Unifying voting theory from Nakamura’s to Greenberg’s theorems

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HIGHLIGHTS

- The source of all paired comparison voting difficulties is identified.
- This paper extends Sen’s theorem avoiding majority vote cycles to all q-rules.
- A new interpretation is given for Nakamura’s number.
- Greenberg’s spatial voting theorem for a core is generalized.

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ABSTRACT

Cycles, empty cores, intransitivities, and other complexities affect group decision and voting rules. Approaches that prevent these difficulties include the Nakamura number, Greenberg’s theorem, and single peaked preferences. The results derived here subsume and significantly extend these assertions while providing a common explanation for these seemingly dissimilar conclusions.

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1. Introduction

Decision and economic issues involving paired comparison decision and voting methods can be plagued by cyclic outcomes, empty cores, and other difficulties. The extensive nature and practical consequences of these problems are captured by Arrow’s Impossibility Theorem (Arrow, 1951), path dependency problems where inferior choices can be selected (McKelvey, 1979), questions raised about court decisions (Cohen, 2010, 2011), and even concerns about perversities in our laws (Katz, 2012).

In response to these difficulties, a variety of seemingly dissimilar results (described below), such as Nakamura’s number, Greenberg’s theorem, and single peaked preferences, describe ways to avoid these problems. But as conditions for applications can differ from what is specified in the theorems, it is necessary to appreciate why the results are true and to obtain stronger conclusions. Unfortunately, an understanding of these approaches can be obscured by their technical mathematical proofs.

Thus, a surprising conclusion developed here is that these results have a common, simple explanation. This is because the culprits causing N-alternative paired comparison mysteries are profiles dominated by what are called “ranking wheel configurations” (denoted by \( \mathcal{RWC}_N \) and introduced in Section 2). By identifying a single source for all of these difficulties, new and stronger conclusions are derived. Two new, general theorems developed here (Theorems 4 and 5), for instance, subsume all of the above conclusions while providing a significantly wider range of applicability.

For notation, the \( N \) alternatives \( \{a_1, \ldots, a_N\} \) are represented by capital letters \( A, B, \ldots \) with specific examples. Each of the \( n \) voters is assumed to have complete, transitive preferences over the specified alternatives where “\( A \succ B \)” represents “strictly preferred to;” e.g., \( A \succ B \) means “\( A \) is strictly preferred to \( B \)”. The symbol “\( \sim \)” refers to a “tied” ranking; e.g., \( A \sim B \) means that in an \( \{A, B\} \) election, neither \( A \) beats \( B \) nor \( B \) beats \( A \). A profile lists the preferences for the \( n \) voters; in what follows, the \( \mathcal{RWC}_N \) terms are components of profiles. (Proofs are in the Appendix.)

1.1. Supermajority voting rules

In supermajority voting, a winning proposition must receive at least a quota of \( q \) votes. In the US Senate, for instance, a vote must receive at least 60 of the 100 possible votes to avoid a filibuster.

Definition 1. A \( q \)-rule with \( n \) voters, denoted by \( (q, n) \), is where \( q > \frac{n}{2} \) and a winning proposition receives at least \( q \) of the \( n \) votes.

While majority vote cycles occur with three alternatives, it is reasonable to expect that \( q \)-rule cycles require more alternatives. This is true, and Nakamura determined the number of alternatives needed to construct them.

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Theorem 1 (Nakamura, 1978). Nakamura’s number for a \((q, n)\)-rule is \(v(q, n) = \lceil \frac{n}{q} \rceil\), where \(\lfloor x \rfloor\) is the smallest integer greater than or equal to \(x\). A \(q\)-rule cycle never occurs with \(N < v(q, n)\) alternatives. With \(N \geq v(q, n)\) alternatives, there are profiles with \(q\)-rule cycles.

A notational term used later is \(\lfloor x \rfloor\), which is the largest integer less than or equal to \(x\). So, \(\lfloor 4.3 \rfloor = 4\) “rounds up” while \(\lfloor 4.3 \rfloor = 4\) “rounds down.”

With the US Senate’s \(q\)-rule, Theorem 1 asserts that “60-vote” cycles can occur with \(v(60, 100) = \lceil \frac{100}{60} \rceil = 3\) alternatives. While a three-fourths rule, requiring 75 votes, is spared three-alternative cycles, it admits \(v(75, 100) = \lceil \frac{100}{75} \rceil = 4\) alternative cycles. In developing a new interpretation for \(v(q, n)\), other results are obtained. For instance, can all supporting examples be identified? When using Theorem 1 to anticipate voting problems, are the cycles likely, or rare enough to be safely ignored? Can we, for example, safely ignore the possibility of a three-alternative 60-vote cycle in the US Senate? To the best of my knowledge, these kinds of issues have not been investigated; results directed toward these concerns are developed here.

Another way to avoid cycles is to restrict profiles. A popular choice is Black’s single peaked condition (Black, 1958) where, for three alternatives, nobody has the alternative ranked. Ward (1965) generalized Black’s condition, and Sen (1966) used Ward’s results to extend Black’s majority vote result to a best possible conclusion for any number of alternatives. Although several authors (e.g., Dummett and Farquharson, 1961, Nakamura, 1975, Pattanaik, 1971, and Salles and Wendell, 1977), used “simple games” to seek extensions to \((q, n)\)-rules, general restrictions holding for all \((q, n)\)-rules had not been developed. This elusive objective finally is resolved here. This best possible result (Theorem 4) for \((q, n)\)-rules includes Nakamura’s and Sen’s results as special cases.

1.2. Spatial voting

Similar difficulties arise in spatial voting where each axis of an Euclidean space represents a different issue; e.g., a \(k\)-issue setting is modeled with points in \(\mathbb{R}^k\). The coordinates of an agent’s ideal point in \(\mathbb{R}^k\) reflect his ideal combination of the level of each issue. Preferences often are modeled in terms of the Euclidean distance between an agent’s ideal point and the points labeled with the issues. Ward (1965) generalized Black’s condition, and Sen (1966) used Ward’s results to extend Black’s majority vote result to a best possible conclusion for any number of alternatives. Although several authors (e.g., Dummett and Farquharson, 1961, Nakamura, 1975, Pattanaik, 1971, and Salles and Wendell, 1977), used “simple games” to seek extensions to \((q, n)\)-rules, general restrictions holding for all \((q, n)\)-rules had not been developed. This elusive objective finally is resolved here. This best possible result (Theorem 4) for \((q, n)\)-rules includes Nakamura’s and Sen’s results as special cases.

Theorem 2 (Greenberg, 1979). For \(k\)-issue spatial voting, let domain \(D \subset \mathbb{R}^k\) be a compact (i.e. closed and bounded), \(k\)-dimensional convex subset. A necessary and sufficient condition for \(C(q, n) \neq \emptyset\) for any positioning of the \(n\) ideal points in \(D\) is if \(k \leq v(q, n) - 2\).

Fig. 1. Creating a \(\mathcal{R}\)W\(\mathcal{C}_N\) profile.

With \(q\)-rules, then, the “single issue” assumption commonly used with voting models (to avoid cycles) can be relaxed to allow up to \(v(q, n) - 2\) issues.

It is important to extend Theorem 2 to more general settings. It is doubtful, for instance, whether Theorem 2 is appropriate to examine consequences of laws. The generalization developed here (Theorem 5) provides the strongest possible extension; e.g., it subsumes Theorems 2 and 4 (hence Nakamura’s and Sen’s conclusions). In this way, all of these major results are extended and unified by one general result.

2. Ranking wheel

What simplifies the analysis is that all of the “trouble creating profiles” now are known; they are strictly caused by \(\mathcal{R}\)W\(\mathcal{C}_N\) terms (introduced below). It has been known since Condorcet’s 1785 article (Condorcet, 1785) that these terms cause cycles and paired comparison problems, but it had not been known whether there were other aspects of a profile that could cause difficulties. This gap is what limited progress; e.g., the need to circumvent this missing information is why the proofs of results (e.g., Theorem 2) can be mathematically technical and involve restrictive assumptions.

In a decomposition of all three-candidate profiles, it was proved (Saari, 1999) that all three-alternative paired comparison difficulties are due to \(\mathcal{R}\)W\(\mathcal{C}_3\) terms, and only these terms. (Others, such as Sen, 1966 and Zwicker, 1991, used these terms to discuss three-candidate transitivity problems.) But the issues described here involve more than three alternatives, which requires the stronger conclusion that \(\mathcal{R}\)W\(\mathcal{C}_N\) terms, and only \(\mathcal{R}\)W\(\mathcal{C}_N\) terms, cause the problems for any \(N \geq 3\); This result is proved in Saari (2000) as part of a decomposition of \(N\)-candidate profiles. (Also see Saari, in press.)

Nothing else can cause pairwise difficulties, so all problems and negative conclusions about paired comparisons are caused by, and reflect properties of, \(\mathcal{R}\)W\(\mathcal{C}_N\) terms. As shown in Saari (2008, Chapter 2), for instance, these \(\mathcal{R}\)W\(\mathcal{C}_N\) terms explain Arrow’s Impossibility Theorem (Arrow, 1951), Sen’s Minimal Liberalism result (Sen, 1970), and other negative paired comparisons assertions. What was not previously explored is whether these terms can explain, unify, and significantly extend the above described results so that they hold for more general settings. This is done here.

2.1. Ranking wheel

To define a \(N\)-alternative “ranking wheel configuration” \(\mathcal{R}\)W\(\mathcal{C}_N\) (Saari, 2008), attach a freely rotating wheel to a surface. Place in an equally spaced manner the numbers 1, 2, …, \(N\) along the wheel’s edge. As illustrated in Fig. 1, list the candidates’ names on the surface in the order determined by a desired initial ranking; e.g., the \(j\)th ranked candidate’s name is positioned next to number \(j\).

This list defines the first ranking in Fig. 1 it is \(A > B > C > D > E > F\). Rotate the wheel to place “1” by the next candidate and read
off the second ranking; in Fig. 1, moving “1” next to B defines the second ranking of \(B > C > D > E > F > A\). Create rankings in this manner until each candidate is in first place precisely once to define \(N\) rankings. The \(RWC\) for Fig. 1 consists of the six rankings
\[
\begin{align*}
& A > B > C > D > E > F, \\
& B > C > D > E > F > A, \\
& C > D > E > F > A > B, \\
& D > E > F > A > B > C, \\
& E > F > A > B > C > D, \\
& F > A > B > C > D > E.
\end{align*}
\]

Similarly, the \(RWC\) defined by \(A > B > C\) is the Condorcet triplet
\[
\begin{align*}
& A > B > C, \\
& B > C > A, \\
& C > A > B.
\end{align*}
\]

while the \(RWC\) defined by the reversed \(C > B > A\) is
\[
\begin{align*}
& C > B > A, \\
& B > A > C, \\
& A > C > B.
\end{align*}
\]

A \(RWC\) (e.g., Eq. (1)) has each candidate in first, second, etc., last place precisely once, which makes it arguable that no candidate is favored: This completely tied outcome is satisfied by all positional rules. (A “positional rule” (Riker, 1982) tallies ballots by assigning specified points to candidates based on their ballot position; e.g., the plurality vote assigns a point only to a voter’s first place candidate.) But this completely tied outcome fails to hold with majority votes over pairs!

To see this with Eq. (1), start with \([E, F]\). The first five rankings have \(E > F\); only the last ranking has \(F > E\), so \(E\) beats \(F\) by a 5:1 vote. As this same phenomenon happens with each adjacent pair of candidates, Eq. (1) defines the cycle
\[
\begin{align*}
& F > A, \\
& A > B, \\
& B > C, \\
& C > D, \\
& D > E, \\
& E > F
\end{align*}
\]

where each outcome has the decisive 5:1 tally. More generally (Proposition 1), \(RWC\) defines a pairwise voting cycle over the \(N\) alternatives (given by adjacent entries) where each has a \((N - 1)\):1 tally: Notice, each tally is but one voter’s choice away from unanimity! These cyclic outcomes reflect both the cyclic way in which \(RWC\) is constructed and the rule’s myopic limitations where, by emphasizing only small portions of available information from \(RWC\), the rule cannot recognize global symmetries that may demonstrate that the outcome should be a tie.

The tallies for non-adjacent alternatives are not as decisive. With Eq. (1), the \(A > C\) tally is 4:2 rather than 5:1, while the \(A \sim D\) tally is the 3:3 tie. The general situation is described in Proposition 1.

