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BASIS FOR BINARY COMPARISONS AND NON-STANDARD PROBABILITIES

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Abstract. To compare outcomes from paired comparisons, such as with non-standard probabilities, a basis is created for the space of all binary interactions. In this manner the source of all transitive and non-transitive (e.g., path dependencies) can be identified.

1. Introduction

In an increasing number of disciplines, ranging from the physical (e.g., quantum mechanics) to the social sciences (e.g., psychology, economics), natural issues arise where standard Kolmogorov probability arguments do not suffice. To provide an unifying structure for some of these approaches, an easily used basis is created here for paired comparisons (whether for probabilities, correlations, or other effects), which separates well-behaved terms satisfying transitivity from those causing path dependencies, cycles, and other mysteries. Information about the paired comparisons (e.g., probabilities) may be known, but what causes the behavior may not. This creates an inverse problem of understanding what kind of modeling supports these outcomes, whether it be the structure of phase space for physical systems, social norms for groups, or brain processing for individuals. The basis helps to identify what may be needed.

This basis is illustrated with an "order effects" result (Wang et. al. [10]) that was motived,[1] in part, by a 1997 Gallup Poll about the perceived honesty of President Clinton and Vice President Gore. The order in which the question was posed, either first about the honesty of Clinton or Gore and then about the other, affected the conclusion. Wang et. al. used a quantum probability model to derive QQ (Quantum Question) equalities that connect differences. This QQ relationship was empirically supported with three examples.

This QQ equality (Sect. 3-3.5) is a potentially valuable contribution for understanding order effects, but verifying the title's claim that their results "... reveal quantum nature of human judgements" requires showing that these equalities always hold and are the only explanation when they do. The first, an empirical issue, is beyond the scope of this paper. As for the second, using the basis developed here, it follows that:

[1]From [10] and Joyce Wang's March 11, 2015, presentation at a Field's Institute conference on "Quantum Probability & the Mathematical Modeling of Decision Making."

(1) QQ need not always hold, so a measure is needed to determine when it does. With the basis, a simple vector analysis identifies the QQ relationship and a measure of when it can, or cannot, hold.
(2) This basis identifies properties and even "hidden variables" involved in the QQ expression.

Notions needed to describe the space of binary interactions are introduced (Sect. 2) with pairwise voting difficulties. Resolving these problems requires identifying what causes them, which is the theme of Sect. 2-2.2. This material is then generalized (Sect. 3) to handle general paired comparison difficulties.

2. Pairwise voting

The voting concerns are motivated with my fictional boast ([6]):

> *Before your next election, tell me who you want to win. For a price, I will visit your group, talk with your colleagues, and design a fair (e.g., all candidates are considered) election method. Your candidate will win.*

To illustrate, suppose a fifteen member department is to select one out of five candidates for a tenure track position. Their preferences are (where "$\succ$" means "strictly preferred to")

(1)

Number	**Ranking**	**Number**	**Ranking**
5	$A \succ B \succ C \succ D \succ E$	5	$B \succ C \succ D \succ E \succ A$
5	$C \succ D \succ E \succ A \succ B$		

These voters *unanimously* prefer $C \succ D \succ E$, so E *cannot* be their top choice. Yet, E can be "convincingly" elected with a method endorsed by *Robert's Rules of Order*—an agenda. This approach shares features of a tournament where, with a specified ordering of the alternatives, the winner of the first two is compared with the next listed candidate, that winner with the next one, etc. The method's outcome is the winner of the last pair. Thus the $< D, C, B, A, E >$ agenda advances the $\{C, D\}$ majority vote winner to a vote with B, that winner with A, and that winner with E to obtain the final outcome.

With Eq. 1 and this agenda, C unanimously beats D to be compared with B, B beats C with a two-thirds vote (the ten voters on the top line) to be compared with A, A beats B with a two-thirds vote (the ten voters in the first column) to be compared with E. E is the overall winner by convincingly beating A with a two-thirds vote (the ten voters in the second row and column).[2]

All votes are unanimous or of landslide (two-thirds) proportions, which (incorrectly) identifies E as the voters' overwhelming choice. Translating this example into probabilities highlights a realistic concern (which extends to general settings) that *a collection of pairwise probabilities can seriously misrepresent the actual structure of an underlying source system.* An accompanying issue is to explain why this is so; a second is to discover how to combine paired outcomes to combat such difficulties (Sect. 2-2.2).

[2]Equation 1 generates a cycle, which creates the path dependency phenomenon where *any candidate* can be elected with an appropriate ordering! To elect D, for instance, use $< C, B, A, E, D >$. For C, use $< B, A, E, D, C >$.

Natural objectives suggested by this example include:

- Find all possible paired comparison outcomes.
- Identify all structures in the source space—the space of voter preferences—that cause such conclusions; this includes cycles and all path dependency concerns.

