 & 1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 & 1 & -1 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline -1 & -1 & 1 & -1 & 1 & 1 \\ \hline 1 & -1 & -1 & 1 & 1 & 1 \\ \hline \end{array}$$

that the only Nash equilibrium is BRBa. The unappealing nature of the BRBa $\mathcal{G}$ outcome (e.g., when compared with TLBa) reflects the strong role played by $\mathcal{G}^B$.

This structure makes it straightforward to determine the number of strict pure Nash equilibria admitted by games and to create examples (construct appropriate $\mathcal{G}^N$ choices). The "strict" modifier refers to the generic setting where, for each $\mathbf{s}$, the j^{th} agent has at most one option for a Nash point; this always happens if an agent's $\mathcal{G}^N$ payoffs differ.

Corollary 1. *Consider the space of $k_1 \times k_2 \times \ldots \times k_n$ games where, without loss of generality, $k_1 \geq k_2 \geq \ldots \geq k_n$. For any integer α satisfying $0 \leq \alpha \leq k_2 k_3 \ldots k_n$, there is a game with precisely α strict pure Nash equilibria. No game can have $\alpha > k_2 k_3 \ldots k_n$ equilibria.*

So, $4 \times 5 \times 6 \times 7$ games allow up to $6 \times 5 \times 4 = 120$ pure Nash points. The proofs of Cor. 1 and Thm. 1, parts 2 and 3, are in Sect. 7.

Proof of Thm. 1, Parts 1, 4: Equations 21–23 follow from the construction; Eq. 21 is obvious. Because κ_j in $\mathcal{G}^K$ is the average of the j^{th} agent's $\mathcal{G}$ entries, the sum of the agent's entries in $[\mathcal{G} - \mathcal{G}^K]$ is zero. This sum is the same if, for each $\mathbf{s}$, each entry in the j^{th} agent's $[\mathcal{G} - \mathcal{G}^K]$ array is replaced with its average, which is $\sum_{\mathbf{s}} \beta_{j;\mathbf{s}}$: Eq. 23 now follows. Similarly, $\beta_{j;\mathbf{s}}$ is the average of the j^{th} agent's $[\mathcal{G} - \mathcal{G}^K]$ entries for a specified $\mathbf{s}$, so the sum of these entries in $\mathcal{G} - [\mathcal{G}^B + \mathcal{G}^K]$ equals zero. This sum is $\sum_{i=1}^{k_j} \eta_{i,j;\mathbf{s}}$, so Eq. 22 holds.

To prove (4) for $n = 2$, the row player's largest $\beta_{1,\mathbf{s}}$ value identifies a specific column, or, if $k_1 \geq 3$, possibly more than one column. Similarly, the column player's largest $\beta_{2,\mathbf{s}}$ value identifies specific rows. The intersection of these rows and columns identify the $\mathcal{G}^B$ cells with the largest values (which, from Eq. 23, must be positive) for each player. These are the Pareto superior term(s). Similarly, there is a $\mathcal{G}^B$ column with the row player's smallest $\beta_{1,\mathbf{s}}$ value, and a $\mathcal{G}^B$ row with the column player's smallest $\beta_{2,\mathbf{s}}$ value. The intersection of this row(s) and column(s) define the Pareto inferior cell(s); each entry in the cell is negative (Eq. 23). With 2×2 games, the Pareto superior and inferior cells are diametrically located.

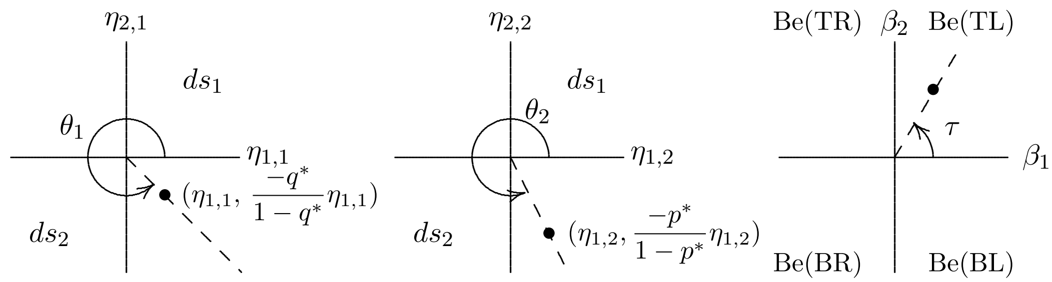
As a specified column and row corresponds to lines that are parallel to different $\mathbb{R}^2$ axes, the proof reflects the geometry that two non-parallel straight lines in $\mathbb{R}^2$ must cross. But n nonparallel straight lines in $\mathbb{R}^n$, $n \geq 3$, need not meet, so $\mathcal{G}^B$ need not have dominant nor inferior entries. What simplifies creating examples is that the $\beta_{j,\mathbf{s}}$ entries are free to be selected subject to Eq. 23. So, adhere to this geometric property of non-intersecting lines as in the following $2 \times 2 \times 2$ game: (For fixed F or Ba, TL is the Pareto superior choice for the first two players, and BR is their Pareto inferior choice. The third player's entries are selected to ensure there is not a superior nor inferior cell.)

$$\mathcal{G}^B: \text{ Front } = \begin{array}{|ccc|ccc|}\hline 2 & 3 & -2 & 1 & 3 & -1 \\ \hline 2 & -1 & 2 & 1 & -1 & 1 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 3 & 2 & -2 & -6 & 2 & -1 \\ \hline 3 & -4 & 2 & -6 & -4 & 1 \\ \hline \end{array}$$

2.6. **A geometric representation for 2×2 games.** With 2×2 games, the notation can be simplified (as in Eq. 24) by dropping the $\mathbf{s}$ parameter. Let $\eta_{i,1}$ be the row player's $\mathcal{G}^N$ entry for the first row, i^{th} column, $i = 1, 2$; then (Eq. 22) $-\eta_{i,1}$ is the player's $\mathcal{G}^N$ entry for the second row, i^{th} column, $i = 1, 2$. Similarly, let $\eta_{i,2}$ be player two's $\mathcal{G}^N$ entry for the first column, i^{th} row, $i = 1, 2$; $-\eta_{i,2}$ is player two's $\mathcal{G}^N$ entry for the second column, i^{th} row, $i = 1, 2$. Similarly, the sum of the two $\beta_{j;\mathbf{s}}$ entires equals zero (Eq. 23), so one is the negative of the other. Let β_j be the j^{th} player's entry in the TL cell, which requires $-\beta_j$ to be the BR entry.

$$\mathcal{G}^N = \begin{array}{|cc|cc|}\hline \eta_{1,1} & \eta_{1,2} & \eta_{2,1} & -\eta_{1,2} \\ \hline -\eta_{1,1} & \eta_{2,2} & -\eta_{2,1} & -\eta_{2,2} \\ \hline \end{array}, \quad \mathcal{G}^B = \begin{array}{|cc|cc|}\hline \beta_1 & \beta_2 & -\beta_1 & \beta_2 \\ \hline \beta_1 & -\beta_2 & -\beta_1 & -\beta_2 \\ \hline \end{array} \tag{24}$$

For a further reduction, plot $\boldsymbol{\eta_j} = (\eta_{1,j}, \eta_{2,j})$, $j = 1, 2$, as points in $\mathbb{R}^2$. The Fig. 1 a, b choices of $\boldsymbol{\eta}_j = (\eta_{1,j}, \eta_{2,j})$ are from $\mathcal{G}_4^N$ with its Nash $p^* = \frac{2}{3}$ and $q^* = \frac{1}{2}$ mixed strategies. Next plot $\boldsymbol{\beta} = (\beta_1, \beta_2)$; the $\boldsymbol{\beta} = (2, 3.5)$ choice in Fig. 1c is from $\mathcal{G}_4^B$.



a. Player 1; strategic **b.** Player 2; strategic **c.** Game $\mathcal{G}_4^B$

Figure 1. Strategic/behavior decomposition of the games $\mathcal{G}_4, \mathcal{G}_5$

All entries on a ray (Figs. 1a,b) emanating from the origin and passing through a specified $\boldsymbol{\eta}_j$ have the same p and q values. (In Eq. 7 terms, multiplying the two bracket values by the same positive scalar yields identical information about whether to play T or B.) Thus the four-dimensional $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ representation for $\mathcal{G}^N$ can be replaced with (θ_1, θ_2) points on the two-dimensional square $[0, 2\pi] \times [0, 2\pi]$; the (θ_1, θ_2) values identify the appropriate rays. (As angles 0 and 2π are identified, the square actually represents a torus T^2. In a significantly different manner, Jordan [6] also obtains a torus.) In Sect. 4, the structure of this square (torus) is developed to characterize the space of all 2×2 games.

Definition 3 collects the Def. 2 classes with the same Nash content. This Fig. 1 construction holds for all games (Sect. 5.4), which extends Def. 3 to all $k_1 \times \dots k_n$ games.

Definition 3. *Two 2×2 games are Nash-content equivalent if and only if their $\mathcal{G}^N$ components define the same θ_1, θ_2 values.*

The β values (Fig. 1c) determine the behavioral positioning (denoted by Be(location)) of the $\mathcal{G}^B$ Pareto superior point. So $\tau \in (0, \frac{\pi}{2})$ (first quadrant) is in Be(TL) indicating that the dominant $\mathcal{G}^B$ point is located at TL. Should the location of this point conflict with that of the Nash point, behavioral considerations in the game's analysis become important.