**Proposition 1.** With a \(RWC\) generated by \(a_1 > a_2 > \cdots > a_N\), the tallies are
\[
\begin{align*}
& N - s : s \quad \text{for } a_s > a_{s+1}, 1 \leq s < \frac{N}{2} \quad \text{and} \\
& a_i \sim a_{j+s} \quad \text{if } s = \frac{N}{2}.
\end{align*}
\]

Here, \(a_{i+k}\) is identified with \(a_k\). Because the tallies tighten for \(s \geq 2\) (Eq. (4)), non-adjacently ranked alternatives in \(RWC\) play no role in what follows.

It is interesting how similar tallies arise by dropping rankings.

**Proposition 2.** Removing \(s\) rankings from a \(RWC\) defined by \(a_1 > a_2 > \cdots > a_N\) causes \(s\) pairs of adjacent candidates to have an unanimous \(a_j > a_{j+1}\) outcome. The common tally for all remaining \(a_j > a_{j+1}\) outcomes is \([\lfloor (N - s) - 1 \rfloor] = 1\).

To illustrate with \(s = 3\) and the \(RWC\) in Eq. (1), by dropping the three rankings on the right-hand side, the \(A > B, C > D,\) and \(E > F\) outcomes now are unanimous while the \(B > C, D > E,\) and \(F > A\) tallies become 2:1 (which are \(RWC\) tallies) rather than 5:1. A profile consisting of two copies of the remaining three Eq. (1) rankings admits a (4, 6)-rule cycle, but not a (5, 6)-cycle. This is because the \(B > C, D > E,\) and \(F > A\) tallies are 4:2, which the (5, 6)-rule treats as ties. For this profile, then, \(C(4, 6) = \emptyset\) and \(C(5, 6) = \{A, C, E\}\).

2.2. Worst case scenarios

If a restriction (e.g., Nakamura’s number, Greenberg’s Theorem) is to prevent cycles, it must handle “worst case scenarios”; i.e., the restriction must exclude profiles with extreme tallies that cause problems. This reality makes it easier to explain and extend conclusions because worst case scenarios must be created with \(RWC\) profiles.

It follows from Proposition 1 that the most dramatic pairwise tallies in a cycle involve adjacently ranked \(RWC\) alternatives; indeed, each \(RWC\) tally for a pair is but one vote away from unanimity. But if a profile includes two different \(RWC\)’s, some pair cannot be adjacent with the same ranking in both \(RWC\)’s. (Each \(N\)-candidate ranking is in only one \(RWC\), there are precisely \((N - 1)!\) ranking wheels.) This lowers the pair’s relative tally in a cycle, so it fails to constitute a worst case setting.

In words, profiles creating worst case scenarios consist strictly of multiples of the same \(RWC\). Therefore, an analysis of paired comparison assertions reduces to examining what happens with these special profiles. As shown starting with the Nakamura number, this approach significantly simplifies the analysis.

3. Supermajority votes: Nakamura’s number

Nakamura sought the maximum number of alternatives, \(N\), with which it is impossible to create a \((q, n)\)-rule cycle. Paired comparison cycles are caused by \(RWC\) profile components, so Nakamura’s concern now is easy to answer; i.e., find the critical \(N\) value so that appropriate multiples of a \(RWC\) never admit \((q, n)\)-rule cycles.

To illustrate by finding this \(N\) for the \((90, 100)\)-rule, we now know that to create a 11-alternative cycle with extreme tallies we must use a \(RWC_{11}\). This choice yields a cycle with 10:1 tallies, so a \(q\)-cycle with \(q = 90\) requires using \(\frac{90}{10} = 9\) copies of the \(RWC_{11}\). As a \(RWC_{11}\) has eleven voters, the nine copies define the preferences for 99 of the 100 voters; the last voter’s preferences in the profile can be anything.

Similarly with \(N = 10\); as the \(RWC_{11}\) cycle has 9:1 tallies, \(\frac{90}{9} = 10\) copies are needed. As each \(RWC_{10}\) has ten voters, this determines the preferences of all 100 voters to create a profile with a cycle. With \(N = 9\), the \(RWC_{10}\) cycle has 8:1 tallies, so \(\frac{80}{2} = 12\) copies are needed. Twelve copies require 12 \times 9 = 108 voters where there are only 100 available, so it is impossible to create such a profile. Thus, to avoid \((90, 100)\)-rule cycles, use no more than \(N = 9\) alternatives.

This computation describes the approach used throughout this paper: Find the multiples of a \(RWC_N\) that solve a specified problem and then determine whether there are enough voters to create the profile. Indeed, generalizing the above, a profile consisting of an integer multiple of \(RWC_N\) terms defines a \((q, n)\)-rule cycle where \(N - 1\) points are assigned to each pair’s winning alternative and \(\frac{q}{N - 1}\) copies of the \(N\)-voter \(RWC_N\) are needed. The full profile cannot have more than \(n\)-voters, so \(N\) must satisfy
\[
\left\lceil \frac{q}{N - 1} \right\rceil N \leq n.
\]
3.1. Fractional parts of a \( RWC_n \)

The above analysis uses integer multiples of \( RWC_k \)'s. Sharper statements follow by using fractional values. To explain this comment by illustrating with the (11, 13) rule, because \( N = 9 \) does not satisfy Eq. (5), it is impossible to construct an 11-cycle using integer multiples of \( RWC_9 \). But, as shown next, such a profile can be created with fractional multiples.

To describe the process, one copy of \( RWC_9 \) yields a cycle with 8 1-tallies, so the winner of each pair needs 3 more points to reach the 11-cycle. According to Proposition 2, including four more ranking from the \( RWC_9 \) adds either three or four points to each winning candidate's tally in the pairs; this creates the desired 11-cycle among all nine candidates. In a real sense, this profile uses \( 1/3 \) copies of \( RWC_9 \). The \( 1/3 \) copies involve 13 voters, each winning candidate in the cycle receives at least 11 votes, so the desired 11-cycle with nine alternatives is constructed.

Stated in general terms, \( k \) copies of a \( RWC_n \) have \( k(N-1) : k \) tallies. To maximize the tally differences and reach quota, choose the largest possible integer \( k \) so that

\[
q = k(N-1) + \gamma, \quad 0 \leq \gamma < N - 1. \tag{6}
\]

To describe the number of copies, let \( \alpha = 0 \) when \( \gamma = 0 \). Otherwise, let \( \alpha = \gamma + 1 \). So \( \alpha \), which satisfies \( 2 \leq \alpha < N \), is the number of extra \( RWC_n \) rankings needed to add \( \gamma \) votes to achieve quota. Thus \( k = \lfloor \frac{q}{N-1} \rfloor \). (Recall, \( \lfloor \frac{q}{N-1} \rfloor \) rounds down.) It remains to ensure there are enough voters (Eq. (7)) to create the profile.

**Proposition 3.** An \( N \)-candidate profile exists with a \( (q, n) \)-cycle if and only if

\[
\left| \frac{q}{N - 1} \right| N + \alpha \leq n \tag{7}
\]

where \( \alpha = 0 \) if \( \frac{q}{N-1} \) is an integer (i.e., if \( q = \lfloor \frac{q}{N-1} \rfloor (N-1) \)); otherwise \( \alpha = q - \lfloor \frac{q}{N-1} \rfloor (N-1) + 1 \geq 2 \). One such profile includes \( \lceil \frac{q}{N-1} \rceil + \frac{\alpha}{N-1} \) copies of \( RWC_n \).

Eq. (7) is the generalized version of Eq. (5). (If \( \alpha = 0 \), then \( \left| \frac{q}{N - 1} \right| = \left| \frac{q}{N-1} \right| \), so the equations agree.) To illustrate Proposition 3 with the \((42, 46)\) rule and \( N = 13 \), as \( k = \lfloor \frac{42}{12} \rfloor = 3 \) and \( \alpha = 42 - 3(12) + 1 = 7 \) (so, each pair's winner needs \( \alpha = 6 \) more votes to create the cycle), the Eq. (7) inequality requires \( 3(13) + 7 \leq 46 \) voters, which is satisfied. Thus a profile that generates the 42-cycle uses three copies of \( RWC_{13} \) plus seven of its rankings, or \( 34 \) copies of \( RWC_{13} \). To further illustrate with an extreme example where \( N = 50 \), here \( k = \lfloor \frac{42}{12} \rfloor = 3 \) and \( \alpha = 42 - 3(49) + 1 = 41 \), so a profile with a 42-cycle uses 43 rankings from a \( RWC_{50} \); it has \( 42 \) copies of \( RWC_{50} \). (The last three voters can be assigned any desired ranking.)

The sharp Proposition 3 results suggest revisiting the (90, 100) rule to determine whether \( N = 9 \) suffices. Here \( k = \lfloor \frac{90}{11} \rfloor = 8 \) and \( \alpha = 90 - 11(8) + 1 = 3 \). As these values fail the Eq. (7) inequality (the construction requires \( 11 \times 3 = 102 \) voters from the available 100), it is impossible to create a 90-cycle with nine alternatives.

The question raised (and solved) by Nakamura now can be answered by using \( RWC_n \)'s. To motivate the assertion, if \( \frac{q}{N-1} \) is an integer for the minimal \( N \) value (so \( \alpha = 0 \)), it follows from Eq. (7) that \( \frac{q}{N-1} \) \( \leq n \), or \( N \geq q/N-1 \). As \( N \) is an integer, \( N = \left\lceil \frac{q}{N-1} \right\rceil \).

In general, and as asserted in Theorem 1, the smallest number of alternatives with a \( (q, n) \)-rule cycle is given by the Nakamura number \( v(q, n) = \left\lceil \frac{q}{N-1} \right\rceil \).

A new interpretation for \( v(q, n) \) follows from its role of describing the profile structure needed to have \( N \)-alternative \( q \)-cycles: Profiles must include copies of \( RWC_n \)'s where \( N \geq v(q, n) \).

Eq. (7) specifies how to create a cycle; e.g., a \( v(q, n) \) candidate \( q \)-cycle is constructed with \( \left\lceil \frac{q}{v(q, n)-1} \right\rceil + \frac{\alpha}{v(q, n)} \) copies of a \( RWC_{v(q, n)} \). If fewer than all \( n \) voters are needed, then either arbitrarily assign rankings to the remaining voters, or replace some \( RWC_{v(q, n)} \) rankings with combinations of other rankings to reduce the number of \( RWC_{v(q, n)} \) copies. All of this is specified in Theorem 3.

**Theorem 3.** A \( (q, n) \)-cycle exists with \( N \) alternatives if and only if \( N \geq v(q, n) \).

To have a \( v(q, n) \) voter profile where a maximum number of voters can be assigned arbitrary preferences, it must have \( \left\lceil \frac{q}{v(q, n)-1} \right\rceil + \frac{\alpha}{v(q, n)} \) copies of the \( RWC_{v(q, n)} \) where \( \alpha = 0 \) if \( \frac{q}{v(q, n)-1} \) is an integer, otherwise \( \alpha = 1 + q - \left\lceil \frac{q}{v(q, n)-1} \right\rceil(v(q, n) - 1) \); arbitrarily select the preferences for the remaining voters. If \( \left\lceil \frac{q}{v(q, n)-1} \right\rceil + \frac{\alpha}{v(q, n)} \) \( v(q, n) \) \( = n \), this approach is the only way to construct a profile. A \( v(q, n) \) voter profile with a \( q \)-cycle has at least \( \left\lceil \frac{2q-n}{v(q, n)-q} \right\rceil \) copies of a \( RWC_{v(q, n)} \).

3.2. Likelihood

Are \( q \)-cycles with \( v(q, n) \) alternatives robust, or are they unlikely to arise? The answers, which follow, suggest exercising caution when using Nakamura's number to anticipate voting difficulties.

**Definition 2.** If \( \frac{n}{n-q} \) is an integer for a \( (q, n) \)-rule, call it a \( v(q, n) \) bifurcation value.

For convenience, Theorem 3 properties are computed for bifurcation values.

**Proposition 4.** At a bifurcation value, \( \frac{n}{n-q} = (n - q) \). A \( v(q, n) \) voter profile causing a \( q \)-rule cycle strictly consists of \( (n - q) \) copies of a \( RWC_{v(q, n)} \).

To motivate what follows, a \((150, 200)\) rule has a \( v(150, 200) = 4 \) bifurcation value. According to Proposition 4, a four-alternative cycle with this rule must use \( n - q = 200 - 150 = 50 \) multiples of \( RWC_4 \). So let each of the \( RWC_4 \) rankings

\[
A > B > C > D, \quad B > C > D > A, \quad C > D > A > B, \quad D > A > B > C
\]

be supported by 50 voters. Here, \( A \) beats \( B \), \( B \) beats \( C \), \( C \) beats \( D \), and \( D \) beats \( A \) where the winning candidate in each election receives precisely 150 of the 200 votes.