These objectives have been answered for any number of alternatives; they have even been addressed for triplets, etc. (Saari [5, 6]). While any collection of paired rankings can emerge from voting (McGarvey [2]), of particular value is to determine all possible tallies. Constraints on all possible three-candidate, pairwise tallies (from complete transitive preferences) are identified in Saari [3, 6]. As indicated next (e.g., Thm. 1; generalized in Thm. 6), answers for $n \geq 3$ alternatives come from profile structures with extreme outcomes.

2.1. **Structure of the source space.** The source space uses standard social choice assumptions: Each voter has a complete (each pair can be compared) transitive ranking of all $n \geq 3$ alternatives. Assign each of the $n!$ strict (i.e., no ties) rankings to a particular $\mathbb{R}^{n!}$ axis. A profile (a list of how many voters prefer each ranking) becomes a point in $\mathbb{R}^{n!}$.

To emphasize differences in tallies, let $P(X,Y) = n(X) - n(Y)$, where $n(X)$, $n(Y)$ are, respectively, X's and Y's vote in an $\{X,Y\}$ paired comparison. So if X receives 60 votes and Y has 40, then $P(X,Y) = 20 = -P(Y,X)$. Outcomes are analyzed by orthogonally dividing $\mathbb{R}^{n!}$ into three components. The first is the *kernel*, consisting of all profiles where, for each X and Y, $P(X,Y) = 0$. The second is the *strongly transitive* space, $\mathcal{ST}^n$, where, beyond transitivity, any set of $k \geq 2$ alternatives $\{X_1, X_2, \dots, X_k\}$ satisfies the demanding

$$P(X_1,X_2) + P(X_2,X_3) + \cdots + P(X_{k-1},X_k) = P(X_1,X_k). \tag{2}$$

A basis for the $(n-1)$-dimensional $\mathcal{ST}^n$ is given in (Saari [5]).

The final subspace is the orthogonal complement of the kernel and strongly transitive spaces. This *Cyclic space*, $\mathcal{C}^n$, consists of all profiles and profile components causing cycles, path dependencies, and all possible complexities that can occur with majority votes.

The central construct for a $\mathcal{C}^n$ basis is what I call a *ranking wheel.* As indicated in Fig. 1, it is a rotating wheel with ranking numbers 1 to n uniformly listed along the wheel's edge. Place the names of the n alternatives on the wall. The ranking wheel number adjacent to a name identifies its ranking position. To obtain the next ranking, rotate the wheel to place "1" by the next alternative. Continue until "1" has been by each name; this defines a set of n rankings. Illustrating with Fig. 1, the set is

$$\begin{array}{lll} A \succ B \succ C \succ D \succ E, & B \succ C \succ D \succ E \succ A, & C \succ D \succ E \succ A \succ B, \\ D \succ E \succ A \succ B \succ C, & E \succ A \succ B \succ C \succ D. & \end{array} \tag{3}$$

Notice, the top row of Eq. 3 was used to create the Eq. 1 example with its cyclic behavior.

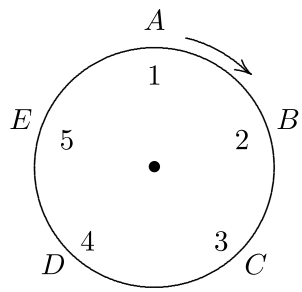


Figure 1. Ranking wheel for $N = 5$.

A ranking wheel array is a Z_n orbit of a ranking. By construction, each candidate is in first, second, ..., last place precisely once, so no candidate is favored: the outcome should be a tie. But by ignoring the full symmetry structure, pairwise outcomes define a cycle. The $D \succ E$ ranking, for instance, appears in the first four Eq. 3 rankings; only in the last is $E \succ D$, which leads to a overwhelming $D \succ E$ victory of 4:1. (So $P(D, E) = 3$.) Reflecting the cyclic action of the ranking wheel, the cyclic majority vote outcome is

$$A \succ B,\ B \succ C,\ C \succ D,\ D \succ E,\ E \succ A, \tag{4}$$

where each has the 4:1 landslide victory. With $n \geq 3$ alternatives, the tallies in this cycle are $(n-1) : 1$, so each tally is only one vote away from unanimity.

Restating Eq. 4 in terms of probabilities, a representative term becomes $P(A, B) = \frac{4-1}{5} = \frac{3}{5}$. Instead of Eq. 2, the example has

$$P(A, B) + P(B, C) + P(C, D) + P(D, E) - P(A, E) = 5 \times \frac{3}{5} = 3.$$

In general, the probabilities associated with a ranking wheel defined by $X_1 \succ \cdots \succ X_n$ significantly violate the strongly transitive constraint Eq. 2 by satisfying $P(X_i, X_{i+1}) = \frac{(n-1)-1}{n} = \frac{n-2}{n}$ and the equality

$$P(X_1, X_2) + P(X_2, X_3) + \cdots + P(X_{n-1}, X_n) - P(X_1, X_n) = n - 2. \tag{5}$$

According to the decomposition, paired comparison outcomes fail the strongly transitive condition (Eq. 2) if and only if ranking wheel components are in the supporting profile. Extremes follow:

Theorem 1. *If k voters have complete transitive preferences over $n \geq 3$ alternatives, and if $P(X_i, X_j) = \frac{n(X_i)-n(X_j)}{k}$, then*

$$P(X_1, X_2) + P(X_2, X_3) + \cdots + P(X_{n-1}, X_n) - P(X_1, X_n) \leq n - 2, \tag{6}$$

where equality holds iff the profile is a multiple of a ranking wheel configuration defined by $X_1 \succ \cdots \succ X_n$.