3. Using the decomposition

The following results indicate ways to use the decomposition.

3.1. Closeness of games. Figure 1 makes it possible to determine if two games are "close" with respect to appropriate conditions. Whether the following seemingly dissimilar games $\mathcal{G}_9$ and $\mathcal{G}_{10}$ are close, for instance, depends on the desired criterion.

$$\mathcal{G}_9 = \begin{array}{|cc|cc|}\hline 13 & -8 & -12 & -16 \\ \hline 7 & 11 & -8 & 13 \\ \hline\end{array}, \quad \mathcal{G}_{10} = \begin{array}{|cc|cc|}\hline 35 & 12 & -24 & 4 \\ \hline -27 & -9 & 16 & -7 \\ \hline\end{array} \tag{25}$$

If the criterion is in terms of Nash content (Def. 3), the answer depends on whether their (θ_1, θ_2) values, (Fig. 1a, b positions) are near each other; "close games" tend to have similar Nash strategies. (Exceptions, as developed in Sect. 5, are points on opposite sides of boundaries to be identified.) The $\mathcal{G}_j^N$ components and their θ_j values are:

$$\mathcal{G}_9^N = \begin{array}{|cc|cc|}\hline 3 & 4 & -2 & -4 \\ \hline -3 & -1 & 2 & 1 \\ \hline\end{array}, \quad \mathcal{G}_{10}^N = \begin{array}{|cc|cc|}\hline 31 & 4 & -20 & -4 \\ \hline -31 & -1 & 20 & 1 \\ \hline\end{array} \tag{26}$$

The closeness of the angles (and the games) becomes obvious by plotting rays passing through $(-\frac{2}{3}, 4)$ and $(-\frac{2}{3.1}, 4)$.

If the criterion involves behavioral aspects, as measured by $\mathcal{G}^B$ terms and τ values, $\mathcal{G}_9$ and $\mathcal{G}_{10}$ are "far apart." For $\mathcal{G}_9^B$, the $\beta_1 = 10, \beta_2 = -12$ values define $\tau = \arctan(-1.2)$ with a Be(BL) Pareto point. In contrast, for $\mathcal{G}_{10}^B$, the $\beta_1 = 4, \beta_2 = 8$ values define $\tau = \arctan(2)$ with the considerably different Be(TL) Pareto point. (This sense of closeness can differ significantly from that in Robinson and Gogorth [11].)

3.2. Strategic behavioral explanations. The decomposition of an asymmetric matching penny game from Goeree and Holt [2] (with $\kappa_1 = 120, \kappa_2 = 60$) is

$$\mathcal{G}_{11} = \begin{array}{|cc|cc|}\hline 320 & 40 & 40 & 80 \\ \hline 40 & 80 & 80 & 40 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 140 & -20 & -20 & 20 \\ \hline -140 & 20 & 20 & -20 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 60 & 0 & -60 & 0 \\ \hline 60 & 0 & -60 & 0 \\ \hline\end{array} + \mathcal{G}_{11}^K \tag{27}$$

with the mixed Nash equilibrium of $p = 0.50, q = 0.125$. The puzzle posed by Goeree and Holt is to explain is why $p = 0.5$ differs so strongly from their experimental data value of $p = 0.96$; the $q = 0.16$ experimental value is somewhat compatible with the Nash value.

A possible behavioral explanation uses that $\mathcal{G}_{11}^N$ captures the loss-gain strategic structure; player one can lose 140 by playing B, but at most 20, with a large possible reward, by playing T. As Luce [8] showed with empirical evidence, if a lottery has two expressions with the same expected outcome (which refines expected value comparisons) where the first is expressed in terms of gains and the second in terms of losses from a given amount, people tend to avoid choices expressed in terms of losses. This proclivity suggests a stronger behavioral tendency to play T than given by Nash predictions. If player two suspects this is the case, R is the appropriate strategy.

3.3. **Tit-for-Tat.** A game's behavioral part both motivates actions other than Nash strategies (Sect. 2.1) and introduces differences between two and multi-player conclusions. To demonstrate with $\mathcal{G}_2$, its decomposition (Eq. 5), with $\kappa_1 = \kappa_2 = 2$, is

$$\mathcal{G}_2 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline \end{array} + \mathcal{G}_2^K.$$

The Be(TL) position of the $\mathcal{G}_2^B$ Pareto superior point makes the TL outcome attractive. But TL conflicts with the Nash BR structure, so strategies to achieve TL must reflect the $\mathcal{G}_2^B$ structure by encouraging cooperation. To see how $\mathcal{G}_2^B$ terms surface in a standard analysis, let player one's tit-for-tat strategy be: play T on step $k = 1$; on step $k > 1$, play T if player two plays L on step $k - 1$, otherwise play B. The goal is to determine player two's best response with discount rate $\delta \in [0, 1)$.

As known, sufficiently large δ values encourage cooperation; what is not fully understood is how the lower bound, δ_0, reflects a tension between strategic and cooperative terms. To explore this issue, if player two cooperates by always playing L, the earnings, $E(L)$, are

$$E(L) = \sum_{k=1}^{\infty} -\delta^{k-1} + \sum_{k=1}^{\infty} 3\delta^{k-1} + \sum_{k=1}^{\infty} 2\delta^{k-1} = \frac{-1}{1-\delta} + [3 + \frac{3\delta}{1-\delta}] + \frac{2}{1-\delta}. \tag{28}$$

The first function on the right side represents losses by not playing the Nash $\mathcal{G}_2^N$ strategy; the second function reflects gains by cooperating to achieve $\mathcal{G}_2^B$ Pareto superior values. The third function plays no substantive role; it captures $\mathcal{G}_2^K$ terms.

In contrast, by always playing the Nash dominant R, the $E(R)$ earnings are

$$E(R) = \frac{1}{1-\delta} + [3 - \frac{3\delta}{1-\delta}] + \frac{2}{1-\delta}. \tag{29}$$

Cooperation requires $\frac{3\delta}{1-\delta} > \frac{1}{1-\delta}$ (from $E(L) > E(R)$) where the left term reflects $\mathcal{G}_2^B$ cooperative rewards and the right captures $\mathcal{G}_2^N$ gains. Cooperation requires $\delta > \delta_0 = \frac{1}{3}$.

Re-expressing this inequality (from a general PD game) identifies conditions on β_j, relative to the discount rate and Nash structure, to encourage cooperation via tit-for-tat.

Theorem 2. *In a 2×2 Prisoner's Dilemma game, where the dominant strategy is BR, tit-for-tat cooperation occurs for a δ value if the behavioral $\mathcal{G}^B$ components satisfy*

$$\beta_1 > \frac{|\eta_{2,1}| + |\eta_{1,1}|}{2} + \frac{1-\delta}{\delta}|\eta_{1,1}|, \quad \beta_2 > \frac{|\eta_{2,2}| + |\eta_{1,2}|}{2} + \frac{1-\delta}{\delta}|\eta_{1,2}| \tag{30}$$

So with $\mathcal{G}_2^N$ and $\delta = \frac{1}{2}$, cooperation with tit-for-tat requires $\beta_1, \beta_2 > 2$. Should β_2 fail to satisfy Eq. 30 but β_1 has a large value, side payments might sustain cooperation. (Analyzing a mixture of R and L strategies yields a similar relationship.)

Multiplayer PD games. A standard two-player result asserts that sufficiently large δ values establish cooperation (e.g., with tit-for-tat, or grim trigger, or ....); i.e., a δ_0 value is found whereby cooperation is ensured for all $\delta > \delta_0$. Computations leading to Eq. 30 show that this conclusion reflects the $\mathcal{G}^B$ structure (Thm. 1, part 4). But this structure does not extend to $n \geq 3$ players, so it is reasonable to anticipate differences.

In particular, *cooperation (e.g., with tit-for-tat, grim trigger, etc.) need not be ensured for all* $\delta > \delta_0$. To illustrate with the three-person PD game $\mathcal{G}_8$ (Sect. 2.4), TLF is preferred to the Nash dominant BRBa strategy. As a standard computation proves, $\delta > \frac{4}{7}$ supports TLF with the grim trigger strategy. But now a difference between two and multiplayer games arises: with a larger $\delta > \frac{4}{5}$, TRF, where player two *defects* to gain an advantage at the expense of the other players, can be sustained.

Experimenting with other $\mathcal{G}^B$ possibilities yields a surprising variety of new situations. For instance, replacing $\mathcal{G}_8^B$ with the following $\mathcal{G}^B$ (which does not have a dominant entry)

$$Front = \begin{array}{|ccc|ccc|}\hline 3 & 0 & 1 & 1 & 0 & 3 \\ \hline 3 & 2 & 0 & 1 & 2 & -4 \\ \hline\end{array}, \quad Back = \begin{array}{|ccc|ccc|}\hline 0 & 2 & 1 & -4 & 2 & 3 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline\end{array}$$

the dominant strategy of the resulting game remains BRBa, where TLF (defining a PD game), BLF, and TLBa are Pareto preferred outcomes [4]. But this game does not have a clear optimal state, and standard computations show that cooperation is impossible to attain with grim trigger.