"Precisely" is emphasized because, according to Theorem 3, should even one of the two hundred voters change her preference ranking in any manner, the cycle disappears. If, for instance, a voter preferring \( A > B > C > D \) interchanges how she ranks just her bottom two candidates to create the \( A > B > D > C \) preferences, \( C \) receives 149 votes in the \( \{C, D\} \) election while \( D \) receives 51. By failing to reach the \( q = 150 \) threshold, the societal ranking is the "tied" \( C \sim D \) outcome leading to the \( D > A, A > B, B > C \) and \( C \sim D \) conclusion that (barely) avoids the cycle and identifies \( D \) as the sole candidate who cannot be beaten; i.e., \( C(150, 200) = \{D\} \).

Thus the \( v(q, n) \) value can represent highly delicate, unrealistic settings rather than robust predictions of what to expect. In this fourth alternative setting, for instance, only in exceedingly rare settings (there are six \( RWC_4 \)'s, so for only six profiles out of the trillions of possibilities) can \( q \)-cycles occur. What must be expected is that 150-cycles will not occur and \( C(150, 200) \neq \emptyset \). In general, at a \( v(q, n) \) bifurcation value, it is highly unlikely to have \( q \)-rule cycles with \( v(q, n) \) alternatives.

Quirks of this behavior are captured by the \((151, 200)\)-rule where the slightly larger \( q \) creates the larger \( v(151, 200) = 5 \), so
four-alternative cycles are avoided. Extreme (151, 200) scenarios (Theorem 3) involve $\mathcal{RWC}_3$ profiles such as:

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<th>Number</th>
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<td>40</td>
<td>$A &gt; B &gt; C &gt; D &gt; E$</td>
<td>40</td>
<td>$B &gt; C &gt; D &gt; E &gt; A$</td>
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<tr>
<td>40</td>
<td>$C &gt; D &gt; E &gt; A &gt; B$</td>
<td>40</td>
<td>$D &gt; E &gt; A &gt; B &gt; C$</td>
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<td>40</td>
<td>$E &gt; A &gt; B &gt; C &gt; D$</td>
<td>40</td>
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(8)

Each pair’s tally in the promised $A > B$, $B > C$, $C > D$, $D > E$, $E > A$ cycle is 160 to 40, which exceeds $q = 151$. Rather than a delicate setting, because $\left\lfloor \frac{151}{200} \right\rfloor = 5$ and $\alpha = 4$, Theorem 3 asserts that the cycle can be created by using $37\frac{1}{2}$, rather than 40, copies of $\mathcal{RWC}_3$. As this multiple generates the $q$-cycle, it follows (Theorem 3) that the preference rankings for the remaining

$n - \left\lfloor \frac{151}{200} \right\rfloor = 200 - 5(37\frac{1}{2}) = 11$ voters can be selected in an arbitrary fashion, and the altered profile still supports the cycle. By admitting a larger number of supporting profiles, a (151, 200)-rule five alternative cycle is more likely to occur than a (150, 200)-rule four alternative cycle.

To compare this statement with the majority vote where

$v(101, 200) = 3$ and $\alpha = 2$, after using 50 $\frac{1}{2}$ copies of $\mathcal{RWC}_3$ to create a cycle, preferences for the remaining 48 voters (almost a quarter of them!) can be selected arbitrarily. By having 48 extra voters, Theorem 3 suggests that a supporting profile may need only

$\left\lfloor \frac{151}{200} \right\rfloor = 200$ copies of $\mathcal{RWC}_3$. This is true, but only if the rest of the profile is so highly restricted that it yields complete ties; e.g., use 97 pairs of reversal pairs such as $A > B > C$, $C > B$ where the ties for all pairs are appropriately broken by the two $\mathcal{RWC}_3$’s.

In general, larger $v(q, n) - \frac{n}{n-q}$ gaps allow larger $n - \left\lfloor \frac{151}{200} \right\rfloor$ differences, which (Theorem 3) admit larger sets of profiles with $q$-rule cycles. So, supporting examples with $v(150 + x, 200) = 5$ for $1 \leq x \leq 10$ involve $\mathcal{RWC}_3$ components, but as $x$ approaches the next bifurcation value ($x = 10$ or $q = 160$), the shrinking $200 - \left\lfloor \frac{151}{200} \right\rfloor$ difference reduces the likelihood of cycles.

Even stronger, if a bifurcation value a "cycle" is not a generic property, where "generic" means that the property is "expected" in that it holds even after the preferences of any agent are slightly changed. To summarize, at $v(q, n)$, bifurcation value, a profile defining a $q$-rule cycle must consist of $n - q$ copies of a $\mathcal{RWC}_3(q, n)$. With any standard probability assumption over profiles, such as a multinomial distribution, cycles are unlikely; the generic property is that $C(q, n) \neq \emptyset$.

4. Supermajority votes: profile restrictions

A way to avoid cycles without restricting the number of alternatives is to regulate which profiles are admissible. According to the above, for a profile restriction to succeed, it must handle the worst case scenarios created by $\mathcal{RWC}_N$ terms.

Interestingly, profile restrictions and Nakamura avoid cycles in the same way: Both temper the strong $\mathcal{RWC}_3$ tallies that create cycles to a more benign $\mathcal{RWC}_{N-3}$ level. Nakamura does this by restricting the number of alternatives. As Proposition 2 shows, dropping $s$ rankings from a $\mathcal{RWC}_N$ also reduces certain tallies to a $\mathcal{RWC}_{N-s}$ level.

4.1. Ward’s conditions

Black’s single-peaked condition (Black, 1958) is a well known profile restriction where, with three candidates, some candidate never is bottom ranked by any voter. The “Condorcet-domain problem” (e.g., Fishburn, 1997, Monjardet, 2009 and Saari, 2009) extends Black’s condition by seeking appropriate restrictions on $N$-alternative rankings so that no matter how many voters are assigned to the admitted rankings, majority vote cycles cannot occur.

“Ward’s conditions” (Ward, 1965) solve the Condorcet-domain problem for three candidates. The following description explaining why this is so uses the geometric representation of profiles introduced in Saari (1995, 2009). Here, a point’s ranking in the Fig. 2(a) equilateral triangle is determined by its distance to each vertex; e.g., a point in the triangle with a dagger in the lower left corner is closest to A, next closest to B, and farthest from C, so it has the A > B > C ranking.

Ward’s constraints are:

1. No voter has a particular candidate top-ranked. The Fig. 2(a) stars represent rankings where C is top-ranked. If no voter has either ranking, Ward’s condition is satisfied. In general, rankings in the two regions sharing the vertex with a candidate’s name have her top-ranked. So if no voter has rankings in either region adjacent to a particular vertex, this condition is satisfied.

2. No voter has a particular candidate middle-ranked. The Fig. 2(a) diamonds are where C is middle-ranked. If no voter has either ranking, Ward’s condition is satisfied. As “middle-ranked” involves diametrically opposite regions, if there are two ranking regions diametrically opposite each other (so the rankings reverse each other) without a single voter, this Ward condition is satisfied.

3. Black’s condition: No voter has a particular candidate bottom-ranked. The Fig. 2(a) daggers represent where C is bottom ranked. “Bottom-ranked” regions are the two regions farthest from the vertex assigned to the candidate, so if no voter has these rankings, Black’s and Ward’s conditions are satisfied.

Ward proved that if a profile satisfies any one of these conditions, a majority vote cycle cannot occur. A different explanation comes from Proposition 2 and Theorem 3: For a profile to have a majority vote cycle, it must include an appropriate multiple of a complete $\mathcal{RWC}_3$. The Eq. (2) $\mathcal{RWC}_3$ triplet in Fig. 2(b) is indicated by stars; the Eq. (3) triplet has bullets. For a profile to contain a full $\mathcal{RWC}_3$ triplet, it must have voters with rankings represented by all three stars, or all three bullets.

A structural explanation of Ward’s conditions is that they succeed by preventing a profile from including a $\mathcal{RWC}_3$ triplet. To illustrate with Fig. 2(b), if, for instance, no voter has B top-ranked, then the two ranking regions sharing the B-vertex are not assumed by any voter. One region has a bullet and the other a star, so this condition prohibits a profile from including either $\mathcal{RWC}_3$ triplet. Similarly, if C never is middle-ranked, then, as indicated in Fig. 2(b), no voter has the ranking with the indicated bullet or the ranking represented by diametrically opposite region with a star, which makes it impossible to include a $\mathcal{RWC}_3$ triplet. The same analysis holds for the never bottom-ranked requirement.

To see why Ward’s conditions are the sharpest possible to ensure a Condorcet domain, if no voter has his preference in a starred region, say the one on the Fig. 2(b) right-hand side, then a profile cannot include the starred $\mathcal{RWC}_3$ triplet. To preclude the profile from including the other $\mathcal{RWC}_3$ triplet, exclude any region with a bullet. For each choice, the Fig. 2(b) arrows indicate which Ward condition is satisfied.

Sen (1966) developed necessary and sufficient conditions for a Condorcet domain for $N \geq 4$ alternatives: When restricting the set of rankings to each triplet, it must satisfy at least one of Ward’s conditions. Sen’s result is subsumed and extended by Theorem 4.

4.2. General conditions

Ward’s conditions ensure a core by prohibiting a profile from including a full $\mathcal{RWC}_3$ triplet (Section 4.1). The elusive objective of finding appropriate profile restrictions for $(q, n)$ rules finally is resolved in the same way.
Theorem 4. For $r \geq 3$ alternatives, a necessary and sufficient condition for a set of rankings to avoid $(q, n)$-rule cycles and have $C(q, n) \neq \emptyset$ independent of how voters are assigned to the rankings, is that, when restricted to each subset of $v(q, n)$ alternatives, at least one ranking is missing from each possible $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$.

By including all $(q, n)$-rules, Theorem 4 subsumes the classical results. Nakamura’s Theorem is a special case because Theorem 4 trivially holds for $r < v(q, n)$; for $r \geq v(q, n)$, cycles can occur. Ward’s result is a special case because, with $v(2, 3) = 3$, Theorem 4 requires dropping at least one ranking from each $\mathcal{R} \mathcal{W} \mathcal{C}_3$, which is Ward’s requirement (Section 4.1). Sen’s result is subsumed because the majority vote requires $v([n+1]/2, n) = 3$; thus Theorem 4 requires applying Ward’s condition to all triplets. Implications for more general $(q, n)$ rules follow immediately from the theorem.

With $r > v(q, n)$ alternatives, one might expect to impose profile restrictions on $\mathcal{R} \mathcal{W} \mathcal{C}_4$ terms rather than the $v(q, n)$ parts. To illustrate why the emphasis must be on $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$, consider $v(18, 24) = 4$ with the $r = 5$ alternatives $A, B, C, D, E$. An extremely severe profile restriction allows at most one ranking to come from each $\mathcal{R} \mathcal{W} \mathcal{C}_5$. The following four rankings (each from a different $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$), for instance, satisfy this tight constraint.

$$E > [A > B > C > D], \quad E > [B > C > D > A],$$
$$E > [C > D > A > B], \quad E > [D > A > B > C].$$

These rankings embed the $\mathcal{R} \mathcal{W} \mathcal{C}_4$ generated by $[A > B > C > D]$, so assigning $\frac{n}{r} = 6$ voters to each ranking creates a $(18, 24)$-rule cycle. With $r > v(q, n)$, then, restrictions on $\mathcal{R} \mathcal{W} \mathcal{C}_4$ structures are insufficient to prevent $(q, n)$-cycles; attention must be focused on all subsets of $v(q, n)$ alternatives to prevent $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$ terms.

An extension of Ward’s condition follows:

Corollary 1. With $v(q, n)$ alternatives, a $q$-rule cycle never occurs if, for some integer $s \in \{1, 2, \ldots, v(q, n)\}$, some alternative is never $s$th ranked.

In $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$, each alternative is ranked in each position once, so Corollary 1 ensures that a profile cannot contain a $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$. While sufficient, Corollary 1 is not a necessary condition for $N \geq 4$. To see why, $v(q, n)$ alternatives define precisely $(v(q, n) - 1)!$ different ranking wheels. To have a restriction that admits the maximum number of rankings, drop one ranking from each $\mathcal{R} \mathcal{W} \mathcal{C}_{v(q, n)}$; this satisfies Theorem 4. But with $(v(q, n) - 1)! > v(q, n)$, or $v(q, n) \geq 4$, there is enough freedom to select the dropped rankings so that, for each ranking position, each candidate is in that position in at least one of the remaining rankings. Thus, Corollary 1 is not satisfied.