Differences from equality in Eq. 6 are due to (e.g., [7]) competing components from other ranking wheel configurations and strongly transitive terms.

Theorem 2. *(Saari* [5, 6]*) With respect to pairwise voting, the space of profiles* $\mathbb{R}^{n!}$ *can be orthogonally decomposed into a kernel, where* $P(X_i, X_j) = 0$ *for all* i, j *and all profiles, the* $(n-1)$*-dimensional* $\mathcal{ST}^n$*, and the* $\frac{(n-1)!}{2}$*–dimensional* $\mathcal{C}^n$ *space spanned by ranking wheel configurations.*

Analyzing methods such as the plurality vote involves appropriate subspaces from the Thm. 2 kernel.

2.2. **Unavoidable problems.** As the ranking wheel is the sole source of all majority and supermajority pairwise voting problems, it can be used to subsume and extend the large social choice/voting theory literature that examines these complexities [7]. To illustrate with the seminal Arrow's Theorem [1], consider the challenge of designing a voting rule for voters with complete transitive preferences over $n \geq 3$ alternatives; the rule must produce a complete, transitive societal (i.e., group) ranking. One method is to use the reductionist approach where, to reduce the complexity, paired comparisons are emphasized; i.e., for each pair of alternatives, design a decision rule. Manifesting the gained simplicity is that only information about how each voter ranks this particular pair is used. (This reductionist step corresponds to Arrow's *Independence of Irrelevant Alternatives*, or IIA.)

To eliminate useless choices, the rules cannot have a constant outcome. (Arrow imposes a Pareto condition whereby if everyone has the same ranking of a pair, then that is the pair's societal outcome. To see how his condition satisfies mine, use two unanimity profiles with different rankings of the pair.) Finally, to capture the intent that the outcome reflects information from more than a single source, not all rules can depend just on the preferences of a single voter; there are settings where a different voter changing preferences can alter the outcome of at least one rule. (This is akin to Arrow's "no dictator" condition.)

The approach seems simple, but the objective is impossible to attain: No such method exists. This is the conclusion of Arrow's result (which played a role in his 1972 Nobel Prize) and my above generalized formulation [3, pp 83-100]. A common branding of Arrow's theorem is that "with three or more alternatives, no voting method is fair," but the above description demonstrates that this interpretation is incorrect. Instead, his theorem is a negative commentary about a *methodology*: Using the reductionist approach with paired comparisons guarantees the existence of situations where the outcome violates transitivity.

This difficulty is strictly caused by the ranking wheel structure. These configurations, which define the basis for the Cyclic subspace, are natural connecting links for pairs. But paired comparisons emphasize only "parts," so they do not recognize, nor use, the configuration's linking structure. Instead, paired comparisons *sever* these connecting links: By doing so, they generate cycles that reflect the ranking wheel circular construction. If a profile is free from Cyclic terms (it is strongly transitive), then links are not cut and Arrow's conditions are satisfied by pairwise voting. So, Arrow's Theorem and all problems described in [7] have the same explanation: paired comparisons distort critical information about the underlying structure of the source space (ranking wheel configurations). All possible pairwise difficulties are caused by severing these natural links. As a result, paired comparison outcomes can provide a distorted image of the source space by ignoring—actually, cutting into pieces—the true underlying structure: Expect this effect to happen elsewhere.

If ranking wheel components, which should create complete ties, cause all of the difficulties, then a natural resolution is to strip a profile of these terms and use what remains: This is equivalent to projecting the profile to the strongly transitive subspace. An easier way to accomplish this objective is with the following:

Theorem 3. *With alternatives $\{X_1, X_2, \ldots, X_n\}$, let*

$$B(X_i) = \sum_{j \neq i} P(X_i, X_j). \tag{7}$$

If profile $\mathbf{p} \in \mathcal{C}^n$, then $B(X_j) = 0$ for all j, which means that the $B(X_i)$ rankings depend only on the strongly transitive portion of a profile. If $\mathbf{p} \in \mathcal{ST}^n$, then the majority vote rankings are complete, transitive and agree with the ranking of $B(X_j)$ values.