3.4. **QRE structures.** A question raised in Sect. 2 was whether a "rationality" value $\lambda > 0$ exists so that $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$. For a brief review, QRE transforms the expected payoffs of a strategy choice into positive weights. The probability of selecting a strategy is proportional to its weight relative to the weights of all other strategic choices. If, for example, the second player's mixed-strategy is q and the first player's expected payoff with strategy s_1 is $\pi_1(q)$, then the weight assigned to s_1 is $w_1 = e^{\lambda\pi_1(q)}$; λ is the QRE parameter. The probability of choosing s_1 is $p_1(\lambda) = \frac{w_1}{w_1+\cdots+w_n}$.

With our Nash decomposition, if player two uses the mixed-strategy q, the expected payoffs for player one of choosing the top or bottom row are

$$EV(T) = q(\eta_{1,1} + \beta_1) + (1-q)(\eta_{2,1} - \beta_1) = q\eta_{1,1} + (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1,$$
$$EV(B) = q(-\eta_{1,1} + \beta_1) + (1-q)(-\eta_{2,1} - \beta_1) = -q\eta_{1,1} - (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1.$$

The corresponding QRE weights are

$$w_T(\lambda) = e^{\lambda[q\eta_{1,1}+(1-q)\eta_{2,1}+q\beta_1-(1-q)\beta_1]}, \quad w_B(\lambda) = e^{\lambda[-q\eta_{1,1}-(1-q)\eta_{2,1}+q\beta_1-(1-q)\beta_1]},$$

so the choice probabilities are functions of λ; e.g., for T, it is $p_T(\lambda) = \frac{w_T(\lambda)}{w_T(\lambda)+w_B(\lambda)}$. Similar expressions hold for $k_1 \times \cdots \times k_n$ games.

This λ dependency makes it reasonable to expect the λ value to strongly influence the structure of $\sim_{QRE,\lambda}$ equivalence classes, which makes the following theorem surprising.

Theorem 3. *For any* $\lambda > 0$*, two* $k_1 \times \ldots \times k_n$ *games* $\mathcal{G}_i$ *and* $\mathcal{G}_j$ *are QRE equivalent,* $\mathcal{G}_i \sim_{QRE,\lambda} \mathcal{G}_j$*, if and only if they are Nash equivalent,* $\mathcal{G}_i \sim_N \mathcal{G}_j$*.*

So, for any $\lambda > 0$, the highly predictable $\mathcal{G}_1$ and the Prisoner's Dilemma $\mathcal{G}_2$ (Eq. 2) satisfy $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$; these games indistinguishable from the perspective of QRE! Clearly, changes in $\mathcal{G}^B$ can significantly alter how players react to a game (as true with $\mathcal{G}_1$ and $\mathcal{G}_2$), but they play *no role* in a QRE analysis. The same holds for the 4×3 games $\mathcal{G}_6$ and $\mathcal{G}_7$. For QRE to distinguish between such games, modification must incorporate $\mathcal{G}^B$ information.

While Thm. 3 is surprising from a game theoretic standpoint, it is expected from an algebraic perspective. Similar to how addition of reals is identified with multiplication through $y = e^{\lambda x}$ for a fixed $\lambda > 0$, the algebraic matrix structure of Nash and QRE remain essentially the same, which is essentially the proof of Thm. 3 (Sect 7).

4. Structure of space of games

Other ways to use the decomposition require the sharper description developed in this section. After describing the partitioning of the space of games (Thm. 4), Fig. 1 is used to describe the $\mathcal{G}^N$ structure of all 2×2 games. This structure is used to find which strategic structures can arise with special games, such as zero sum and identical play.

4.1. **Complexity measures.** The complexity of games is reflected by the huge dimensions associated with the decomposition's subspaces.

Theorem 4. *The space of n person $k_1 \times \cdots \times k_n$ games can be identifies with $\mathbb{R}^\Gamma$ where $\Gamma = n[k_1 k_2 \dots k_n]$. This space is decomposed as*

$$\mathcal{G} = (\mathcal{G}^N, \mathcal{G}^B, \mathcal{G}^K) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{B}} \times \mathbb{R}^n$$

where

$$\mathcal{N} = k_1 k_2 \dots k_n[n - \sum_j \frac{1}{k_j}], \qquad \mathcal{B} = k_1 k_2 \dots k_n[\sum_j \frac{1}{k_j}] - n. \tag{31}$$

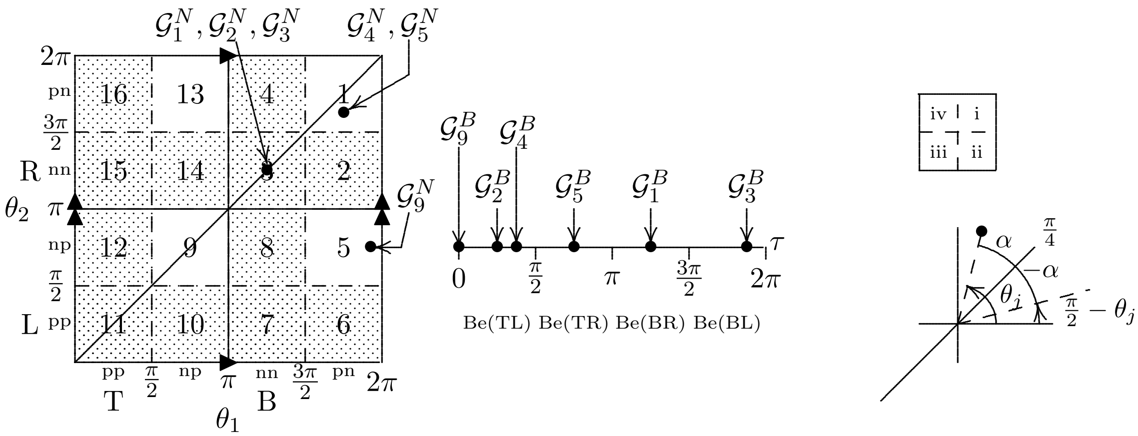
The space of $4 \times 5 \times 6 \times 7$ games is identified with the $4[4 \times 5 \times 6 \times 7] = 3360$ dimensional space $\mathbb{R}^{3360}$. Because $\mathcal{N} = 4 \times 5 \times 6 \times 7[4 - \{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\}] = 2722$, the Nash structure depends upon 2722 independent variables that define up to $6 \times 5 \times 4 = 120$ pure Nash points (Cor. 1). A wide selection of other game theoretic features are introduced by appropriate choices of the $\mathcal{B} = 4 \times 5 \times 6 \times 7[\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}] - 4 = 634$ behavioral $\mathcal{G}^B$ variables.

With 2×2 games, Thm. 4 becomes

- The four dimensional $\mathcal{G}^N$ strategic subspace is defined by $\{\eta_{i,j}\}_{i,j=1,2}$.
- The two dimensional $\mathcal{G}^B$ behavioral subspace defined by $\{\beta_j\}_{j=1}^2$.
- The two dimensional $\mathcal{G}^K$ kernel subspace spanned by $\{\kappa_j\}_{j=1}^2$.

4.2. **A further simplification: 2×2 games.** For $k_1 \times \cdots \times k_n$ games, the crucial $\mathcal{G}^N$ information (Thm. 4) is reduced (Eqs. 22, 23) from $\mathcal{N}$ to $\mathcal{N} - n$ dimensions, and for $\mathcal{G}^B$ from $\mathcal{B}$ to $\mathcal{B}-1$ dimensions. For 2×2 games (Fig. 1), the $\mathcal{G}^N$ information (Def. 3) is reduced from four to two dimensions, while for $\mathcal{G}^B$ from two to one. This permits representing the information with the Fig. 2 geometric diagrams.

Using Fig. 1, plot (θ_1, θ_2) on a $[0, 2\pi] \times [0, 2\pi]$ square (Fig. 2a); each axis is divided into the four Fig. 1a, b quadrants to define 16 smaller squares. Games in each smaller square have different strategic behaviors. For instance, if θ_j defines a ray in the first quadrant, then $\eta_{1,j}$ and $\eta_{2,j}$ are both positive, so the j^{th} agent has a dominant strategy (Eq. 24). This region is denoted in Figs. 1a, b, with ds_1; in Fig. 2a, the appropriate dominant strategy (T for $j = 1$; L for $j = 2$) is listed by the axis and represented by shading the corresponding column or row. Similarly, if θ_j is in the third quadrant, then $\eta_{1,j}$ and $\eta_{2,j}$ are negative so the $-\eta_{1,j}$ and $-\eta_{2,j}$ values in the other row (column) are positive; again, the j^{th} agent has a dominant strategy (denoted in Figs. 1a,b, by ds_2 and captured in Fig. 2a by B or R). This square (torus) suffices for a strategic analysis. The entries along the Fig. 2a axes, such as *pn*, identify the quadrant signs of $(\eta_{1,j}, \eta_{2,j})$; e.g., *pn* on the horizontal axis means that $\eta_{1,1} > 0$ and $\eta_{2,1} < 0$.



a. Strategic $\mathcal{G}^N$ decomposition **b.** Behavioral $\mathcal{G}^B$ **c.** Rotating

Figure 2. Decomposition of games

For a mixed strategy, θ_j must be in the second or fourth quadrants (Figs. 1a, b) where $\frac{\pi}{2} < \theta_j < \pi$ or $\frac{3\pi}{2} < \theta_j < 2\pi$, $j = 1, 2$. In Fig. 2a, these are the np or pn intervals. The lines separating Fig. 2a squares (the edges mentioned in Sect. 3.1) correspond to a transition between a dominated and mixed strategy. In each Fig. 2a shaded square, and only in these squares, *some agent has a dominant strategy.*

For $\mathcal{G}^B$, all points along a Fig. 1c ray passing through $\boldsymbol{\beta} = (\beta_1, \beta_2)$ have an identical Pareto structure for the players; differences involve positive scalar multiples. These terms can be represented by angle τ as in Fig. 1c or 2b. Each quadrant represents a different Pareto superior position.