To illustrate, $v(q, n) = 4$ alternatives define precisely the $(4 - 1)! = 6$ $\mathcal{R} \mathcal{W} \mathcal{C}_4$’s generated by $A > B > C > D, A > B > D > C, A > D > B > C$ and their reversals $D > C > B > A, C > D > B > A$ and $C > B > D > A$. For each $\mathcal{R} \mathcal{W} \mathcal{C}_4$ drop the generating ranking: As all $\mathcal{R} \mathcal{W} \mathcal{C}_4$’s are involved, the remaining rankings cannot contain a $\mathcal{R} \mathcal{W} \mathcal{C}_4$, so Theorem 4 is satisfied. To see why the remaining rankings do not satisfy Corollary 1, notice that after dropping $A > B > C > D$, $A$ is in second, third, and fourth place, respectively, in one of the three remaining rankings from the generated $\mathcal{R} \mathcal{W} \mathcal{C}_4$. Thus $A$ is in first, second, third, fourth place, respectively, in remaining rankings of at least the fourth, first, first, and first $\mathcal{R} \mathcal{W} \mathcal{C}_4$. $B$ is, respectively, in the first, third, first, and first $\mathcal{R} \mathcal{W} \mathcal{C}_4$. $C$ is, respectively, in the first, first, second, and first $\mathcal{R} \mathcal{W} \mathcal{C}_4$, and $D$ is, respectively, in the first, first, first, and second $\mathcal{R} \mathcal{W} \mathcal{C}_4$.

5. When a core always exists

While Theorem 4 subsumes Nakamura’s and Sen’s results, it fails to include Greenberg’s result (Theorem 2). This flaw is remedied by the following surprisingly general assertion, Theorem 5, which includes both settings.

Let $\mathcal{D}$ consist of a specified set of alternatives and a specified set of complete transitive rankings of the alternatives; each agent can select any admissible ranking. Once each voter selects an admissible ranking, the core consists of the alternatives specified in $\mathcal{D}$ that cannot be beaten with a $(q, n)$ rule. The goal is to impose appropriate conditions on $\mathcal{D}$ so that the rankings defined by $(q, n)$ are complete and acyclic, or that $C(q, n) \neq \emptyset$.

To review differences, with an acyclic group ranking, some alternative cannot be beaten, so $C(q, n) \neq \emptyset$. There are choices of $\mathcal{D}$, however, where $C(q, n) \neq \emptyset$ but the group ranking contains cycles. (Let the admissible rankings over four alternatives $\{A, B, C, D\}$ always have a top-ranked. Here $A \in C(\frac{n+1}{2}, n)$, so the core is nonempty, but, as shown below, the outcome could have a cycle involving $\{B, C, D\}$.) So, a nonempty core means there cannot be a top-cycle (a cycle among terms that beat all others), while the stronger acyclic condition prevents cycles of any type.

The $\mathcal{D}$ choice of alternatives and rankings is explicit in Theorem 4. As Theorem 4 imposes conditions on $\mathcal{D}$ to ensure that the group outcome always is complete and acyclic, it also follows that $C(q, n) \neq \emptyset$. With Theorem 2 and spatial voting, the specified domain $\mathcal{D}$ determines $\mathcal{D}$ by defining both the admissible alternatives and the rankings. (A voter’s ideal point defines the voter’s ranking where preferred alternatives have smaller Euclidean distances to the ideal point.) Here the core consists of all alternatives that cannot be beaten, so a core point identifies an alternative in $\mathcal{D}$ and, in this setting, a complete transitive ranking of the alternatives in $\mathcal{D}$.

The following result provides a convenient way to analyze problems.

Theorem 5. a. For a specified $\mathcal{D}$, a necessary and sufficient condition for $C(q, n) \neq \emptyset$ for any selection of $n$ rankings from $\mathcal{D}$ is if it is impossible to construct a setting with $\beta = v(q, n)$ rankings from $\mathcal{D}$ where $C(\beta - 1, \beta) = \emptyset$.

b. For a specified $\mathcal{D}$ and $q$-rule, a necessary and sufficient condition for the $q$-rule outcome to be complete and acyclic for any $n$ rankings from $\mathcal{D}$ is if it is impossible to construct a setting with $\beta = v(q, n)$ rankings from $\mathcal{D}$ with a $(\beta - 1, \beta)$ rule cycle.

To illustrate Theorem 5, let the admissible rankings over four alternatives $\{A, B, C, D\}$ always have $A$ top-ranked. Because $v(60, 100) = 3$, to find whether it always is true that $C(60, 100) \neq \emptyset$, it suffices (Theorem 5(a)) to determine whether, with these preference rankings, it always is true that $C(2, 3) \neq \emptyset$. As $v(2, 3) = 3$, only two profiles have the $\mathcal{R} \mathcal{W} \mathcal{C}_3$ structure; one is
A \triangleright [B \triangleright C \triangleright D], A \triangleright [C \triangleright D \triangleright B], A \triangleright [D \triangleright B \triangleright C], while the other reverses the rankings in the brackets. For both profiles, A \in C(2, 3), so (Theorem 5(a)) it always true that C(60, 100) \neq \emptyset. But the profile’s paired rankings are A \triangleright B, A \triangleright C, A \triangleright D with the (2, 3)-rule cycle B \triangleright C, C \triangleright D, D \triangleright B, so (Theorem 5(b)) there are q = 60 rule outcomes that are not acyclic.

What makes Theorem 5 unexpectedly general is that no conditions are imposed on the structure of D. Instead, D’s structure is determined by what it means for a model to rank alternatives and to make paired comparisons. What makes Theorem 5 possible is that the conclusion depends on properties of tallying ballots from complete transitive preferences (i.e., Propositions 1, 2) rather than the structure of D.

Theorem 5 significantly simplifies the analysis by replacing (q, n) problems with simpler (β − 1, β) settings. Because ν(β − 1, β) = β is a bifurcation value, the analysis reduces to determining whether the D structure permits constructing a \( \mathcal{R} \mathcal{W} \mathcal{E}_\beta = \mathcal{R} \mathcal{W} \mathcal{E}_{\nu(q, n)} \). For instance, to determine a limit on the number of alternatives that always has \( C(200, 250) \neq \emptyset \), because \( \nu(200, 250) = 5 \), Theorem 5(a) reduces the analysis to the much simpler task of finding the number of alternatives for which it is true that \( C(4, 5) \neq \emptyset \). A \( \mathcal{R} \mathcal{W} \mathcal{E}_\beta \) requires five alternatives, so (Propositions 1, 2) the answer is four; thus four or fewer alternatives ensure that \( C(200, 250) \neq \emptyset \).

The same argument shows that Nakamura’s result is a special case of Theorem 5(b). Here D is the set of N alternatives; any complete transitive ranking is admissible. To find a restriction on the value of N (that is, a restriction on D) to ensure acyclic outcomes, we must (Theorem 5(b)) find the N value so that a \( \mathcal{R} \mathcal{W} \mathcal{E}_{\nu(q, n)} \) cannot be constructed. As this wheel requires \( \nu(q, n) \) alternatives, if \( N < \nu(q, n) \), a \( \mathcal{R} \mathcal{W} \mathcal{E}_{\nu(q, n)} \) cannot be constructed, so a \( \nu(q, n) \) cycle does not exist. In turn (Theorem 5(b)), a q-rule cycle cannot be constructed, which is Nakamura’s result.

Similarly, to find profile restrictions to ensure that \( C(200, 250) \neq \emptyset \) with ten alternatives, Theorem 5(b) reduces to the simpler (4, 5)-rule. A (4, 5) cycle is prevented only if the profile does not contain a \( \mathcal{R} \mathcal{W} \mathcal{E}_c \) when restricted to any choice of five of the ten alternatives; thus, this is the answer for both (4, 5) and (200, 250). In this manner (replace (4, 5) with (\( \nu(q, n) − 1 \), ν(q, n))), Theorem 4 is included in Theorem 5(b).

The following Proposition 5 provides a convenient way to determine whether the core is nonempty. (Proposition 5 was a main tool used in Saari (1997), but it may have been known earlier.) First, recall that a Pareto point for coalition C is a point that, if altered in any manner, creates a poorer outcome for some member in C. Denote the set of all Pareto points for coalition C by \( \mathcal{P}(C) \). With (q, n)-rules and Euclidean preferences in spatial voting, \( \mathcal{P}(C) \) is the convex hull defined by the ideal points of agents in C. (If the region does not admit straight connected lines, the hyperplanes connecting the ideal points can be replaced with unique connecting surfaces to define the \( \mathcal{P}(C) \) faces.) A minimal winning coalition \( \mathcal{P}(C) \) contains q voters.

**Proposition 5.** For (q, n), the core equals

\[
C(q, n) = \cap \mathcal{P}(C)
\]

where the intersection is over all minimal winning coalitions.

Proposition 5 (as with Theorem 5) imposes no requirements on the underlying space; e.g., the structure of the Pareto sets \( \mathcal{P}(C) \) is determined by the particular setting. To illustrate, the Fig. 3(a) choice of D is an open annulus representing an island with a lake in its middle. For nonstandard preferences, suppose each agent prefers to be near either a particular point or the lake, whichever is closer; i.e., each agent’s preferences are determined by the minimal distance of a proposal to the agent’s ideal point or the lake’s boundary. For each coalition, \( \mathcal{P}(C) \) is the convex hull of the ideal points of coalition members and the lake’s boundary. This could be a disjoint union, but each \( \mathcal{P}(C) \) includes the lake’s boundary, so (Proposition 5) the core is nonempty.

Greenberg’s result (Theorem 2) follows from Proposition 5 and Theorem 5(a). According to Theorem 5(a), issues about the core reduce to determining whether \( \beta = \nu(q, n) \) ideal points can be positioned to have an empty core. This is simple to do if the space has dimension \( k \geq \beta - 1 \); e.g., let the ideal points be vertices of an equilateral \( \beta \)-gon. With \( \beta = 3 \), for instance, the Pareto sets are the three edges of the defined triangle; no point is common to all three edges, so \( C(\beta - 1, \beta) = \emptyset \). Similarly with \( \beta = 4 \), let the ideal points be vertices of a tetrahedron; the Pareto sets are the four faces and no point belongs to all four faces. The same assertion holds for any \( \beta \).

With \( k \leq \beta - 2 \) issues, these objects collapse dimensions so the \( \beta \)-gon cannot be constructed; e.g., vertices of a triangle become points on a line. With \( \beta = 3 \) points on a line, the edges must have a common intersection, so it always true that \( C(\beta - 1, \beta) = \emptyset \). If \( \beta = 4 \) points in a plane define a quadrilateral, each Pareto set includes one of the two diagonals, so their intersection is the core. (If three points define a triangle and the fourth is in it, the interior point is the core point.) Indeed, it is not overly difficult to show that in \( \mathbb{R}^k \), where \( k \leq \beta - 2 \), the Pareto sets (defined by \( \beta - 1 \) points, so each uses “all but one point”) must have a point in common; this is a core point (Proposition 5). Carrying out the details would prove Theorem 2.

This argument provides a simple way to show how Theorem 2 follows from Theorem 5(a), but it obscures the \( \mathcal{R} \mathcal{W} \mathcal{E}_\beta \) role (the vertices of the \( \beta \)-gon), and it cannot handle more delicate theorems involving acyclic outcomes that are based on Theorem 5(b).

So, Proposition 6 is needed; its proof essentially mimics the intuition described in the previous paragraphs. While the rankings in Proposition 6 represent Euclidean preferences (because these preferences are so often used), they can be generalized to continuous, convex utility functions with an ideal point as a bliss point.
Proposition 6. It is impossible to position $N$ ideal points and $N$ alternatives $(a_j^N)_{j=1}^N$ in $\mathbb{R}^{N-2}$ so that the associated rankings define a $\mathcal{RWC}_N$. This can be done in $\mathbb{R}^{N-1}$.