This $B(X_i)$ approach is equivalent to the Borda Count [6], which tallies ballots by assigning $n - j$ points to the j^{th} positioned candidate. To indicate why Eq. 7 serves as a projection, notice from Eq. 3 that

$$P(A, B) = \frac{4-1}{5}, \quad P(A, C) = \frac{3-2}{5}, \quad P(A, D) = \frac{2-3}{5}, \quad P(A, D) = \frac{1-4}{5}, \tag{8}$$

which sum to zero. As this relationship holds for all candidates and any n, the $B(X_i)$ tally drops a profile's Cyclic components to retain information only about the strongly transitive terms.

3. Space of paired outcomes

To address all ways of making paired comparisons (Saari [8]), use $d_{i,j} \in (-\infty, \infty)$ to compare the $\{X_i, X_j\}$ pair. Similar to the $P(X_i, X_j)$ function, let

$$d_{i,j} = -d_{j,i}, \ d_{j,j} = 0, \ \text{and } d_{i,j} > 0 \text{ implies that } X_i \succ X_j. \tag{9}$$

All of the $n(n-1)$ $d_{i,j}$ values are determined by the $\binom{n}{2}$ $\{d_{i,j}\}_{i<j}$ terms (Eq. 9), so vector

$$\mathbf{d}^n = (d_{1,2}, d_{1,3}, \ldots, d_{1,n}; d_{2,3} \ldots d_{2,n}; d_{3,4} \ldots; d_{n-1,n}) \in \mathbb{R}^{\binom{n}{2}} \tag{10}$$

catalogues all needed information. Changes in the first index are indicated by semicolons.

No interpretation is assigned to $d_{i,j}$; it can represent marginal probabilities, $P(X_i, X_j)$ values, correlation indices, or whatever paired measure is desired. The $d_{i,j}$ could be restricted to the three $\{-1, 0, 1\}$ values, the interval $[-1, 1]$ (e.g., marginal probabilities), or $\mathbb{R}$ (e.g., differences in vote tallies). The $d_{i,j}$ values are observables, so a goal is to develop $\mathbb{R}^{\binom{n}{2}}$ structures to assist in determining properties of an associated source. Most of what is described in Sect. 2 extends.

3.1. A basis for $\mathbb{R}^{\binom{n}{2}}$.

The $\mathbb{R}^{\binom{n}{2}}$ basis identifies which $\mathbf{d}^n$ components lie in well-behaved (transitive) and ill-behaved subspaces.

3.1.1. *The strongly transitive space $\mathcal{ST}^n$.* Assuming the role of Eq. 2 is the *strongly transitive* requirement

$$d_{i,j} + d_{j,k} = d_{i,k}, \quad \text{for all } i, j, k = 1, \dots, n. \tag{11}$$

As with Eq. 2, this expression resembles sums of signed distances where the distance from i to j plus that from j to k equals the signed distance from i to k. In this way, it represents a strong version of transitivity.

Equation 11 restricts $\mathbf{d}^n$ to a $(n-1)$-dimensional[3] *strongly transitive plane:*

$$\mathcal{ST}^n = \{\mathbf{d} \in \mathbb{R}^{\binom{n}{2}} \mid d_{i,j} + d_{j,k} = d_{i,k} \text{ for all } i, j, k = 1, \dots, n\}. \tag{12}$$

A $\mathcal{ST}^n$ basis follows:

Definition 1. *For each $i = 1, \dots, n$, let $\mathbf{B}_i^n \in \mathbb{R}^{\binom{n}{2}}$ be where $d_{i,j} = 1$ for $j \neq i$, $j = 1, \dots, n$, and $d_{k,j} = 0$ if $k, j \neq i$. $\mathbf{B}_i^n$ is called the "X_i basic vector."*

To illustrate with $n = 4$ where $\mathbf{d}^4 = (d_{1,2}, d_{1,3}, d_{1,4}; d_{2,3}, d_{2,4}; d_{3,4})$, the $n - 1 = 3$ basic vectors are

$$\mathbf{B}_1^4 = (1, 1, 1; 0, 0; 0), \quad \mathbf{B}_2^4 = (-1, 0, 0; 1, 1; 0), \quad \mathbf{B}_3^4 = (0, -1, 0; -1, 0; 1). \tag{13}$$

To explain the $\mathbf{B}_3^4$ negative components, Def. 1 mandates $d_{3,1} = d_{3,2} = d_{3,4} = 1$. But the $\mathbf{d}^4$ representation (Eq. 9) requires using $d_{1,3} = -d_{3,1} = -1$ and $d_{2,3} = -d_{3,2} = -1$. Because $\mathbf{B}_4^4 = (0, 0, -1; 0, -1; -1)$, it follows that $\sum_{j=1}^4 \mathbf{B}_j^4 = \mathbf{0}$, which means that the basic vectors define a three-dimensional space.