Information for the $\mathbb{R}^8$ space of 2×2 games is reduced to points in a square ($\mathcal{G}^N$ information) times an interval ($\mathcal{G}^B$ information) defining a three-dimensional cube. The $0 = 2\pi$ endpoints are identified, so it is the product of three circles, or the three-torus T^3.

4.3. **Symmetric, identical, and zero-sum games.** To show how to use Fig. 2a to determine which $\mathcal{G}^N$ structures accompany different kinds of games, start with 2×2 symmetric games. It is easy to show that a symmetric game's $\mathcal{G}^N$ point must be on Fig. 2a diagonal. But the diagonal misses several Fig. 2a regions, which limits the Nash strategic structures for symmetric games.

In this geometric way, the strategic structure for specified classes of games can be determined. In particular, we prove that (Sect. 2.3) with limited exceptions, zero-sum (sum of entries in a cell equals zero) and identical play (all entries in a cell are the same) games cannot be in the same $\mathcal{G}^N$ equivalence class. By identifying the precise 2×2 games that are best response equivalent (Def 1) to either class of games, we extend a comment in Monderer and Shapley [10] that ... every nondegenerate 2-person 2×2 game is best response equivalent in mixed strategies either to a [identical play] or to a zero-sum game."[1]

Theorem 5. *If $\mathcal{G}$ is a $k_1 \times k_2$ zero-sum game, then $\kappa_1 = -\kappa_2$. If $\mathcal{G}$ is identical play, then $\kappa_1 = \kappa_2$. For zero-sum, the β values are*

$$\beta_{1;j} = -\frac{1}{k_1}\sum_{i=1}^{k_1} \eta_{j,2;i} \quad \beta_{2,j} = -\frac{1}{k_2}\sum_{i=1}^{k_2} \eta_{j,1;i} \tag{32}$$

and, for each i and j,

$$k_1[k_2\eta_{i,1;j} - \sum_{s=1}^{k_2} \eta_{i,1;s}] = -k_2[k_1\eta_{j,2;i} - \sum_{s=1}^{k_1} \eta_{j,2;s}] \tag{33}$$

For identical play, change all Eqs. 32, 33 "-1" signs to $+1$.

An interesting peculiarity about zero-sum and identical play games comes from Eq. 32; each agent's β values are uniquely defined by the *other agent's* strategic terms.

As shown below for 2×2 games, about half of the $\mathcal{G}^N$ behaviors (Fig. 2a) occur in zero-sum games, and (almost) the other half occur with identical play, so these games capture most of what can happen. But the $\mathcal{G}^N$ structure of these games satisfy Eq. 33, which (Eq. 22) imposes $(k_1 - 1)(k_2 - 1)$ constraints on $\mathcal{G}^N$ choices. It will follow for $k_1, k_2 \geq 3$ that not all $k_1 \times k_2$ $\mathcal{G}^N$ structures occur with zero-sum or identical play games.

Proof: For zero sum games, the sum of payoffs in the (i, j) cell equals zero, so

$$\eta_{i,1;j} + \beta_{1;j} + \eta_{j,2;i} + \beta_{2;i} = -[\kappa_1 + \kappa_2] = K. \tag{34}$$

Summing over $i = 1, \dots, k_1$, $j = 1, \dots, k_2$, yields (Eqs. 22, 23) $k_1k_2K = 0$, or $\kappa_1 = -\kappa_2$.

Setting $K = 0$, holding j fixed and summing Eq. 34 over all i leads to $k_1\beta_{1;j} = -\sum_{i=1}^{k_1} \eta_{j,2;i}$, which is the first term in Eq. 32; the second term is found in the same way. Equation 33 emerges by substituting these β values into Eq. 34. For identical play games, Eq. 34 becomes $\eta_{i,1;j} + \beta_{1;j} - [\eta_{j,2;i} + \beta_{2;i}] = -[\kappa_1 - \kappa_2] = K$.

In this manner, appropriate conditions can be derived for the $\mathcal{G}^N, \mathcal{G}^B, \mathcal{G}^K$ terms for any $k_1 \times \ldots \times k_n$ zero sum or identical play game. □

Corollary 2. *If $\mathcal{G}$ is a 2×2 zero-sum, or identical play game, its $\mathcal{G}^N$ component satisfies, respectively,*

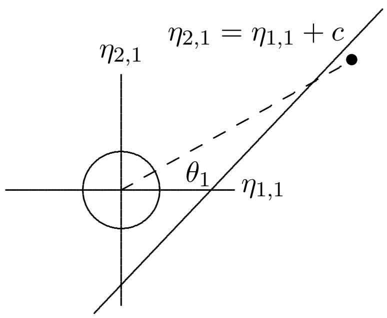
$$\eta_{1,1} - \eta_{2,1} = \eta_{2,2} - \eta_{1,2}; \quad \eta_{1,1} - \eta_{2,1} = \eta_{1,2} - \eta_{2,2}. \tag{35}$$

If the appropriate Eq. 35 equality is satisfied, there is a zero-sum, identical play game in its equivalence class uniquely defined, respectively, by

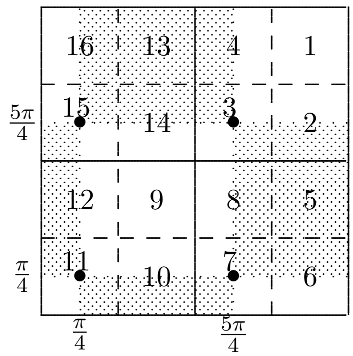
$$\beta_1 = -\frac{\eta_{1,2} + \eta_{2,2}}{2}, \beta_2 = -\frac{\eta_{1,1} + \eta_{2,1}}{2}; \quad \beta_1 = \frac{\eta_{1,2} + \eta_{2,2}}{2}, \beta_2 = \frac{\eta_{1,1} + \eta_{2,1}}{2} \tag{36}$$

[1] Our thanks to a referee for calling this [10] quote to our attention. Other material in the section was motivated by a question the referee raised.

The $\mathcal{G}^N$ component of a zero-sum (or identical play) game typically is not of that type. (For instance, $\eta_{1,1} = 9, \eta_{2,1} = 8, \eta_{2,2} = 2$, $\eta_{1,2} = 1$ satisfy the zero sum part of Eq. 35.) For 2×2 games, a necessary and sufficient condition for $\mathcal{G}^N$ to be zero-sum is $\eta_{1,1} = \eta_{2,2} = -\eta_{2,1} = -\eta_{1,2}$, which is a "matching pennies" game located at the center points of squares 5 and 13. Here Eq. 36 forces $\beta_1 = \beta_2 = 0$, so the $\mathcal{G}^B$ component plays no role when $\mathcal{G}^N$ has a zero-sum structure. Similarly, $\mathcal{G}^N$ has an identical play structure if and only if $\eta_{1,1} = \eta_{1,2} = -\eta_{2,1} = -\eta_{2,2}$, which is a "choosing sides" coordination game located at the midpoints of squares 1 and 9; similarly, $\beta_1 = \beta_2 = 0$. .



a. Player 1; strategic



b. $\mathcal{G}^N$ Zero sum games

Figure 3. Strategic characteristics of zero sum games

The $\mathcal{G}^N$ structures that admit a 2×2 zero-sum game, given by the shaded Fig. 3b regions, essentially fill half of the Fig. 2a square. The unshaded region are $\mathcal{G}^N$ structures that allow identical play games. Included in both sets are the four bullets, which admit both zero-sum and identical play games. A direct computation shows that the boundaries between shaded and unshaded regions correspond to degenerate games (from the [10] quote).

Theorem 6. *For 2×2 zero-sum games, if both $\boldsymbol{\eta_j} = (\eta_{1,j}, \eta_{2,j})$, $j = 1, 2$, are on the same side of the $\eta_{1,j} = \eta_{2,j}$ dividing diagonal, there is an identical play (but no zero-sum) game in the $\mathcal{G}^N$ equivalence class. If they are on opposite sides, there is a zero-sum (but no identical play) game in the $\mathcal{G}^N$ equivalence class. If both are on the $\eta_{1,j} = \eta_{2,j}$ diagonal (the four Fig. 3b bullets), there is a zero-sum and an identical play game in the $\mathcal{G}^N$ equivalence class. If one $\boldsymbol{\eta_j}$ is on the diagonal and the other is not, there are no zero-sum, nor identical play games in the equivalence class.*

With the exception of games with $\mathcal{G}^N$ structures in squares 1 and 9, each 2×2 game is best response equivalent to some zero-sum game; with the exception of games in squares 5 and 13, each 2×2 game is best response equivalent to some identical play game.