An example illustrating both Propositions 5 and 6 is the median voter theorem where voters have their preferences along a line. Because $v(\frac{N}{2}+1, n) = 3$, the majority vote analysis reduces (Theorem 5) to determining whether a $\mathcal{RWC}_3$ can be constructed on a line. According to Proposition 6, it cannot, so $C(\frac{N+1}{2}, n) \neq \emptyset$. Also in this three voter setting, $\mathcal{P}(C)$ for each minimal winning coalition is the smallest interval containing the ideal points for both coalition members. Geometry immediately shows that the median point must be in each $\mathcal{P}(C)$, so (Proposition 5) $C(2, 3) \neq \emptyset$. It now follows from Theorem 5(a) that $C(\frac{N+1}{2}, n) \neq \emptyset$.

According to Proposition 6, while a $\mathcal{RWC}_3$ can be created with $\beta$ ideal points, this never can be done if $k \leq \beta - 2$. Therefore, a nonempty core is guaranteed as long as $k \leq v(q, n) - 2$, which makes Theorem 2 a special case of Theorem 5(a).

By not imposing added restrictions on $D$, Theorem 5 is more inclusive than Theorem 2; e.g., rather than a compact, convex set as required by Theorem 2, $D$ can be just about anything. For instance, change Fig. 3(a) to a solid island (no interior lake), but do not include the boundary with the ocean. This set is not compact, so Theorem 2 is not applicable. But according to Proposition 6 and Theorem 5(a) with the $v(22, 30) = 4$ value, we have that $C(22, 30) \neq \emptyset$ no matter where the ideal points are located.

This example and Theorem 2 suggest that, perhaps, only $D$’s dimension is needed. To prove this is false, consider a $(40, 60)$-rule with the Fig. 3(b) one-dimensional triangle $D$ where only the three edges are included; e.g., a group wishes to select a sidewalk-booth spot on a triangular block. As $v(40, 60) = 3$, the problem reduces to determining whether three points (Theorem 5(a)) can be positioned in $D$ so that $C(2, 3) = \emptyset$. To do so, place an ideal point at each vertex; this defines a $\mathcal{RWC}_3$, so three ideal points can be positioned to make the core empty. Indeed, a minimal winning coalition consists of two points, so each triangle edge is a Pareto set. As the intersection of these three legs is empty, $C(2, 3) = \emptyset$ (Proposition 5), which means that sixty ideal points can be positioned on $D$ so that $C(40, 60) = \emptyset$. (Similar examples can be created where $D$ consists of a finite number of points, so $D$ is zero-dimensional.)

To indicate how Theorem 5 includes settings that differ from those addressed by Theorem 2, return to the US Senate filibuster example where $v(60, 100) = 3$. Let the voter preferences be given by the Euclidean distance from a voter’s ideal point to each alternative and let $D$ be the Fig. 3(c) triangle where the three alternatives are located at the vertices. As specified in Theorem 2, $D$ is closed, convex, and its dimension of two exceeds $v(60, 100) - 2 = 1$. Thus one might suspect from Theorem 2 that the 100 ideal points can be positioned so that $C(60, 100) = \emptyset$, which would unleash the consequences of the McKelvey chaos theorem. But as the $X \sim Y$ indifference lines (the perpendicular bisector of the $X - Y$ edge; denoted by $l_X Y$) show by allowing only four of the six rankings, it is impossible to position three ideal points in $D$ to create a cycle. (Moving from left to right, the strict rankings associated with ideal points in these regions are $A > B > C$, $B > A > C$, $B > C > A$, and $C > B > A$, where $B$ never is bottom ranked, thus, whatever the positioning of the ideal points, one of Ward’s conditions is satisfied.) Thus (Theorem 5(a)) it always is true that $C(60, 100) \neq \emptyset$. According to Theorem 5(b), all votes are acyclic.

To explain, the Fig. 3(c) setting differs from Theorem 2 by providing added information: By specifying the choice of the proposals, it imposes restrictions on preferences no matter where the ideal points are located. When models have added information about the alternatives and/or voter preferences, Theorem 5 is the appropriate choice to determine whether a core must be nonempty, or rankings must be acyclic.

6. Summary

The fact that all paired comparison difficulties caused by $\mathcal{RWC}_N$ terms, and only by $\mathcal{RWC}_N$ terms in a profile (Saari, 2000) is the key that unlocks several long standing mysteries. This result makes it easier to understand why problems occur, unite seemingly dissimilar conclusions, and extend several standard results. A contribution made here is to unify and significantly extend certain well known conclusions into a single new result (Theorem 5) that has a much wider range of applicability.

Appendix

Proof of Proposition 1. To tally $(a_j, a_{j+1})$ for the $\mathcal{RWC}_N$ generated by $a_1 > a_2 > \cdots > a_N$, as $a_{j+1}$ is ranked above $a_j$ in precisely $s$ of the $\mathcal{RWC}_N$ rankings, the outcome for $s < \frac{N}{2}$ is $a_j > a_{j+1}$ with tally $N - s$; s. If $s = \frac{N}{2}$, the outcome is a tie.

Proof of Proposition 2. Let $a_{j+1}$ be the top-ranked candidate in a removed ranking; only in this ranking is $a_{j+1}$ ranked above $a_j$ in $\mathcal{RWC}_N$, so $a_j$ beats $a_{j+1}$ with an unanimous vote. There are precisely $s$ such pairs. For all other pairs of adjacently ranked candidates, $a_{j+1}$ beats $a_j$ precisely once in the $N - s$ remaining rankings, so $a_j > a_{j+1}$ has a $(N - s) - 1$ : 1 tally. These are the tallies expected from a $\mathcal{RWC}_N$.

Proof of Proposition 3. This result summarizes the computations prior to the statement of Proposition 3.

Proof of Proposition 4. These computations are used in the proof of Theorem 3, so Proposition 4 is proved first. At a bifurcation value, both $\nu(q, n) = \frac{n}{n-q}$ and $\frac{n}{\nu(q,n)-1} = \frac{n(1-q)}{n-q} = n - q$ are integers. Similarly, $\frac{2q-n}{2n-\nu(q,n)} = \frac{(2q-n)(n-q)}{2n(1-q)} = \frac{(2q-n)(n-q)}{2n(1-q)} = n - q$.

Finally, with Eq. (7), $\alpha = 0$ and the number of needed voters is $\frac{q}{n-2(n-q)} v(q, n) = (n-q) \frac{n}{n-q} = n$.

Proof of Theorem 3. To show that a $q$-cycle exists if and only if $N \geq v(q, n)$, it must be shown that Eqs. (6) and (7) are satisfied only for these $N$ values. It follows from the above computations that $v(q, n)$ satisfies these equations if $\alpha = 0$. So, assume $\alpha \geq 2$.

By substituting the value of $\alpha$ and Eq. (6) into Eq. (7), Eq. (7) becomes

$$\frac{q}{N-1} + \frac{q}{N-1} (N-1) + 1 \leq n$$

or

$$\frac{q}{N-1} \leq n - q - 1.$$  \hspace{1cm} (10)

Clearly, $N = 1$ is not a possibility. Each $N > 1$ can be expressed as $N = \frac{n+1}{n-q}$, for some integer value $y \neq 0$ where $-q < y < n$. So, $N \geq v(q, n)$ if and only if $y > 0$.

With this expression, $\frac{q}{N-1} = \frac{q}{N-1} = \frac{q}{N-1} (n-q)$ for integer $y \neq 0$. As the integer $\frac{q}{N-1} (n-q) < (n-q)$ if and only if $y > 0$, Eq. (10) is satisfied if and only if $N \geq v(q, n)$. This proves the first assertion.

For the second conclusion, $\frac{q}{\nu(q,n)} + \frac{a}{\nu(q,n)}$ copies of the $\mathcal{RWC}_N$ creates a $q$-cycle, so the preferences for remaining voters can be arbitrarily selected. Removing any ranking from the specified copies of $\mathcal{RWC}_N$ drops the tally for some pair below the $q$ threshold. To compensate without using $\mathcal{RWC}_N$ rankings, at least two other rankings with appropriate properties must be added; e.g., the lost point for the $\{A, B\}$, $\{B, C\}$, $\{C, D\}$ winners by dropping $A > B > C > D$ can be recovered by using $A > B > D > C$ and $C > D > B > A$. This reduces the number of voters whose
determining whether there are enough voters to have the construction. So assume when restricting a given set of rankings to any subset of \( v(q, n) \) alternatives, at least one ranking from each \( RWC_{v(q, n)} \) is missing. With \( r = v(q, n) \) alternatives, the conclusion follows from Proposition 2 or Theorem 3. With \( r > v(q, n) \) alternatives, if a \( q \)-cycle can be created, it uses a \( RWC_r \). If this \( RWC_r \) includes all of its rankings, then because a \( RWC_{v(q, n)} \) can be constructed from the \( RWC_r \) by ignoring \( r - v(q, n) \) variables (Lemma 1(a)), it would contradict the hypothesis. Thus, not all rankings in a \( RWC_r \) are admitted.

According to Lemma 1(b), to avoid including rankings that would create some \( RWC_{v(q, n)} \), at least \( r - v(q, n) + 1 \) of the rankings that define a \( RWC_r \) must be removed. (At most \( v(q, n) - 1 \) remain; e.g., a \( N \)-candidate Condorcet domain avoiding majority vote cycles has no more than two rankings from each \( RWC_r \), for a total of no more than \( 2((N - 1)!\) rankings.) Thus the sharpest construction (Proposition 2) to create a cycle has tallies that are at best \( v(q, n) - 2 : 1 \). To meet quota, \( k \frac{2}{N} \) copies of what remains of the \( RWC_r \) must be used where \( q = (v(q, n) - 2) + y \). If \( y = 0 \), then \( \alpha = 0 \); otherwise \( \alpha = y + 1 \); in either case \( k = \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor \).

To have enough voters to construct this profile, it must be that
\[
q \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor + (v(q, n) - 1) + \alpha \leq n. \tag{11}
\]
As with the proof of Theorem 3, there are two cases: \( \alpha = 0 \) and \( \alpha \geq 2 \). If \( \alpha = 0 \), then \( \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor \) is an integer, so solving the resulting Eq. (11) for \( v(q, n) \) yields the inequality \( v(q, n) \geq \frac{1}{\frac{2}{N} + \frac{(v(q, n) - 1)}{N}} \). This contradiction proves that Eq. (11) cannot be satisfied.

For \( \alpha > 2 \), by replacing \( \gamma = \alpha - 1 \), the condition on the number of voters becomes
\[
q \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor + (q + 1 - \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor) \leq n, \tag{12}
\]
or
\[
q \left\lfloor \frac{2}{v(q, n) - 2} \right\rfloor \leq n - q - 1. \tag{13}
\]
But this is Eq. (10) with \( N = (v(q, n) - 1) \). As shown above, Eq. (10) holds only for \( N \geq v(q, n) \), which completes the proof.

**Proof of Theorem 5.** The proofs for Theorem 5(a) and (b) follow the same theme: determine whether there are enough voters to create a \( q \)-cycle. If it is possible to create a \( RWC_{v(q, n)} \) (so either \( C(v(q, n) - 1, v(q, n)) = \emptyset \), or the group ranking has a cycle), then, because \( N = v(q, n) \) satisfies Eq. (7), there are at least \( \left\lfloor \frac{2}{v(q, n) - 1} \right\rfloor v(q, n) + \alpha \) voters in the society. According to the above, by using \( \left\lfloor \frac{2}{v(q, n) - 1} \right\rfloor \) and \( \frac{2}{v(q, n) - 1} \) copies of the \( RWC_{v(q, n)} \), a \( q \)-cycle is formed. Thus, depending on the setting, either \( C(q, n) = \emptyset \), or the group ranking contains a cycle. Preferences assigned to the remaining voters do not affect the conclusion.

The proof of the converse direction (i.e., if it is impossible to create a \( \beta \)-rule cycle for \( \beta = v(q, n) \), then it is impossible to create a \( q \)-rule cycle) essentially repeats the above proof for Theorem 4. Namely, this condition means that whatever \( RWC_r \) is used, the tallies in a cycle have at least \( v(q, n) - 2 : 1 \) tallies. The rest of the proof is as given in the above proof of Theorem 4.

**Proof of Proposition 5.** By definition of a Pareto point, if \( p \in P(C) \) for a minimal winning coalition \( C \), then there is no alternative that \( C \) could select over \( p \). If \( p \in \cap P(C) \) for all minimal winning coalitions, \( p \) cannot be beaten, so \( p \in C(q, n) \). Conversely, if \( p \in C(q, n) \), it cannot be beaten by any alternative. But if \( p \notin P(C) \) for minimal winning coalition \( C \), then \( C \) can elect an alternative
to $p$. Thus, $p \in \mathcal{P}(C)$ for all minimal winning coalitions. This completes the proof.