A direct computation proves that vectors in this space satisfy Eq. 9. Illustrating with

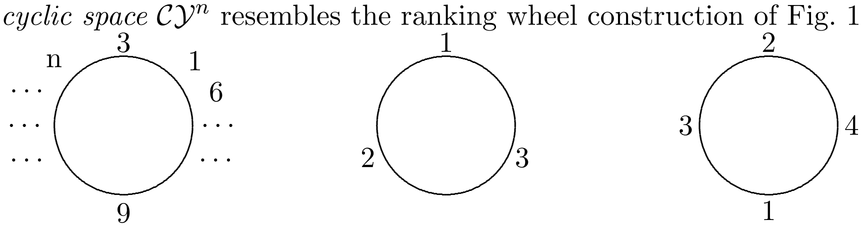
$$\mathbf{d}^4 = a_1\mathbf{B}_1^4 + a_2\mathbf{B}_2^4 + a_3\mathbf{B}_3^4 = (a_1 - a_2, a_1 - a_3, a_1; a_2 - a_3, a_2; a_3),$$

it must be shown that $d_{i,j} + d_{j,k} = (a_i - a_j) + (a_j - a_k)$ equals $d_{i,k} = (a_i - a_k)$, which is immediate.

Theorem 4. [8] *A basis for $\mathcal{ST}^n$ is given by $\{\mathbf{B}_j^n\}_{j=1}^{n-1}$.*

While $\{\mathbf{B}_j^n\}_{j=1}^{n-1}$ is a basis for $\mathcal{ST}^n$, applications emphasize appropriate $\mathcal{ST}^n$ subsets.

3.2. **$\mathcal{CY}^n$; the $\mathcal{ST}^n$ normal space.** The $\mathcal{ST}^n$ plane of strongly transitive entries and its normal space (denoted by $\mathcal{CY}^n$ for "cyclic") define a $\mathbb{R}^{\binom{n}{2}}$ coordinate system. A basis for this *cyclic space* $\mathcal{CY}^n$ resembles the ranking wheel construction of Fig. 1.



a. General case **b.** $n = 3$ **c.** $n = 4$ case

Figure 2. Cyclic arrangements of data

[3]The $d_{i,k}$ values satisfying Eq. 11 can be determined from $\{d_{j,j+1}\}_{j=1}^{n-1}$.

As indicated in Fig. 2, list, in any order, the n indices along the edge of a circe. Moving clockwise about the circle, each integer is preceded by an integer and followed by a different one. In Fig. 1a, for instance, 1 precedes 6 and follows 3.

Definition 2. *Let π be a specified permutation, or listing, of the indices $1, 2, \ldots, n$ around a circle. Define $\mathbf{C}^n_\pi \in \mathbb{R}^{\binom{n}{2}}$ as follows: If j immediately follows i in a clockwise direction, then $d_{i,j} = 1$. If j immediately precedes i, then $d_{i,j} = -1$. Otherwise $d_{i,j} = 0$. Vector $\mathbf{C}^n_\pi$ is the "cyclic direction defined by π".*

With Fig. 2a, $d_{1,6} = 1$ as 6 follows 1, while $d_{1,3} = -1$ because $d_{3,1} = 1$. Figure 2b defines $\mathbf{C}^3_{(1,3,2)} = (-1, 1; -1)$ and Fig. 2c defines $\mathbf{C}^4_{(2,4,1,3)} = (0, 1, -1; -1, 1; 0)$. The $\mathbf{C}^n_\pi$ vectors have obvious properties: Rotating a listing defines the same vector, so there are $\frac{n!}{n} = (n-1)!$ different $\mathbf{C}^n_\pi$'s. Reversing a listing defines $-\mathbf{C}^n_\pi$. (For instance, $\mathbf{C}^4_{(1,4,2,3)} = (0, -1, 1; 1, -1; 0)$ is the negative of $\mathbf{C}^4_{(3,2,4,1)} = (0, 1, -1; -1, 1; 0)$.)

To prove that each $\mathbf{C}^n_\pi$ is orthogonal to each $\mathbf{B}^n_j$, notice that in $\mathbf{B}^n_j$, all $d_{j,k} = 1$ and all other entries are zero. But $\mathbf{C}^n_\pi$ has only two non-zero $d_{j,k}$ values, where one equals -1 and the other equals 1, so $\mathbf{C}^n_\pi$ and $\mathbf{B}^n_j$ are orthogonal. A simple proof that the $\mathbf{C}^n_\pi$ vectors span the normal space for $\mathcal{ST}^n$ is in [8]. Applications may emphasize specific Cyclic $\mathcal{CY}^n$ subsets.