Proof: To derive Fig. 3b, setting each side of an Eq. 35 equality equal to a constant defines two sets of linear equations

$$\eta_{2,1} = \eta_{1,1} + c, \eta_{2,2} = \eta_{1,2} - c; \text{ and } \eta_{2,1} = \eta_{1,1} + c, \eta_{2,2} = \eta_{1,2} + c \tag{37}$$

Because of scaling (Fig. 1), it suffices to select $c = -1, 0, 1$, and to determine which equations are satisfied by $\boldsymbol{\eta_j}$ terms. With $c = 0$, both sets of equations are satisfied iff both $\boldsymbol{\eta_j}$ are on the diagonal. The four possibilities (given by the signs of $\eta_{1,j} = \eta_{2,j}$) define the four bullet points in Fig. 3b.

If one $\boldsymbol{\eta_j}$ is on the diagonal and the other is not, Eqs. 35, 37 cannot be satisfied, so the $\mathcal{G}^N$ class does not include zero-sum or identical play games. If neither $\boldsymbol{\eta_j}$ is on the diagonal, then either both are on different sides of the diagonal, or both on the same side. If on different sides, then select the $c = -1, 1$ value for which the first set of Eq. 37 is on the same side as the corresponding $\boldsymbol{\eta_j}$ point. To find the appropriate scaling (Fig. 1) that satisfies Eqs. 35, 37, draw a ray from the origin through the point. Where this ray intersects the Eq. 37 line is the scaling ensuring a zero-sum game in the equivalence class. The Fig. 3b region is determined by the corresponding θ_j values. As indicated in Fig. 3a, the line below the diagonal has θ_j ranging from $\frac{\pi}{4} - \pi$ to $\frac{\pi}{4}$, while the θ values for the companion line above the diagonal vary between $\frac{\pi}{4}$ and $\frac{\pi}{4} + \pi$; the end-points (which represent points at infinity) cannot be achieved. These constraints, where θ_1 is in one interval while θ_2 is in the other, define the shaded Fig. 3b regions.

Similarly, if both $\boldsymbol{\eta_j}$ are on the same side of the diagonal, then carry out the same construction for the second set of equations in Eq. 37. This defines the unshaded region where identical play games are in the equivalence classes. The boundary between shaded and unshaded corresponds to where one $\boldsymbol{\eta_j}$ is on the diagonal and the other is not.

To define *the "best response equivalence" classes,* player one has a dominant strategy in both completely shaded Fig. 2a columns; for the other two columns, the player's mixed strategy remains the same along a vertical line. For player two, the same holds for completely shaded rows and horizontal lines. Therefore all games in each of squares 3, 7, 11, 15 are best response equivalent; all games on each vertical line in each of squares 2, 6, 10, 14, and all games on each horizontal line in each of squares 4, 8, 12, 16, are best response equivalent. Finally, all games represented by a point in each of squares 1, 5, 9, 13 are best response equivalent. Combining these comments with Fig. 3b, it follows that with the exception of the games in squares 1 and 9, each game is best response equivalent to some zero-sum game. Similarly, with the exception of games in squares 5 and 13, each game are best response equivalent to some identical play game. □

Identical play and zero-sum games capture almost everything that strategically happens for 2×2 games. This assertion does not extend beyond 2×2 games.

Corollary 3. *For $k_1 = k_2 \geq 3$, there exist open sets of $k_1 \times k_2$ games that do not have zero-sum nor identical play games in their $\mathcal{G}^N$ equivalence set.*

Proof: The proof follows from the $(k_1 - 1)(k_2 - 1) \geq 4$ linear constraints required by Eq. 33. Here, $\boldsymbol{\eta_j}$ is a vector consisting of the j^{th} agent's $\mathcal{G}^N$ components.

As above, set each side of a constraint equal to zero, the resulting planes divide $\mathbb{R}^{\mathcal{N}}$ (Thm. 4) into $2^{(k_1-1)(k_2-1)} \geq 8$ open regions. For a $\mathcal{G}^N$ to have an identical play game in its equivalence class, both $\boldsymbol{\eta_j}$'s must be on the same side of each hyperplane (Eq. 33); for a zero-sum, both $\boldsymbol{\eta_j}$'s must be on opposite sides. But the geometry admits open sets of $\boldsymbol{\eta_j}$

that are on the same side of some hyperplanes and on the opposite side of other planes. With obvious modifications, the assertion holds for k_1, k_2 where $(k_1-1)(k_2-1) \geq 2$. □

5. Characterizing all 2×2 games

It remains to identify which games fall in each of the $16 \times 4 = 64$ small cubes defined by the Fig. 2a square and the $\mathcal{G}^B$ line segment (Fig. 2b), and to define new connections. What simplifies the process are results in the technical Sect. 5.3 showing that, rather than all 16 (Fig. 2a) squares, only the Nash structure (Sect. 5.1) of squares {1, 2, 3, 5} are needed. In Sect. 5.2, features of the games in each cube are described.

5.1. **The Nash structure of the square.** The Nash structure of each Eq. 2a square. follows from the $\eta_{i,j}$ signs (along the Fig. 2a axes) and the $\mathcal{G}^N$ form (Eq. 24). According to Sect. 5.3, it suffices to describe the squares {1, 2, 3, 5}.

Theorem 7. *The Nash structures of games in the Fig. 2a torus are as follows:*

1. *Games in square 1 ($\eta_{1,j} > 0$ and $\eta_{2,j} < 0$ for $j = 1, 2$) have the two pure strategies TL and BR, and a mixed strategy.*
2. *Games in square 2 ($\eta_{1,1} > 0, \eta_{2,1} < 0$ and $\eta_{i,2} < 0$, $i = 1, 2$) have a dominant R strategy for player 2. Because $\eta_{2,1} < 0$, the first player plays B.*
3. *Games in square 3 ($\eta_{i,j} < 0$) have BR as the dominant Nash strategy.*
4. *Games in square 5 ($\eta_{1,1} > 0, \eta_{2,1} < 0$ and $\eta_{1,2} < 0, \eta_{2,2} > 0$) have no pure strategies and one mixed strategy.*

By using ± 1 to define canonical examples, the center points of squares 1, 2, 3, and 5 are characterized, respectively, by $\mathcal{G}^N_{12}$, $\mathcal{G}^N_{13}$, $\mathcal{G}^N_{2}$, and $\mathcal{G}^N_{14}$ where:

$$(38) \qquad \mathcal{G}^N_{12} = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ \hline \end{array}, \ \mathcal{G}^N_{13} = \begin{array}{|cc|cc|}\hline 1 & -1 & -1 & 1 \\ \hline -1 & -1 & 1 & 1 \\ \hline \end{array}, \ \mathcal{G}^N_{14} = \begin{array}{|cc|cc|}\hline 1 & -1 & -1 & 1 \\ \hline -1 & 1 & 1 & -1 \\ \hline \end{array}$$

5.2. **The cube structures.** To design all 2×2 games with a variety of desired properties, add appropriate $\mathcal{G}^B$ terms to a desired Nash structure (Thm. 7). The $\mathcal{G}^B$ Pareto term is either in conflict, or in agreement, with the Nash structure. For a specified $\mathcal{G}^N$, $\mathcal{G}^N + \mathcal{G}^B$ defines a two parameter (β_1, β_2) family of games with the same Nash structure.

Illustrating with square 3, this family starts with the $\mathcal{G}^N_2$ game and its dominant strategy; it branches off in one direction to create a Prisoner's Dilemma, while in a different direction to suggest using side payments. Square 2 has surprisingly similar structures. Square 1 replaces the dominant strategies with two pure and one mixed strategy; in one direction the parameterized family separates the equilibria, and the another creates interest in non-equilibria outcomes. All pure strategies are dropped in square 5.

5.2.1. *Square 3.* These games have a dominant BR Nash strategy.

Cube $(3, Be(BR))$*:* The $\mathcal{G}^B$ Pareto superior point in this cube reinforces the Nash dominant strategy, so, as true with $\mathcal{G}_1$, the Nash outcome coincides with expected behavior.

Cube $(3, Be(TL))$*:* Games in this cube have $\beta_1, \beta_2 \geq 0$ and the general form

$$\mathcal{G}^N + \mathcal{G}^B = \begin{array}{|cc|cc|}\hline -|\eta_{1,1}| + \beta_1 & -|\eta_{1,2}| + \beta_2 & -|\eta_{2,1}| - \beta_1 & |\eta_{1,2}| + \beta_2 \\ \hline |\eta_{1,1}| + \beta_1 & -|\eta_{2,2}| - \beta_2 & |\eta_{2,1}| - \beta_1 & |\eta_{2,2}| - \beta_2 \\ \hline \end{array}. \tag{39}$$

Thus Eq. 39 defines a two parameter family of games starting from $\mathcal{G}^N$ ($\beta_1 = \beta_2 = 0$) to a Prisoner's Dilemma. The characterization involves which matrix entries dominate, so set equal each player's TL and BR entries to find the transition values

$$\beta_1^* = \frac{|\eta_{1,1}| + |\eta_{2,1}|}{2}, \quad \beta_2^* = \frac{|\eta_{1,2}| + |\eta_{2,2}|}{2}. \tag{40}$$

The transition values with $\mathcal{G}_2^N$ are $\beta_1^* = \beta_2^* = 1$.