To motivate the proof of Proposition 6, any two points, corresponding to propositions, define a line segment; the segment’s perpendicular bisector defines points that are indifferent to the propositions. It also defines two strict rankings each of which is on a particular side of the bisector. For instance, placing three points on a line segment defines three perpendicular bisectors; all are perpendicular to the line so they are parallel. In particular, they define four strict, transitive rankings.

Now place the three points in a space to define a triangle. Here the perpendicular bisectors are not parallel; they intersect to define six transitive rankings as in Fig. 2. So when dropping a dimension, two rankings are dropped; it turns out that each ranking comes from a different $\mathcal{RWC}_3$. So, while a cycle can be constructed with ideal points in the plane, it cannot be constructed with ideal points on the line. The difference is the geometry of the perpendicular bisectors.

The general proof is the same. The number of strict rankings defined by $N$ alternatives placed at vertices of a $N$-gon in a $(N - 1)$-dimensional space is $N!$. Dropping a dimension restricts the orientation of the perpendicular bisectors, which prevents them from intersecting in certain ways; this drops some rankings $(N - 1)!$ of them). The main part of the proof shows that each $\mathcal{RWC}_N$ loses a ranking. The proof is by contradiction; after assuming a complete $\mathcal{RWC}_N$ can be constructed, paths connecting ideal points are developed to create a contradiction.

**Proof of Proposition 6.** Position alternatives $\{a_i\}_{i=1}^N$ to define a $(N - 1)$-dimensional simplex; each $a_i$ defines a unique vertex. Each edge is defined by two vertices $(a_i, a_j)$: construct a plane orthogonal to this edge that passes through its midpoint. This plane, $l_{ij}$, identifies all points where (with Euclidean preferences), $a_i \sim a_j$. (See Fig. 4(a) for the $A \sim B$, or $l_{AB}$, plane.) These planes intersect to define $N!$ open regions; points in each region define a particular strict ranking of the $N$ alternatives and each of the $N!$ rankings has its own region. To create a $\mathcal{RWC}_N$, place the voter ideal points in the appropriate region. Each set of $(N - 1)$ ideal points defines a $(N - 2)$-dimensional simplex (a face of the $(N - 1)$-dimensional simplex). This face is the Pareto set for this minimal winning coalition. As the faces do not have a common intersection, it follows (Proposition 5) that the core is empty.

**Induction argument:** Different induction arguments to complete the proof use the fact that an ideal point changes $(X, Y)$ ranking only if $i$ crosses $l_{XY}$. Start with $N = 3$ alternatives $A, B, C$; their positions define the $l_{XY}$ indifference lines. Suppose a $\mathcal{RWC}_3$ generated by $A > B > C$ is created where the alternatives and ideal points are on a line. By symmetry, we can assume $A$ is to the right of $C$.

If this construction is possible, it is possible to move on the line among the ideal points $l_{AB}, B = C, l_{BC}, C = A, l_{CA}, A = B$ (or vice versa). A path from $l_{AB}, B = C$ to $l_{BC}, C = A$ can be assumed to miss $l_{AB}, C$. This is because if $l_{AB}, C$ is crossed in one direction (to change the $B \succ C$ ranking to $C \succ B$), the path must recross $l_{BC}$ to preserve $B \succ C$. Thus $l_{AB}, B = C$ and $l_{BC}, C = A$ are separated by $l_{AB}, C$ and $l_{LI}, C$ in that order (Fig. 4(b)), and $l_{JI}, C$ is either on the left of $l_{AB}, C$ or to the right of $l_{AB}, C$ (the two Fig. 4(b) dashed lines). If it is to the right of $l_{AB}, C$, then $l_{BI}, C$ is between $l_{AB}, C$ and $l_{BI}, C$ to the right.

To go from $l_{AB}, B = C$ to $l_{BI}, A = B$, the path must cross $l_{AB}, C$ and $l_{BI}, C$ in that order, and never cross $l_{AB}, B$. But either $l_{AB}, C$ choice requires $l_{AB}, B$ to be crossed with this second path; this contradicts the assertion that constructing a $\mathcal{RWC}_3$ on a line is geometrically impossible. With an added dimension (Fig. 2(b)), however, it is easy to construct such paths without crossing a forbidden $l_{XY}$.

With the induction hypothesis, assume the result holds for $N$ alternatives; it must be shown that it is impossible to construct a $\mathcal{RWC}_{N+1}$ defined by $a_1 > \cdots > a_N > a_{N+1}$ with $N+1$ alternatives and ideal points in $\mathbb{R}^{N-1}$.

Assume this is false; i.e., assume that $\{a_i\}_{i=1}^{N+1}$ points and $\{l_{ij}\}_{i=1}^{N+1}$ ideal points can be positioned in $\mathbb{R}^{N-1}$ to construct a $\mathcal{RWC}_{N+1}$. Label the $\mathcal{RWC}_{N+1}$ rankings by $k = 1, \ldots, N + 1$ with respective ideal points given by $l_{ik}$. (With $N + 1 \leq 4$ alternatives in Fig. 4(c), the region with $l_i$ has the ranking $A \succ B \succ C \succ D$ and the region with $l_k$ represents the last $D \succ A \succ B \succ C$.) By ignoring any one of the $N + 1$ alternatives, the resulting rankings include a $\mathcal{RWC}_N$ (Lemma 1), so (induction hypothesis) a geometric representation of these $N$ alternatives must define a $(N - 1)$-dimensional simplex (for $N = 3, 4$, a triangle). This simplex is defined by $N$ alternatives; all $(N + 1)$ ideal points are in appropriate sectors.

If $a_i$ is dropped, the $\binom{N}{2}$ indifference planes meet in a common, the “not-$a_i$” hub denoted by $\tilde{a}_i$. (If they did not meet, they would define more than $N$! open sectors, so some represent nontransitive rankings. For $N = 3$ and dropping D, the three solid indifference lines in Fig. 4(c) meet at point $D$ and define six sectors representing the six strict transitive rankings when $D$ is not considered.) With $N + 1$ alternatives, there are $N + 1$ “not-$a_i$” hubs, which must be created (induction hypothesis) so that the $a_i \sim a_{N+1}$ indifference plane, $l_{ij}$, must meet $l_{ik}$ and $l_{jk}$ for each $x, y \neq k$.

(With $N = 3$, $l_{ik}$ must meet $l_{ijb}$ and $l_{ijk}$ at, respectively, $C$ and $B$.)

The structure for $N$ alternatives is established by ignoring $a_{N+1}$, next, determine where to locate the $l_{i+1,a}$ planes as required by changes in rankings. For each $l_{i+j, a} = 1, \ldots, N$, point $l_{i+j}$ is on one side of this plane, and all other ideal points are on the other (Proposition 1). What builds to a contradiction is that both $l_{i+j}$ and $l_{i+k}$ are on one side of $l_{N+1,a}$, and the other $N - 1$ points are on the other side.

Points $i$ and $l_{i+1}$ are in “sector 1” defined by the $N - 1$ boundaries $\{l_{i+1,a}\}_{i=1}^{N-1}$; the other ideal points are in other sectors defined by $\mathcal{RWC}_N$. (For $N = 3$, $l_{ij}$ and $l_{ik}$ are in the $A \succ B \succ C$ sector 1. $l_i$ is in $B \succ C \succ A$ and $l_j$ is in $C \succ A \succ B$. Sector 1 is bounded by $l_{i, j}$ and $l_{i, c}$. Changes in only the $a_{N+1}$ ranking occur in each of the $N!$ sectors; they are given by how the $l_{i+j, k}$ planes intersect with the sector, and the intersections create at most $N + 1$ regions. (For $N = 3$, the $l_{ij}$ planes create at most four regions in any sector; or with five or more, the extra regions would represent nontransitive rankings.) In sector 1, $l_i$ and $l_{i+1}$ have, respectively, $a_{N+1}$ and top and bottom ranked, so any path in this sector connecting $l_i$ and $l_{i+1}$ must cross all $\{l_{i+j, k}\}_{j=0}^{N-1}$ planes. (In Fig. 4(c), to move from $D \succ A \succ B \succ C$ to $A \succ B \succ C \succ D$, reverses rankings with each alternative, so each $l_{ij}$ meets sector 1.) Except for $i$, all ideal points have $a_{N+1} \succ a_i$, so points $\{l_{i+j, k}\}_{j=1}^{N-1}$ are on the same $l_{i+1, k}$ side as $a_{N+1}$; thus $l_{i+1}$ is on the hub side. In Fig. 4(c), $l_{i, D}$ meets $l_{ij}$ to identify $C$, and $l_{i, C}$ for $B$. All but $i$ have $D \succ A$, so $\{l_{i+j, k}\}_{j=2}^{N-1}$ are on the $D$ side of $l_{i, D}$.

Moving from $l_{i+1}$ to $l_i$, the first permissible change reverses $a_{N+1} \succ a_i$, so no other $l_{ij}$ can enter the sector 1 region as defined by $l_{i+1}$ and contains $l_{i, k}$. The ordering of how to reverse $a_{N+1}$ dictates how the $l_{i+j, k}$ planes intersect sector 1; moving from $a_{N+1}$ out, they follow the $k = 1, \ldots, N$ order. Point $i$ is outside (i.e., away from $a_{N+1}$) of the last $l_{N+1, k}$ plane; i.e., $i$ is on the side of $l_{N+1}$ away from the hub. While these $l_{N+1, k}$ planes cannot meet in the interior of the sector (or they would create more than $N$ regions), some must intersect on the boundaries to create various hubs; e.g., $l_{i+2, 1}$ meets $l_{i+1}$ on the $l_{2, 1}$ boundary, and for $N > 3$, this line connects with at least two hubs.

Consider the path from $l_{N+1}$ (near $a_{N+1}$) to $l_{i+1}$ in sector 2. This path crosses $l_{i+1}$, $l_{i+3}$, $\ldots, l_{i, N}$ in that order (to move $a_{i+1}$ to the bottom plane). Similarly, at some position after crossing $l_{i, a}$, the path must cross $l_{i+1, a}$ (to move $a_{N+1}$ to its final position) in the $l_{i+1, 2}$, $l_{i+3, 3}$, $\ldots, l_{i, N}$ order (to reverse rankings with the next
adjacent alternative). Thus the \( \{I_{N+1,k}\}_{k=2}^N \) planes meet sector 2, but they cannot meet each other in the interior of the sector. At the final stage, \( i_3 \) is on the \( I_{k+1,N} \) side away from \( a_{N+1} \); i.e., \( i_1 \) and \( i_2 \) are on the \( I_{k+1,N} \) side away from \( a_{N+1} \).

Each \((N - 1)\)-dimensional plane \( I_{N+1,k} \) is uniquely defined by \((N - 1)\) of the \((N + 1)\) possible hub points; only \( a_{N+1} \) and \( a_k \) are excluded. These hub points are determined by the intersection of certain edges of sectors \( k, k = 1, \ldots, N - 1 \), with either \( I_{N+1,1} \) or \( I_{N+1,2} \); e.g., \( a_2 \) is the intersection of the sector 2 edge \( a_1, a_2, \ldots, a_N \) with \( I_{N+1,1} \). In general, \( a_2 \) is the intersection of the \( k \)th-sector’s edge \( a_{k+1} \sim a_{k+2} \sim \cdots \sim a_n \sim a_1 \sim \cdots \sim a_{N-1} \) with \( I_{N+1,1} \) if \( k \neq 1 \) and \( I_{N+1,2} \) if \( k = 1 \). Thus, \( I_{N+1,N} \) is uniquely defined by \( \{a_k\}_{k=1}^{N-1} \). (In Fig. 4(c), \( A \) is far to the right where the dashed \( I_{2,D} \) would meet the solid \( I_{2,C} \) while \( B = I_{2,0} \cap I_{2,C} \). As \( A \) uniquely define \( I_{2,C} \) (the dotted line), it follows that there must be two ideal points on each side, which is the desired contradiction. The rest of the proof leads to the same conclusion after handling the higher dimensional nature of \( I_{N+1,N} \).

Now consider the portion of the region \( a_1 \succ a_2 \succ \cdots \succ a_{N-1} \) (with boundaries \( I_{2,1}, I_{2,2}, \ldots, I_{N-2,N-1} \)) on the \( a_{N+1} \) side of \( I_{N+1,1} \). This region, \( R \), contains the ideal points \( i_1, i_{N+1} \). However, \( I_{N+1,N} \) does not meet \( R \) because in sector 1, \( I_{N+1,N} \) is separated from \( R \) by \( I_{N+1,1} \), and none of its other defining hub points have this ranking. This means that \( i_1, i_{N+1} \) are on one side of \( I_{N+1,N} \) while at least \( i_2 \) are on the other. As this contradicts the tallies from Proposition 1, the proof is completed. (With an added dimension, the extra defining point for \( I_{N+1,N} \) would separate \( i_2, i_{N+1} \).)