Theorem 5. [8] *The $\mathcal{ST}^n$ normal bundle, denoted by $\mathcal{CY}^n$, is spanned by the set of all $\mathbf{C}^n_\pi$ vectors. Thus $\mathbb{R}^{\binom{n}{2}}$ is orthogonally decomposed into the $(n-1)$-dimensional subspace $\mathcal{ST}^n$ and the $\binom{n-1}{2}$-dimensional $\mathcal{CY}^n$.*

3.3. **Properties.** As with $P(X_i, X_j)$, non-transitive outcomes require $\mathbf{C}^n_\pi$ terms. Extreme settings are defined by these components. For instance, Eq. 5 defined the maximum sum of marginal differences, which come from a ranking wheel configuration. Similarly with $\mathbf{C}^n_\pi$ defined by the listing [1, 2, ..., n], we have the following:

Theorem 6. *If $d_{i,j} \in [-1, 1]$, then the maximum deviation from satisfying the strongly transitive condition is*

$$d_{1,2} + d_{2,3} + \cdots + d_{n-1,n} - d_{1,n} = n - 2 \tag{14}$$

which is defined by $\mathbf{C}^n_{(1,2,\ldots,n)}$. Any vector $\mathbf{d}^n$ that is not a multiple of $\mathbf{C}^n_{(1,2,\ldots,n)}$ will have a strict inequality.

The Eq. 14 behavior requires a specific $\mathbf{C}^n_\pi$, which could represent strictly cyclic inputs or the aggregation of strongly transitive inputs satisfying the ranking wheel structure. Different choices of inputs lead to differences in the dimension of the source space. Notice, Eq. 14 holds independent of what the $d_{i,j}$ terms model; whether it is marginal probabilities, correlations, etc.

Assuming the $B(X_j)$ role (Eq. 7) is the *Borda assignment rule* (BAR) defined as

$$\bar{b}_j = \sum_{i=1}^{n} d_{i,j}, \quad j = 1, \ldots, n. \tag{15}$$

Similar to $B(X_j)$, BAR (Eq. 15) eliminates cyclic components. Indeed, the $\overline{b}_j$ value of any $\mathbf{C}^n_\pi$ is zero because $\mathbf{C}^n_\pi$ has only two non-zero $d_{j,k}$ terms where one is positive and the other negative, so they cancel in the $\overline{b}_j$ summation. Thus $\overline{b}_j$ serves as a projection of $\mathbf{d}$ from $\mathbb{R}^{\binom{n}{2}}$ to its strongly transitive $\mathcal{ST}^n$ space. The way in which the $\overline{b}_j$ functions handle strongly transitive terms is specified in the following:

Theorem 7. [8] *For any $n \geq 3$ (where $a_n = 0$), let $\mathbf{d}^n = \sum_{j=1}^{n-1} a_j \mathbf{B}^n_j + \sum_k c_k \mathbf{C}^n_k$. Then $\sum_{j=1}^n \overline{b}_j = 0$, and*

$$\overline{b}_i - \overline{b}_j = n(a_i - a_j) \quad \text{for all } i, j \leq n. \tag{16}$$

Thus the BAR ranking for $\mathbf{d}^n$ reflects the ordering of the coefficients in its strongly transitive component; cyclic terms are ignored. If $a_i > a_j$, then independent of any cyclic terms, it must be that $\overline{b}_i > \overline{b}_j$.

3.4. **An interesting phenomenon:** It is reasonable to expect that dropping an alternative has a minimal effect on a ranking. For instance, with a $X_1 \succ X_2 \succ X_3 \succ X_4$ ranking, if alternative X_4 is dropped, then the $X_1 \succ X_2 \succ X_3$ outcome would be anticipated. But this need not be the case! The sole source of this effect are cyclic components.

Theorem 8. [8] *Let $\Pi^n : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n-1}{2}}$ be the projection mapping defined by dropping the k^{th} alternative. Then*

$$\Pi^n(\mathbf{B}^n_j) = \mathbf{B}^{n-1}_j, \quad j \neq k. \tag{17}$$

But $\Pi^n(\mathbf{C}^n_\pi)$ is the sum of a multiple of $\mathbf{C}^{n-1}_{\pi^*}$ and non-zero strongly transitive terms, where π^* is the listing obtained from π by removing index k. Thus $\Pi^n(\mathbf{d}^n) \in \mathcal{ST}^{n-1}$ iff $\mathbf{d}^n \in \mathcal{ST}^n$.

Proving Eq. 17 is a simple computation. To illustrate the cyclic term assertion, notice that $\mathbf{C}^4_{(1,2,3,4)} = (1, 0, -1; 1, 0; 1)$. Dropping X_4 removes all $d_{j,4}$ terms from $\mathbf{C}^4_{(1,2,3,4)}$ to create $\mathbf{d}^3 = (1, 0; 1)$, where

$$\mathbf{d}^3 = (1, 0; 1) = (\frac{1}{3}, \frac{2}{3}; \frac{1}{3}) + (\frac{2}{3}, \frac{-2}{3}; \frac{2}{3}) = [\frac{2}{3}\mathbf{B}^3_1 + \frac{1}{3}\mathbf{B}^3_2] + \frac{2}{3}\mathbf{C}^3_{(1,2,3)}. \tag{18}$$