(1) If $\beta_j < \beta_j^*$, $j = 1, 2$, the dominant BR Nash strategy also is a Pareto point. An example with $\beta_1 = \beta_2 = 0.5$ is the first Eq. 42 game.
(2) If $\beta_j > \beta_j^*$ for only one player, this player find attractive an outcome different from that of BR. Similar to $\mathcal{G}_3$, a sufficiently large β_j can suggest cooperative strategies with side payments as illustrated in the second Eq. 42 game with $\beta_1 = 4, \beta_2 = 0$. More generally, if

$$\beta_j > \frac{|\eta_{2,j}| - |\eta_{1,j}|}{2} + |\eta_{2,i}| \tag{41}$$

(where "i" is the other player), the sum of terms in another matrix entry is larger than that of BR, which makes a cooperative solution attractive.
(3) If $\beta_j > \beta_j^*$, $j = 1, 2$, the game is a Prisoner's Dilemma.

$$\begin{array}{|cc|cc|}\hline -0.5 & -0.5 & -1.5 & 1.5 \\ \hline 1.5 & -1.5 & 0.5 & 0.5 \\ \hline \end{array}, \quad \begin{array}{|cc|cc|}\hline 3 & -1 & -5 & 1 \\ \hline 5 & -1 & -3 & 1 \\ \hline \end{array} \tag{42}$$

Cubes $(3, Be(TR))$, $(3, Be(BL))$*:* In these cubes the $\mathcal{G}^B$ component emphasizes components that differ from the Nash structure, which can make certain entries more attractive than the Nash dominant strategy. The symmetry, where both $Be(TR)$ and $Be(BL)$ emphasize off-diagonal terms, means that games in these cubes have similar characteristics, so it suffices to describe only cube $(3, Be(TR))$, which, with Eq. 39, has $\beta_1 < 0$ and $\beta_2 > 0$. A transition value $\beta_2 > |\eta_{2,1}| + \frac{|\eta_{2,2}| - |\eta_{1,2}|}{2}$ (agreeing with Eq. 41) is where the sum of TR entries exceeds that of BR, which suggests using a cooperative bargaining solution.

5.2.2. *Square 2.* Player two with a dominant R strategy. Because $\eta_{2,1} < 0$, player one plays B leading to a BR outcome.

Cube $(2, Be(BR))$*:* The structure is similar to that of cube $(3, Be(BR))$ where the behavioral term adds support to the Nash outcome.

Cube $(2, Be(TL))$*:* With only minor differences from Eq. 39, the general form of a game in this cube (where $\beta_1, \beta_2 > 0$) is

$$\mathcal{G}^N + \mathcal{G}^B = \begin{array}{|cc|cc|}\hline |\eta_{1,1}| + \beta_1 & -|\eta_{1,2}| + \beta_2 & -|\eta_{2,1}| - \beta_1 & |\eta_{1,2}| + \beta_2 \\ \hline -|\eta_{1,1}| + \beta_1 & -|\eta_{2,2}| - \beta_2 & |\eta_{2,1}| - \beta_1 & |\eta_{2,2}| - \beta_2 \\ \hline \end{array}. \tag{43}$$

While this Eq. 43 parameterized family (where $\mathcal{G}^B$ terms add appeal to TL in conflict with the Nash BR) does *not* include the Prisoner's Dilemma, the features are similar. The only difference from Eq. 40 is the transition value $\beta_1^* = \frac{|\eta_{2,1}|-|\eta_{1,1}|}{2}$, so cooperation becomes attractive to player one at a smaller β_1 value. If both $\beta_j > \beta_j^*$, strategies such as tit-for-tat become applicable.

Cubes $(2Be(TR))$, $(2, Be(BL))$*:* Essentially the same as for cubes $(3, Be(TR))$ and $(3, Be(BL))$.

5.2.3. *Square 1.* All 2×2 games with two pure and one mixed strategy belong to this square. Thanks to the $\mathcal{G}^N$ symmetry, where the pure strategies are at TL and BR, there are essentially two classes of games defined by $\mathcal{G}^B$; they support, or conflict with, the Nash solutions. Examples are the following $\mathcal{G}^N + \mathcal{G}^B_{15}$ and $\mathcal{G}^N + \mathcal{G}^B_{16}$ games

$$\mathcal{G}^N = \begin{array}{|cc|cc|}\hline 3 & 3 & -3 & -3 \\ \hline -3 & -3 & 3 & 3 \\ \hline \end{array},\ \mathcal{G}^B_{15} = \begin{array}{|cc|cc|}\hline 4 & 5 & -4 & 5 \\ \hline 4 & -5 & -4 & -5 \\ \hline \end{array},\ \mathcal{G}^B_{16} = \begin{array}{|cc|cc|}\hline -4 & 5 & 4 & 5 \\ \hline -4 & -5 & 4 & -5 \\ \hline \end{array}.$$

located at the center of square 1. The $\mathcal{G}^B_{15}$ structure creates a distinction between the two $\mathcal{G}^N$ pure strategies (making TL more appealing than BR); with $\mathcal{G}^B_{16}$, only the TR cell has positive entries.

Cubes $(1, Be(TL))$ *and* $(1, Be(BR))$*:* In this setting, $\mathcal{G}^B$ adds support to one of the pure strategies to make one Nash outcome Pareto superior to the other. The "Stag Hunt" characterizes this behavior; a $\kappa_1 = \kappa_2 = 2$ example is

$$\mathcal{G}_{SH} = \begin{array}{|cc|cc|}\hline 4 & 4 & 0 & 2 \\ \hline 2 & 0 & 2 & 2 \\ \hline \end{array} = \mathcal{G}^N_{12} + \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & 1 \\ \hline 1 & -1 & -1 & -1 \\ \hline \end{array} + \mathcal{G}^K \tag{44}$$

Because $\mathcal{G}^N_{12}$ does not distinguish between TR and BL, TL becomes the $\mathcal{G}_{SH}$ Pareto superior choice strictly because of the $Be(TL)$ choice of $\mathcal{G}^B$. This $\mathcal{G}^B$ component plays a crucial role in achieving the Stag Hunt's defining features:

(1) There is a Pareto dominant Nash outcome.
(2) But, there is a risk-free option that provides an incentive to deviate from the Pareto dominant choice.

A definition for a "risk-free strategy" is that its outcome remains the same independent of what the other player does. So, when player one plays B and player two plays R, it must be that $g_{3,1} = g_{4,1}$ and $g_{2,2} = g_{4,2}$, which become, respectively, $-\eta_{1,1} + \beta_1 = -\eta_{2,1} - \beta_1$ and $-\eta_{1,2} + \beta_2 = -\eta_{2,2} - \beta_2$. Thus, the risk-free condition requires

$$\beta_j = \frac{\eta_{1,j} - \eta_{2,j}}{2}, \quad j = 1, 2. \tag{45}$$

Because $\mathcal{G}^N$ is in square 1, $\eta_{1,j} > 0, \eta_{2,j} < 0$, so both β_j's are positive in Be(TL). These positive β_j values, which designate TL as the Pareto superior outcome, ensures the Stag-Hunt structure. By including the $\mathcal{G}^K$ variables, it follows from Eq. 45 that the stag-hunt structure holds for a portion of a six-dimensional subspace of $\mathbb{R}^8$.

"Pure coordination games" are in these cubes; one choice, with the $Be(TL)$ values of $\beta_1 = \beta_2 = 0.5$, along with $\kappa_1 = \kappa_2 = 1.5$, is

$$\mathcal{G}_{PC} = \begin{array}{|cc|cc|}\hline 4 & 4 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 \\ \hline\end{array} = \begin{array}{|rr|rr|}\hline 2 & 2 & -1 & -2 \\ \hline -2 & -1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|lr|rr|}\hline 0.5 & 0.5 & -0.5 & 0.5 \\ \hline 0.5 & -0.5 & -0.5 & -0.5 \\ \hline\end{array} + \mathcal{G}^K_{PC}.$$

To have equal non-equilibrium payoffs, the $\mathcal{G}^B$ values must be determined by $\mathcal{G}^N$ values. This conditions requires $-\eta_{1,j} + \beta_j = \eta_{2,j} - \beta_j$, or that β_j equals the average $\frac{1}{2}(\eta_{1,j} + \eta_{2,j})$. This β_j dependency of $\eta_{i,j}$ means that one equilibrium is made Pareto superior to the other by the $\mathcal{G}^N$ location in either 1i or 1iii (Fig. 2c). The 1i corner has $\eta_{1,j} > |\eta_{2,j}|$, $j = 1, 2$, so TL is the Pareto superior outcome; $\mathcal{G}^N$ in 1iii with $|\eta_{2,j}| > \eta_{1,j}$ with BR as the Pareto superior choice; $\mathcal{G}^K$ allows assigning zeros to non-equilibrium terms. The following captures the $\mathcal{G}^B$ and $\mathcal{G}^K$ values as determined by these conditions and the $\mathcal{G}^N$ choice:

$$\beta_j = \frac{1}{2}(\eta_{1,j} + \eta_{2,j}), \quad \kappa_j = \eta_{1,j} - \beta_j = \frac{1}{2}(\eta_{1,j} - \eta_{2,j}), \quad j = 1, 2. \tag{46}$$