References

What makes voting paradoxes so intriguing is that “voting” appears to be conceptually simple. If so, then why do these puzzles arise? Following the “picture is worth a thousand words” adage, the approach adopted here (starting with Saari [8, 12]) to explain these mysteries uses simple geometry. With nothing more complicated than an equilateral triangle or an ordinary cube (i.e., a child’s playing block), geometry provides new insights that offer a broader perspective about why these paradoxes arise – and how to avoid them.

1. Does a voting outcome mean what we think it does?

Suppose two proposals designed to help teachers are defeated: Proposition $S$, an attempt to increase salaries, received only 40% of the vote, while Proposition $B$, an effort to improve benefits, received 45%. A natural interpretation of this discouraging conclusion is that it represents the public’s strong, negative attitude about teachers because for both proposals, a solid majority of the voters, 55%, voted against them, an added 5% voted for improved benefits but against a salary increase, and only 40% wanted to improve salaries and benefits.

Is this interpretation correct? It may not be because the same election tallies occur if, instead, a vast majority of the voters strongly support the teachers where a full 85% vote to improve conditions for the teachers. But with budgetary constraints, the community can afford only one option. As such 40% voted for a salary increase but against benefits, 45% voted to improve benefits, but against a salary increase, and only 15% voted against both.

In other words, a source of voting puzzles is that radically different scenarios (called profiles), which specify how many voters prefer which options, can share the same election outcome. In this example designed by Katri Sieberg [17], the outcome accurately represents the voters’ wishes for the first profile, but it violates their intent with the second one. The voting theory issue is to understand how and why this can arise.

Sieberg and I developed an elementary geometric approach to explain these mysteries that arise in voting [17] and engineering [18]. We did so by showing how to find all possible profiles (i.e., all possible scenarios) that yield the same specified election outcome.

This is easy to accomplish should there be only a single issue, say the salary. The reason is that both the voters’ preferences and an interpretation of a “Yes-No” pairwise vote can be represented by the same point in the Fig. 1a unit interval. If point $y_S$ (the subscript identifies the proposal) represents the proportion of “Yes” votes, then $y_S = 0$ is the left
endpoint with an unanimous negative vote, while \( y_S = 1 \) is the right endpoint with an unanimous positive vote. The Fig. 1a point of \( y_S = .40 \), which is closer to the left endpoint denoting rejection, means that only 40% of the voters are for Proposal \( S \), and 60% against. As the point represents both the profile (i.e., a listing of how many voters prefer each option) and its outcome, the interpretation is clear: These voters strongly reject \( S \).

![Diagram](image_url)

**Figure 1.** Finding all profiles

1.1. **Set of all profiles.** The Fig. 1b unit square represents the two-issue case; the election outcomes are represented by the point \((y_S, y_B) = (0.40, 0.45)\). The lesson learned from the above second scenario is that, in general, the point representing the election outcomes need not represent the profile! Ideally, to appreciate what the outcomes might mean, all possible profiles supporting these tallies should be identified. A way to do so is indicated in Fig. 1c.

As represented by the Fig. 1c shaded region, draw the cone consisting of all lines with endpoints on the left and right edges and that pass through the specified election outcome. This is easy to do; just find the cone’s boundaries: They are the two extreme lines where each hits one of the square’s vertices. Each line in the cone is called a “profile line.” As the name suggests and as indicated below, each profile line represents a supporting profile.

The profile cone’s boundary edges are the extreme supporting profiles. The Fig. 1c upward slanting cone edge represents Sieberg’s first scenario (the right endpoint is the vertex \((1, 1)\) where all of these voters vote for \( S \) and \( B \), while the extreme downward sloping profile line corresponds to the second scenario (the right endpoint is the vertex \((1, 0)\) where all of these voters support \( S \) \((y_S = 1)\) but none support \( B \) \((y_B = 0)\)).

As Sieberg and I proved [17], for any specified pair of election tallies, any profile line in its profile cone represents a supporting profile, and any supporting profile defines a profile line. Thus, the sought after set of all possible profiles is given by the profile cone.

Details of how to identify a profile line with a standard profile description are in [17], but to illustrate the scheme, consider the Fig. 1c dashed profile line with endpoints \((0, \frac{1}{6})\) and \((1, \frac{7}{8})\). (Two points define a line, so an endpoint and the tally define the other endpoint.) The \( y_S = 0.40 \) outcome (the dotted vertical line in Fig. 1c) means that the 40% of voters who favor \( S \) are represented on the square’s right edge, while the 60% against \( S \) are represented on the left edge. The zero in \((0, \frac{1}{6})\) requires the profile line’s endpoint to be on the left edge; the second component of \( \frac{1}{6} \) describes how the 60% of voters against \( S \) are divided with respect to Proposal \( B \). As this component means that one-sixth of them favor \( B \), while five-sixths are against, it follows that
the proportion of all voters against both \( S \) and \( B \) is \( \frac{5}{6} \times 0.60 = 0.50 \), while
proportion against \( S \) but for \( B \) is \( \frac{1}{6} \times 0.60 = 0.10 \).
A similar computation for the \((1, \frac{7}{8})\) endpoint completes the profile by showing that
the proportion of all voters supporting both \( S \) and \( B \) is \( \frac{7}{8} \times 0.40 = 0.35 \) and
the proportion supporting \( S \) but against \( B \) is \( \frac{1}{8} \times 0.40 = 0.05 \).
Conversely, a given profile identifies a point on each of the square’s vertical edges; the
associated profile line connects these points. To illustrate with the first scenario, all voters
voting for \( S \) also voted for \( B \), so the point on the \( y_S = 1 \) right edge is the top vertex
\((1, 1)\) where \( y_B = 1 \). The left edge corresponds to the 60% of the voters voting against \( S \);
of these voters, 5%/60% voted for \( B \), so the point on the \( y_S = 0 \) edge is \( y_B = \frac{1}{12} \). The
corresponding profile line is the line segment connecting \((0, \frac{1}{12})\) with \((1, 1)\).

1.2. New results. A wealth of new results follows by knowing how to find all possible
supporting profiles. To explore what can happen, just place the election outcome in various
places in the square, draw the cone, and then determine properties of the associated profile
cone. The following offers a sample of conclusions developed in [17]:

- Whatever the strict (i.e., no ties) election outcome for two proposals, there always
  are some voters who support both outcomes. To extract this assertion from the
  geometry, start with the one vertex that agrees with the outcome; the other vertex
  on this vertical edge has voters who agree with one outcome but not the other. As
  the geometry keeps this second vertex out of the profile cone, all profile lines are
  kept a distance from this second vertex. This gap, this distance, defines a minimal
  number of voters in a supporting profile who agree with both outcomes.

  Illustrating with Sieberg’s example, all profile lines have their left endpoint below
  \((0, \frac{3}{4})\), so at least \( 1 - \frac{3}{4} = \frac{1}{4} \) of the voters on the left edge support both outcomes.
  Thus in any profile at least \( \frac{1}{4} \times 0.60 = 0.15 \) of all voters support both outcomes.

- If neither outcome is a tie, the profile cone has another gap along a vertical edge; in
  Fig. 2.c, this gap on the left edge is between the vertex \((0, 0)\) and \((0, \frac{1}{12})\) – the cone’s
  bottom boundary. Gaps have meanings; this opening keeps profile lines away from
  the vertex representing the successful election outcomes, so it identifies the maximal
  percentage of all voters who support both winning conclusions. Here the assertion
  is that with the specified tallies, no profile has more than \((1 - \frac{1}{12})60\% = 55\% \) of all
  voters supporting both successful outcomes.

- Sieberg’s second scenario proves that most voters could disagree with at least one
  outcome. But is this an anomaly, or does it reflect a general concern? A way
to address this kind of question is to find the percentage of supporting profiles
with this property; here the property is that most voters disagree with at least one
outcome: Answers follow directly from the geometry. To illustrate with Fig. 1c,
the 40\% of all voters who are represented on the right edge disagree with at least
one outcome. It remains to find enough voters on the left edge who disagree with
one outcome; i.e., they voted for \( B \).
With 60% of all voters being on the left edge, to obtain at least 10% more of all voters, the left endpoint must be above where 10%/60% = \( \frac{1}{6} \) of the voters on the left vote for B. This left endpoint of \((0, \frac{1}{6})\) defines the Fig. 1c dashed line, so each profile in the broad region of profile lines with a left endpoint above \( \frac{1}{6} \) has the property that most voters disagree with at least one outcome.

What a surprise! The dominating size of this geometric region proves that most profiles have most voters against at least one outcome! To provide numbers, the edge length of this region (the difference between the cone’s top endpoint and \( \frac{1}{6} \)) is \( \frac{3}{4} - \frac{1}{6} = \frac{7}{12} \), while the left-edge length of the cone (the difference between the cone’s top and bottom endpoints) is \( \frac{3}{4} - \frac{1}{12} = \frac{2}{3} \), so the fraction of profile lines where most people disagree with at least one outcome is the surprisingly large \( \frac{7/12}{2/3} = \frac{7}{8} \); that is, 87.5% of all profiles lines have most voters disagreeing with at least one outcome! (Different probability distributions can be used on the edge; e.g., knowing the number of voters permits using binomial or normal distributions; both yield values larger than 87.5%.)

- This result raises other concerns; e.g., while the outcomes for Sieberg’s example seem conclusive, doubts are raised by learning that a whopping 87.5% of supporting profiles have most voters disagreeing with at least one outcome. How strong must the election tallies be to ensure that most profiles have most voters supporting both outcomes? Answers (which are surprising) come from the geometry; just position the outcome point so that the profile cone has desired properties.
- This geometry extends to “bundled voting,” where a package of alternatives are bundled into one bill. Is it possible (or even likely) for most voters to dislike aspects of a passed bill? For a specified outcome, how likely is this to occur?
- Some elections involve many proposals; e.g., Californian voters expect to vote on several initiatives. To extend the geometry to these settings, replace the unit square with a higher dimensional unit cube. Here, the geometry provides surprises. Contrary to the above first bulleted point, for instance, with three or more proposals it could be that nobody agrees with all of the actual election outcomes. Ongoing questions, then, include examining the likelihood that, say, not a single marked ballot in a specified California election agrees with all actual election outcomes!

2. Pairwise vs. Positional Outcomes

The geometry of an equilateral triangle provides a convenient way to answer questions about positional voting. This is where a ballot is tallied by assigning a specified number of points to a candidate depending on her position on the ballot. With the plurality vote, for instance, the top-positioned candidate receives one point and all others receive zero. The antiplurality vote has the opposite flavor; it is where the bottom-positioned candidate receives zero points and all others receive one. (This “voting for all but one” is equivalent to voting against one. Thus this method is the same as a plurality vote to determine who the voters view as the most inferior candidate, which is the source of its name.) The Borda
Count for three candidates assigns to the top, middle, and bottom positioned candidate, respectively, two, one, and zero points.

Three candidate positional elections are characterized by the assigned weights

\[ w = (w_1, w_2, 0), \]  

where, in tallying a ballot, \( w_j \) points are assigned to the \( j^{th} \) positioned candidate. So the plurality ballot is defined by \((1,0,0)\), while the Borda Count by \((2,1,0)\).

To introduce issues and terminology, consider the following example (which, only for identification purposes, is called the initial example) where all we know about voter preferences is that the pairwise votes over the three candidates \( \{A,B,C\} \) are

\[
A \succ B \quad (\text{that is, } A \text{ is preferred to, or beats, } B) \quad \text{with a 70:30 tally}, \\
A \succ C \quad \text{with a 60:40 tally, and} \\
B \succ C \quad \text{with a 55:45 tally}.
\]

By beating all other candidates, \( A \) is called the Condorcet winner. Conversely, poor \( C \) loses all paired comparisons, so she is the Condorcet loser.

The challenge is to determine all possible positional outcomes that can accompany specified paired comparison outcomes. For instance, even though the Condorcet loser \( C \) in this example seriously lost all paired comparison elections, could she be the plurality winner by beating both \( A \) and \( B \)? (Yes.) Could second-ranked \( B \) be the plurality winner? (No.) Who could be the Borda, or antiplurality winner? What about other positional methods?