As Eq. 18 illustrates, dropping alternatives from cyclic terms creates cyclic terms *plus* strongly transitive terms. (The $\mathcal{ST}^3$ term in Eq. 18 defines the $X_1 \succ X_2 \succ X_3$ ranking.) These unexpected $\mathcal{ST}^n$ components can cause a rule's ranking (definitely its weights) to differ. The BAR ranking for $3\mathbf{B}^4_1 + 2\mathbf{B}^4_2 + \mathbf{B}^4_3 - 9\mathbf{C}^4_{(1,2,3,4)}$ is $X_1 \succ X_2 \succ X_3 \succ X_4$, for instance, but by dropping X_4 (see Eq. 18) it becomes the reversed $X_3 \succ X_2 \succ X_1$. What adds concern to this feature is that *all major decision rules are affected by $\mathcal{ST}^n$ data components.*

Theorem 9. [8] *If $\mathbf{d}^{n+k} \in \mathcal{ST}^{n+k}$, then the rankings defined by $d_{i,j}$ terms and the BAR outcome for $\mathbf{d}^{n+k}$ agree with the outcome obtained after projecting $\mathbf{d}^{n+k}$ to any subset of n alternatives. (The projection is in $\mathcal{ST}^n$.) Conversely, if $\mathbf{d}^n \in \mathcal{ST}^n$ is the projection of $\mathbf{d}^{n+k}$, then $\mathbf{d}^{n+k} \in \mathcal{ST}^{n+k}$.*

If a BAR ranking changes after dropping an alternative, then $\mathbf{d}^{n+1}$ *has* $\mathcal{CY}^{n+1}$ *components. With a sufficiently strong* $\mathcal{CY}^{n+1}$ *component, the rankings for* n *and* $n+1$ *alternatives can differ in any desired manner. However, the* $\overline{b}_i$ *value for* $(n+1)$ *alternatives is* $1/(n-1)$ *times the sum of the* $\overline{b}_i$ *values over all subsets of* n *alternatives.*

This difficulty, strictly caused by $\mathcal{CY}^n$ terms, can be expected to affect *all* major paired comparison rules. But at least BAR retains some level of regularity because (Thm. 9) the averaged outcome over all ways to drop an alternative remains consistent. The dimension of $\mathcal{CY}^n$ exceeds that of $\mathcal{ST}^n$ for $n \geq 5$, which means that these difficulties can become commonplace with larger n values.

3.5. **A QQ example.** To conclude, the basis is illustrated with the 1997 Gallup poll information (as reported in [10]) concerning the perceived honesty of President Clinton and Vice President Gore. The four alternatives are

- X_1 is a positive opinion of Clinton, X_2 is a positive opinion of Gore,
- X_3 is a negative opinion of Clinton, X_4 is a negative opinion of Gore.

Let $s_{i,j}$ be the fraction of people with X_i as the first outcome and X_j as the second. For instance, $s_{1,4} = 0.0447$ is the reported fraction of people who view Clinton as honest (X_1) and Gore as dishonest (X_4) when asked in that order. In contrast, $s_{4,1} = 0.0255$ is the fraction with these opinions when asked about Gore first and Clinton second. To capture order effects, let $d_{i,j} = s_{i,j} - s_{j,i}$. Thus $d_{1,4} = 0.0447 - 0.0255 = 0.0192$.

Vector $\mathbf{d}$ equals $\mathbf{0}$ if there are no order effects, so how $\mathbf{d}$ deviates from zero captures ways in which the order matters. The Gallup information [10] defines the context vector

$$\mathbf{d} = (-0.0726, 0, 0.0192; 0.0224, 0; 0.0786). \tag{19}$$

The zeros represent unavailable, but presumably zero values for changed opinions about the same person; e.g., $d_{2,4}$ is where Gore is viewed as honest and then dishonest. These hidden variables play a subtle role in the QQ equalities. The BAR values for $\mathbf{d}$ are $\overline{b}_1 = -0.0534, \overline{b}_2 = 0.0950, \overline{b}_3 = 0.0562, \overline{b}_4 = -0.0978$ defining the $X_2 \succ X_3 \succ X_1 \succ X_4$ ranking. The strongly positive opinion of Gore is reflected by how the $X_2 \succ X_4$ inequality is separated by $X_3 \succ X_1$ and its negative assessment of Clinton.