Cubes $(1, Be(TR))$ *and* $(1, Be(BL))$: The "Battle of the Sexes" resides in these cubes with a description similar to the coordination game. The main exception is that each player prefers a different Nash outcome as illustrated in the following $(\kappa_1, \kappa_2 = 1.5)$

$$\begin{array}{|cc|cc|}\hline 4 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 4 \\ \hline\end{array} = \begin{array}{|rr|rr|}\hline 2 & 1 & -1 & -1 \\ \hline -2 & -2 & 1 & 2 \\ \hline\end{array} + \begin{array}{|lr|rr|}\hline 0.5 & -0.5 & -0.5 & -0.5 \\ \hline 0.5 & 0.5 & -0.5 & 0.5 \\ \hline\end{array} + \mathcal{G}^K,$$

which has a $Be(BL)$ behavioral component. Also, the positioning of $\mathcal{G}^N$ in square 1 played a role. An analysis similar to that of Eq. 46 determines the β_j values needed to achieve such a setting.

Another feature of games in this cube is how the $\mathcal{G}^B$ component directs attention to non-equilibrium solutions. To indicate the differences, to create a Stag Hunt game, add a $\mathcal{G}^B$ with $\beta_1 = \beta_2 = 1$ to $\mathcal{G}^N_{12}$ (to emphasize one Nash choice over the other); this is the first game of Eq. 47.) But by adding a sufficiently large $Be(TR)$ component of $\beta_1 = -10, \beta_2 = 10$ to $\mathcal{G}^N_{12}$, the resulting game is the second one in Eq. 47; the two games are Nash indistinguishable, but the second game attracts attention to TR.

$$\begin{array}{|lr|rr|}\hline 2 & 2 & -2 & 0 \\ \hline 0 & -2 & 0 & 0 \\ \hline\end{array}, \quad \begin{array}{|rr|rr|}\hline -9 & 11 & 9 & 9 \\ \hline -11 & -11 & 11 & -9 \\ \hline\end{array} \tag{47}$$

5.2.4. *Square 5.* All 2×2 games with a single mixed strategy, which include variants of "Matching Pennies," belong to square 5. As true with the above, a sufficiently strong $\mathcal{G}^B$ component added to $\mathcal{G}_{14}$ can direct attention to a particular $\mathcal{G}$ entry.

5.3. **A reduction.** The argument showing why only certain Fig. 2a squares need to be considered corresponds to changing a game by interchanging columns and/or rows; e.g., exchanging p and $(1-p)$, and/or q and $(1-q)$. This symmetry permits focussing attention on squares {1, 2, 3, 5} rather than all 16, which reduces the need to examine what happens in only $4 \times 4 = 16$ small cubes, rather than all 64 of them.

Interchanging the top and bottom rows of $\mathcal{G}^N$ (Eq. 24) changes $\boldsymbol{\eta}_1 = (\eta_{1,1}, \eta_{2,1})$ to $(-\eta_{1,1}, -\eta_{2,1}) = -\boldsymbol{\eta}_1$. In angular terms, θ_1 is moved to $\theta_1 + \pi$. The $\boldsymbol{\eta}_2 = (\eta_{1,2}, \eta_{2,2})$ term is changed to $(\eta_{2,2}, \eta_{1,2})$, which interchanges the "x" and "y" coordinates. A (x, y) to (y, x) change is a reflection about the $y = x$ line, which is the ray $\theta = \frac{\pi}{4}$. If (as in Fig. 2c) angle θ differs from $\frac{\pi}{4}$ by angle α (so $\alpha = \theta - \frac{\pi}{4}$), the reflected point is on the ray that differs from $y = x$ by angle $-\alpha$; i.e., the new angle is $\frac{\pi}{4} - \alpha = \frac{\pi}{4} - (\theta - \frac{\pi}{4}) = \frac{\pi}{2} - \theta$. This is formalized in the following proposition:

Proposition 1. *If the rows of a $\mathcal{G}^N$ matrix represented by (θ_1, θ_2) are interchanged, the new matrix is represented by $(\theta_1+\pi, \frac{\pi}{2}-\theta_2)$. If the columns of a (θ_1, θ_2) matrix are interchanged, the new matrix is $(\frac{\pi}{2} - \theta_1, \theta_2 + \pi)$.*

To use Prop. 1, the Fig. 2a squares are further subdivided as indicated in Fig. 2c. So, if the game matrix of a point is in square 4i, then (θ_1, θ_2) is in the upper-right corner of square 4. Interchanging the rows (Prop. 1) forms an $(\theta_1 + \pi, \frac{\pi}{2} - \theta_2)$ matrix. The $\theta_1 + \pi$ value positions the new matrix in the right-hand side of one of the {11, 12, 15, 16} squares (the first column of Fig. 2a). To determine which one, notice that the vertical distance from the horizontal $\frac{\pi}{2}$ line (Fig. 2a) to the top-right region in square 4 is over $2\frac{1}{2}$ squares; this is the $\theta_2 - \frac{\pi}{2}$ distance. To find the new point with value $\frac{\pi}{2} - \theta_2$, go down this "more than $2\frac{1}{2}$ squares" distance from the top horizontal 2π line (identified with the bottom horizontal line); it is in the lower right section of square 12. Thus, changing the rows of a matrix represented by a point in square 4i defines a unique point in 12ii.

Points (i.e., games) that are identified with each other can be determined with the dynamic of exchanging columns, then rows, then columns, and then rows to return to the starting point. Illustrating with square 4, a point in $4i$ is mapped to point in $8iv$. The full sequence is $4i \to 8iv \to 16iii \to 12ii \to 4i$.

Proposition 2. *Starting with a matrix (θ_1, θ_2), perform the following operations in the same manner: Interchange columns, then rows, then columns, then rows. These operations define points in the following regions:*

$$4i \to 8iv \to 16iii \to 12ii \to 4i, \tag{48}$$

$$2i \to 10iv \to 6iii \to 14ii \to 2i, \quad 3i \to 7iv \to 11iii \to 15ii \to 3i \tag{49}$$

$$1i \to 9iv \to 1iii \to 9ii \to 1i, \quad 5i \to 13iv \to 5iii \to 13ii \to 5i. \tag{50}$$

Similar sequences arise by starting in any other sector of the starting square. According to Eq. 49, analyzing a game in square 2 is the same as doing so with a game in squares 6, 10, or 14. Similarly, analyzing all games in square 3 describes what happens in squares {3, 7, 11, 15}, and square 4 handles squares {4, 8, 12, 16}. Square 1, however, only handles squares {1, 9} while square 5 only handles squares {5, 13}. This dynamic shows that it suffices to examine the game structures only in squares {1, 2, 3, 4, 5}. This dynamic, for instance, identifies the four Fig. 3b bullets. It also shows that the two matching pennies games located at the center of squares 5 and 13 define the same game properties.

A second reduction uses the Fig. 2a diagonal line: Interchanging row and column players flips the Fig. 2a square about this diagonal. As an illustration, in square 4, player one has the dominant strategy of B, and player two reacts accordingly; in square 2, player two has the dominant strategy of R, and player one reacts accordingly. This symmetry reduces the $\mathcal{G}^N$ analysis to the squares $\{1, 2, 3, 5\}$, which is then combined with the $\mathcal{G}^B$ properties from the four Fig. 2b regions. (As the diagonal passes through squares 1 and 3, only the half of these squares below the diagonal needs to be examined.)

5.4. **General games.** For completeness, this concluding technical subsection shows that a scaling similar to Fig. 1 applies to all games, which extends Def. 3 to all games. For the j^{th} agent in a $k_1 \times \cdots \times k_n$ game and each $\mathbf{s}$, a $k_j - 1$ dimensional vector has all $\eta_{i,j,\mathbf{s}}$ strategic information (Eq. 22). Let $\boldsymbol{\eta_j}$, $j = 1, \dots, n$, collect all of these vectors over all $\mathbf{s}$ choices to define a $\frac{k_1 k_2 \dots k_n}{k_j}[k_j - 1]$ dimensional vector. The scaling argument reduces the relevant information to a point on $S^{\frac{k_1 k_2 \dots k_n}{k_j}[k_j-1]-1}$; all $\mathcal{G}^N$ information is represented by a point in the product of these spheres, which has dimension $\mathcal{N} - n$. Illustrating with 2×3 games, the row player's $\mathcal{G}^N$ information is captured by $\boldsymbol{\eta_1} = (\eta_{1,1}, \eta_{2,1}, \eta_{3,1}) \in \mathbb{R}^3$ where the first $\eta_{i,1}$ subscript identifies the $\mathcal{G}^N$ column. After scaling, the relevant information is a S^2 point. (To create a figure similar to Fig. 2a, represent the sphere as a square where its top and bottom edges collapse, respectively, to the North and South poles.) With the $\eta_{i,2,\mathbf{s}}$ notation, $\mathbf{s} = T, B$, the column player's $\mathcal{G}^N$ information is $\boldsymbol{\eta_2} = (\eta_{1,2,T}, \eta_{2,2,T}, \eta_{1,2,B}, \eta_{2,2,B}) \in \mathbb{R}^4$, which the scaling reduces to a point on S^3. Thus all strategic information for a 2×3 game is represented by a point in $S^2 \times S^3$. A geometric description similar to (but more complicated than) Fig. 2a can be created.