A constraint on $\mathbf{d}^4$ comes from the modeling. The sum of the probabilities with Clinton first must equal unity, $[s_{1,2} + s_{1,4}] + [s_{3,2} + s_{3,4}] = 1$, with the same for Gore, $[s_{2,1} + s_{2,3}] + [s_{4,1} + s_{4,3}] = 1$, so

$$d_{1,2} + d_{1,4} - d_{2,3} + d_{3,4} = 0. \tag{20}$$

The QQ equalities assert that YY path effects are countered by NN effects with $d_{1,2} + d_{3,4} = 0$, and YN terms match NY values with $d_{1,4} = d_{2,3}$. According to Eq. 20, should either equality hold, then so does the other. In particular, the QQ equations hold if $\mathbf{d}^4$ is orthogonal to $\mathbf{C}_{1,2,34}$, which is

$$0 = < \mathbf{d}^4, \mathbf{C}_{(1,2,3,4)} > = d_{1,2} + d_{2,3} + d_{3,4} - d_{1,4}. \tag{21}$$

As proved next, Eq. 21 is a necessary and sufficient condition for the QQ expressions.

Theorem 10. *If* $\mathbf{C}_{(1,2,3,4)}$ *is orthogonal to* $\mathbf{d}^4$ *(Eq. 21), then* $\mathbf{d}^4$ *has the idealized QQ vector form*

$$\tilde{\mathbf{d}} = (-\alpha, 0, \beta; \beta, 0; \alpha), \tag{22}$$

where the QQ equalities hold. Conversely, if QQ holds, then so does Eq. 22, and its decomposition is

$$\tilde{\mathbf{d}} = \frac{\beta}{2}\mathbf{B}_1^4 + \frac{\alpha+\beta}{2}\mathbf{B}_2^4 + \frac{\alpha}{2}\mathbf{B}_3^4 - \frac{\alpha}{2}\mathbf{C}^4_{(1,2,4,3)} + \frac{\beta}{2}\mathbf{C}^4_{(1,4,2,3)}, \tag{23}$$

so $\tilde{\mathbf{d}}$ *is orthogonal to* $\mathbf{C}^4_{(1,2,3,,4)}$. *The Eq. 22 BAR values are* $\overline{b}_1 = \beta - \alpha, \overline{b}_2 = \beta + \alpha, \overline{b}_3 = \alpha - \beta, \overline{b}_4 = -\alpha - \beta$.

Proof: Solving Eqs. 20, 21 leads to Eq. 22. □

To show how Thm. 10 involves the hidden variables $d_{1,3} = d_{2,4} = 0$ equations, use the decomposition

$$\mathbf{d}^4 = a_1\mathbf{B}_1^4 + a_2\mathbf{B}_2^4 + a_3\mathbf{B}_3^4 + c_1\mathbf{C}^4_{(1,2,3,4)} + c_2\mathbf{C}^4_{(1,2,4,3)} + c_3\mathbf{C}^4_{(1,4,2,3)}$$

with these two equations to obtain that $c_2 = \frac{1}{2}[a_1 - a_2 - a_3)], c_3 = \frac{1}{2}[a_1 + a_2 - a_3]$. Substituting into Eq. 20 leads to the $a_1 - a_2 + a_3 = 0$ equality, which defines the form

$$\mathbf{d}^4 = (-2a_3 + c_1, 0, 2a_1 - c_1; 2a_1 + c_1, 0; 2a_3 + c_1). \tag{24}$$

According to Eq. 24, the QQ equalities hold iff $c_1 = 0$ iff $\mathbf{d}^4$ and $\mathbf{C}_{1,2,3,4}$ are orthogonal. In other words, a measure of the QQ relationship is how closely Eq. 21 is satisfied. (So an answer to (i) on the first page is that a QQ setting occurs iff there is a $(Clinton_{yes}, Gore_{yes}) + (Gore_{yes}, Clinton_{no}) + (Clinton_{no}, Gore_{no}) + (Gore_{no}, Clinton_{yes})$ canceling relationship.) Indeed, the decomposition of $\mathbf{d}$ (from Eq. 19) is

$$\mathbf{d} = 0.0111\mathbf{B}_1^4 + 0.0482\mathbf{B}_2^4 + 0.0385\mathbf{B}_3^4 + [0.0023\mathbf{C}^4_{(1,2,3,4)} - 0.0378\mathbf{C}^4_{(1,2,4,3)} + 0.0104\mathbf{C}^4_{(1,4,2,3)}] \tag{25}$$

with its small $\mathbf{C}^4_{(1,2,3,4)}$ component. The cyclic $\mathbf{C}^4_{(1,2,3,4)}$ term, then, measures whether QQ holds. When it does, this term (which *never appears* with an idealized QQ vector (Thm. 10)) represents noise.

4. Summary

Guided by conclusions discovered in pairwise voting, a basis can be created to examine paired comparisons. An advantage of this straightforward basis is that, as it involves familiar linear algebra and vector analysis concepts, only a minimal learning curve is needed to use it. As the basis can handle all paired relationships, these simple tools recapture conclusions based on more complicated assumptions and structures, while offering simpler alternative interpretations. For instance, the QQ equalities are based on the "law of reciprocity" from quantum theory. According to Thm. 10, the effect of this reciprocity law is related to, and can be replaced by, the orthogonality of the two vectors $\mathbf{C}^4_{(1,2,3,4)}$ and $\mathbf{d}^4$.

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