A similar computation holds for $\mathcal{G}^B$. For the j^{th} player in an n person $k_1 \times \cdots \times k_n$ game, all but one of the $\beta_{j,\mathbf{s}}$ terms captures this agent's $\mathcal{G}^B$ information (Eq. 23). So let $\boldsymbol{\beta}_j \in \mathbb{R}^{\frac{k_1 k_2 \dots k_n}{k_j} - 1}$, $j = 1, \dots, n$, be the selected $\beta_{j,\mathbf{s}}$ terms. All $\mathcal{G}^B$ information is a point in the product of these spaces. A scaling reduces the $\mathcal{G}^B$ information to a point in $S^{\mathcal{B}-1}$. With a $2 \times 2 \times 2$ game, then, the $\mathcal{G}^B$ information is given by a point in S^8. Combined with the $\mathcal{G}^N$ representation, all $(\mathcal{G}^N, \mathcal{G}^B)$ information is given by a point in the 17-dimensional $[S^3 \times S^3 \times S^3] \times S^8$. For a 2×3 game, the $(\mathcal{G}^N, \mathcal{G}^B)$ information is given by a point in the eight-dimensional $[S^2 \times S^3] \times S^3$.

6. Summary

The decomposition of $k_1 \times \cdots \times k_n$ games simplifies the analysis by uniquely separating a game's Nash equilibria information from the behavioral content, which promotes non-strategic kinds of analysis. This decomposition reduces the complexity of the space of games and provides tools that can be used in a variety of ways. As illustrated by using symmetric, zero-sum, and identical play games, this approach can identify all possible Nash structures that can accompany a particular class of games. The decomposition also shows why standard tit-for-tat results for two-player games can change with more players, while providing new insights how the behavioral terms influence the strategy and cooperation. This unified framework also can be used to identify, and then analyze algebraically equivalent games, which is illustrated here by showing that all QRE games have the same strategic structure of the Nash analysis.

This decomposition identifies the similarities between many different types of applications of game theory, and provides a new language with which to discuss games. Rather than focusing on a single game, the decomposition advances a simple way to examine the entire space of games. This not only permits obtaining more general conclusions about classes of games, but it permits doing so in a way that simplifies the discussion and reveals essential features that may not have been previously known.

7. Proofs

Uniqueness of decomposition: As shown next, the described decomposition is unique. (The results were discovered by using representation theory based on symmetry properties of games; subsequently the simpler approach used here was discovered.) Start with what is required to ensure that a $k_1 \times \cdots \times k_n$ game does *not* contain any Nash equilibria information. By using the usual fixed point theorem approach to find Nash equilibria, this setting is the degenerate case where all possible strategies are fixed points.

This degenerate setting occurs for the j^{th} player if and only if the player's arrays defined by specified pure strategies of the other players are identical; e.g., in $k_1 \times \cdots \times k_n$ games, each row for the row player is the same. If a game does not satisfy this degenerate condition, then the equality of expected values over all possible mixed strategies of other players no longer holds. For purposes of being able to use Eq. 23 (needed to prove several of the results), this space is divided into the $\mathcal{G}^K$ and the $\mathcal{G}^B$ components. What remains are the $\mathcal{G}^N$ components with all Nash information. $\square$

Proof of Thm, 1: Parts 1 and 4 are proved above.

Part 2: In expected value computations for each of player j's strategies, terms from $\mathcal{G}^B$ and $\mathcal{G}^K$ have the same value, so they play no role in comparisons. Only terms from $\mathcal{G}^N$ give different expected values.

Part 3: For a cell to be a Nash point, it must have the largest value for each agent's choice. According to Eq. 22, these values must be positive, which is the stated necessary condition. For $k_j = 2$, the j^{th} player's other entry must be negative, so this is the best response. But for $k_j \geq 3$, more than one entry in the j^{th} player's array can be positive; the Nash point is the choice where, for each player, this is the largest value.

In computing $\mathcal{G}^N$ for the j^{th} player, there are $\frac{k_1 \dots k_n}{k_j}$ choices of strategies for the other players. The defined array has k_j terms, where the average is computed and then the difference of each term from this average, is computed. Thus the j^{th} player has $\frac{k_1 \dots k_n}{k_j}[k_j+1]$ computations. The sum over all n players is $k_1 k_2 \dots k_n[n + \frac{1}{k_1} + \cdots + \frac{1}{k_n}]$. $\square$

Proof of Cor. 1: Constructing games with the maximum number of strict pure Nash points makes it clear how to construct games of any smaller number. To show that a $k_1 \times k_2$ game can have k_2 strict pure Nash equilibria, it suffices to create a $\mathcal{G}^N$ with these properties; the only constraint on selecting $\eta_{i,j;\mathbf{s}}$ terms is Eq. 22. To be strict, for each j and $\mathbf{s}$, one $\eta_{i,j;\mathbf{s}}$ term is the largest. Only ordinal information is being used, so it suffices to assume that one value is positive and the rest are negative.

There are k_1 first-column cells to place player one's positive $\eta_{i,1;1}$ entry; let it be $\eta_{i^*,1;1}$. To make this $(i^*, 1)$ cell a Nash point, let it have player two's positive entry. These choices specify positive signs of player one's first column entry and player two's i^* row.

For the induction step, assume for up to player two's strategy j, $j < k_2$ that j Nash points are created. This identifies signs for player one's entries for the first j columns (positive in all Nash cells, negative otherwise), and signs for player two's entries for j rows. For player two's strategy $(j+1)$, there remain $k_1 - j > 0$ rows where the sign of player one's entry is not specified. Select one of these $k_1 - j$ rows to be the Nash cell by containing both player's positive entry; this completes this induction step. By construction, it is impossible to have more than k_2 Nash points because all of the second player's positive entries have been assigned.

For the second induction step, assume that $\alpha = k_2 k_3 \dots k_j$ strict Nash equilibria can be created for $k_1 \times \dots \times k_j$ games, but no games can have more equilibria. It must be shown that this assertion extends to $k_1 \times \dots \times k_j \times k_{j+1}$ games. For the $(j+1)^{th}$ player's first strategy, select one of these $k_1 \times \dots \times k_j$ games with the maximum number of Nash points. Assign the $(j+1)^{th}$ agent's positive value for each Nash point, which makes it a Nash point for this $k_1 \times \dots \times k_{j+1}$ game. Other choices of the $(j+1)^{th}$ agent are negative. But, for the $k_{j+1} - 1$ the assignment of signs are not made. So for the k^{th} strategy, construct one of the $k_1 \times \dots \times k_j$ games with the maximum number of Nash points. This can only be done for each strategy, which leads to the conclusion. □

Proof of Thm. 2: This standard tit-for-tat computation solves for β_j, rather than δ values.

Proof of Thm. 3: The proof is first given for $k_1 \times k_2$ games and then shown how it extends to $k_1 \times \dots \times k_n$ games. For player two's mixed strategy $(q_1, q_2, \dots, q_{k_2})$, player one must compute k_1 expected values. For the i^{th} strategy, the weight is

$$w_i(\lambda) = e^{\lambda[\sum_{s=1}^{k_2} q_s(\eta_{i,1;s}+\beta_{1;s})]} = e^{\lambda \sum_{s=1}^{k_2} q_s(\eta_{i,1;s}} e^{\lambda \sum_{s=1}^{k_2} \beta_{1;s}}$$

Each term in the probability assigned to the i^{th} strategy, $\frac{w_i(\lambda)}{\sum w_j(\lambda)}$, has the $e^{\lambda \sum_{s=1}^{k_2} \beta_{1;s}}$ multiple, so it cancels out leaving the probability

$$\frac{e^{\lambda \sum_{s=1}^{k_2} q_s \eta_{i,1;s}}}{\sum_{t=1}^{k_1} e^{\lambda \sum_{s=1}^{k_2} q_s \eta_{t,1;s}}}$$

showing that the relevant QRE game structure is the same as for the Nash equilibrium: any two games sharing $\eta_{i,j}$ values are the same for all $\lambda > 0$ values. The extension to general games follows by replacing the s in these equations with $\mathbf{s}$.

Proof of Thm. 4: The proof involves a computation. The j^{th} player faces $\frac{k_1 \dots k_n}{k_j}$ strategies from the other strategies. In each of these arrays, there are $k_j - 1$ independent η values (because of Eq. 22), which means the j^{th} agent has $\frac{k_1 \dots k_n}{k_j}[k_j - 1]$ independent η values.

Summing over all players yields the $\mathcal{N}$ value. A similar computation (using Eq. 23) hold for the $\mathcal{B}$ computation. $\square$

Proofs of Thms. 5, 6, Cors. 2, 3, Props. 1, 2: Given above.

Proof of Thm. 7: Standard computation.

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