Clearly, these are important issues. But, except for the Borda Count, they are not addressed in the literature. (We know, for instance, that the Borda Count always ranks a Condorcet winner over a Condorcet loser.) To fill this gap concerning what happens with other rules, Tomas McIntee and I [16] developed the following geometric approach to answer all three-candidate questions of this kind. As true in the previous section, the key is to find all possible profiles supporting specified paired comparison outcomes.

**Figure 2.** Computing tallies

2.1. **Geometric profile representation.** The approach relies on a geometric way to represent profiles and to compute tallies that I developed in [8]. (Also see [11, Chap. 5]; Hannu Nurmi [6] nicely used my approach to analyze actual elections.) Start by assigning each candidate to a vertex of an equilateral triangle. Rankings are assigned to points in the triangle according to how close they are to each vertex. For instance, all points on the vertical line in Fig. 2a are equal distance from the \( A \) and \( B \) vertices, so they correspond to
indifference, or a tie, represented by \( A \sim B \). Points in the large shaded triangle to the left of this line have \( A \succ B \) while those in the large triangle to the right have \( B \succ A \).

After drawing similar indifference lines for each pair (i.e., perpendicular bisectors for each edge), there are thirteen regions. The six small open right triangles correspond to strict rankings; e.g., points in the lower left corner triangle (with “26”) are closest to \( A \) and next closest to \( B \), so they have the \( A \succ B \succ C \) ranking. Points on the six line segments involve indifference; e.g., points on the short vertical segment meeting the \( C \) vertex have the ranking \( C \succ A \sim B \). The center point is complete indifference, or \( A \sim B \sim C \).

As an aside, notice how this Fig. 2a geometry captures the transitivity conditions. The intersection of the \( A \succ B \) and \( B \succ C \) large triangles is the sole \( A \succ B \succ C \) ranking region, which reflects the “\( A \succ B \) and \( B \succ C \) requires \( A \succ C \)” transitivity condition. Similarly, the intersection of the \( B \sim C \) line with the \( C \succ A \) right triangle is the sole line segment representing \( C \sim B \succ A \), which is the “\( B \sim C \) and \( C \succ A \) requires \( B \succ A \)” transitivity condition. But the intersection of the \( A \succ B \) and \( C \succ B \) triangles has the three regions \( A \succ C \succ B \), \( A \sim C \succ B \), and \( C \succ A \succ B \). This indeterminacy plays a central role in proving Arrow’s Theorem (e.g., see Saari [8, pp. 83-99], [10, pp. 217-227], and Sect. 3.3).

A profile (which now specifies how many voters have each strict transitive ranking of the candidates) is geometrically represented by placing each number in the appropriate ranking region. With Fig. 2a, then, 26 voters prefer \( A \succ B \succ C \), 5 prefer \( A \succ C \succ B \), 39 prefer \( C \succ A \succ B \), 1 prefers \( C \succ B \succ A \), zero prefer \( B \succ C \succ A \), and 29 prefer \( B \succ A \succ C \). (If voters can have ties, place these numbers on the appropriate regions.) As shown next, this geometry significantly simplifies computing pairwise and positional tallies!

2.2. Tallies. All voters preferring \( A \) to \( B \) are in the shaded Fig. 2a left triangle; those preferring \( B \succ A \) are in the right triangle. Thus to compute the \( \{A, B\} \) tally, just add the numbers in each triangle; the \( A \succ B \) tally of 70:30 is listed below the horizontal edge. The pairwise tallies for all paired elections are similarly found and listed by the appropriate edge. Incidentally, notice that this Fig. 2a profile supports the initial example.

The plurality vote counts how many voters have each candidate top-positioned. According to the geometry, these numbers are located in the two regions that share the candidate’s vertex; e.g., in Fig. 2b, the number of voters voting for \( A \) is the sum of numbers in the two shaded regions. For Fig. 2b, these tallies define the outcome \( C \succ A \succ B \) with the 40:31:29 tally, which proves that even a badly beaten Condorcet loser could be a strong plurality winner!

It remains to compute the outcomes for the infinite number of other positional methods! To do so, scale each \( \mathbf{w} = (w_1, w_2, 0) \) so that its tally is an “add-on” to the plurality tally. Namely, divide each term by \( w_1 \) to obtain \( \mathbf{w}_s = (1, s, 0) \), where \( s = \frac{w_2}{w_1} \) is the “second place value.” For instance, \( \mathbf{w}_0 = (1, 0, 0) \) is the plurality vote, \( \mathbf{w}_{\frac{1}{2}} = (1, \frac{1}{2}, 0) \) is the normalized Borda Count, and \( \mathbf{w}_{\frac{3}{7}} = (1, \frac{3}{7}, 0) \) is a normalized form of \((7, 3, 0)\). To convert a normalized tally into a tally for the original method, multiply each normalized value by \( w_1 \).

With normalized voting vectors, a candidate’s \( \mathbf{w}_s \) tally is
VOTING MYSTERIES: A PICTURE IS WORTH A THOUSAND WORDS

{the candidate’s plurality tally} plus \( s \) times the number of voters who have the candidate second-ranked.

Thus, candidate A’s \( w \) tally is her plurality tally plus \( s \) times the sum of values in the two Fig. 2b regions with an arrow, or \( 31 + s(29 + 39) = 31 + 68s \). All tallies are similarly computed and listed by the appropriate Fig. 2b vertex. They are

\[(2) \quad A : 31 + 68s, \quad B : 29 + 27s, \quad C : 40 + 5s.\]

A wealth of information about how election outcomes can change with the voting rule now follows: For instance, it always is true that \( 31 + 68s > 29 + 27s \), so A always beats B with any positional method. Similarly, by solving \( 31 + 68s > 40 + 5s \), it follows that when \( s > \frac{9}{63} = \frac{1}{7} \), A also beats C, so A is the \( w \) winner. Solving \( 29 + 27s > 40 + 5s \) shows that once \( s > \frac{11}{22} = \frac{1}{2} \) (the Borda Count), the ranking is \( A \succ B \succ C \). This profile, then, has five \( w \) rankings where either the Condorcet winner A or loser C are winners, but never B:

\[(3) \quad C \succ A \succ B \text{ for } 0 \leq s < \frac{1}{7}, \quad C \sim A \succ B \text{ for } s = \frac{1}{7}, \quad A \succ C \succ B \text{ for } \frac{1}{7} < s < \frac{1}{2}, \quad A \succ B \succ C \text{ for } s = \frac{1}{2}, \quad A \succ B \sim C \text{ for } \frac{1}{2} < s \leq 1.\]

2.3. Never affecting paired comparison rankings. Key for what follows is to determine what part of a profile causes the pairwise tallies. To do so, it is useful to use the difference in pairwise tallies. So, let \( P(X, Y) \) be \( X \)’s tally in a paired comparison minus \( Y \)’s tally. Illustrating with the initial example, \( P(A, B) = 70 - 30 = 40 = -P(B, A) \), \( P(A, C) = 60 - 40 = 20 \), and \( P(B, C) = 55 - 45 = 10 \). For convenience assume that \( P(A, B), P(B, C) \geq 0 \), which means that A either beats or ties with B and B either beats or ties with C.

The crucial step is to find what we call the essential profile [16]; it is the profile with the smallest number of voters that supports the specified \( P(X, Y) \) values. Should a profile be given, a natural way to find its essential profile is to drop all profile components that never affect \( P(X, Y) \) values; i.e., drop profile components for which \( P(X, Y) = 0 \). This \( P(X, Y) = 0 \) condition always holds for reversal pairs such as \( A \succ B \succ C \) and \( C \succ B \succ A \) because they have a tie for each pair. A surprise is that reversal terms, and only reversal terms, have this property for three-candidate elections (Saari [9, 11], [12, Chap. 4])! This result about reversal terms is central to a theory that explains all possible, single profile, three candidate positional and pairwise difficulties. (A nice exercise is to compare the symmetries of the Fig. 2a triangle with the components from this decomposition.)

Reversal pair entries are diametrically opposite in my triangle representation for profiles, such as with the two Fig. 2b regions with arrows. So with a given profile and each pair of diametrically opposite regions, subtract the smaller value from both. With Fig. 2a, for instance, drop 29 pairs of \( B \succ A \succ C \), \( C \succ A \succ B \) to leave 10 voters with \( C \succ A \succ B \) preferences and zero with \( B \succ A \succ C \). Carrying out this program for all three reversal pairs creates the essential profile for Fig. 2a, which is given by the first triangle in Fig. 2c.

This “subtraction” construction means that, in an essential profile, at least one region for each reversal pair has a zero. This leaves four possible configurations; three have a large right-triangle filled with zeros, as with the Fig. 2c essential profile. The remaining
choice, with zeros in every other region in a cyclic fashion about the triangle, violates

\[ P(A, C) \geq \min(P(A, B), P(B, C)). \]

This condition is violated with a cycle because \( P(A, C) < 0 \). To violate it with a transitive outcome, \( P(A, C) \) – the tally difference between the Condorcet winner and loser – must be smaller than the tally differences for both \( P(A, B) \) and \( P(B, C) \).

As an aside and as developed in [16], the essential profile can be computed directly from \( P(X, Y) \) values. If they satisfy Eq. 4, subtract the largest \( P(X, Y) \) value from each tally.

For the initial example, then, subtract \( P(A, B) = 30 \) to yield \( A \succ B \) by 40:0, \( B \succ C \) by 25:15, and \( A \succ C \) by 30:10. The unique supporting profile can be computed from algebra. If Eq. 4 is not satisfied, the computation is slightly different.

2.4. Set of all profiles. As proved in [16],

- The essential profile is unique.
- By knowing the essential profile, it is easy to find all possible supporting profiles!

Knowing all possible supporting profiles, of course, must be expected to lead to many new results. As illustrated with Fig. 2c, to find all of these profiles, just add reversal pairs, in all possible ways, to the essential profile.

The second Fig. 2c triangle identifies all reversal pairs; e.g., the variable \( \alpha \) represents the number of \( \{A \succ B \succ C, C \succ B \succ A\} \) pairs, etc. The initial example’s essential profile has forty of the hundred voters, so sixty voters, or thirty reversal pairs need to be added back. This means that all supporting profiles for the initial example are obtained (as illustrated by Fig. 2c) by adding to the essential profile all possible reversal pairs satisfying

\[ \alpha + \beta + \gamma = 30, \quad \alpha, \beta, \gamma \geq 0. \]

By knowing all profiles and by computing tallies (as with the Eq. 3 listing), all possible positional outcomes associated with these pairwise tallies can be determined. To illustrate with Fig. 2c, the essential profile’s A:B:C tallies are 30 + 10s : 25s : 10 + 5s, and the reversal pairs’ tallies are \( \alpha + \beta + 2\gamma s : \beta + \gamma + 2\alpha s : \alpha + \gamma + 2\beta s \). So the general A:B:C tallies are

\[ 30 + \alpha + \beta + (10 + 2\gamma)s : \beta + \gamma + (25 + 2\alpha)s : 10 + \alpha + \gamma + (5 + 2\beta)s \]

with the Eq. 5 side condition.

To illustrate how to discover new results, let me prove my earlier assertion that \( B \) cannot be the plurality winner (\( s = 0 \)) for the initial example. The proof reduces to showing it is impossible to find \( \alpha, \beta, \) and \( \gamma \) values satisfying Eq. 5 so that \( \beta + \gamma > 30 + \alpha + \beta \) (\( B \) beats \( A \)) and \( \beta + \gamma > 10 + \alpha + \gamma \) (\( B \) beats \( C \)). The first inequality requires \( \gamma > 30 + \alpha \), which is impossible with Eq. 5. Of course an \( A \sim B \) plurality tie is possible with \( \gamma = 30 \) and \( \alpha = \beta = 0 \), but with these values, the second inequality anoints \( C \) as the plurality winner.

2.5. New results. Knowing all profiles permits deriving many new results. Start with the Borda \( s = \frac{1}{2} \) value for Eq. 6 with the \( A \succ B \sim C \) tally of \( 35 + (\alpha + \beta + \gamma) : 12.5 + (\alpha + \beta + \gamma) : 12.5 + (\alpha + \beta + \gamma) \). The \( (\alpha + \beta + \gamma) \) term is common to each tally, so it never impacts the Borda ranking outcome. As this tally shows (and as known to be true in general [12]), Borda rankings are not affected, in any way, by reversal terms. The Borda ranking is
completely determined by the essential profile; it is completely determined (for any number of candidates) by \( P(X, Y) \) values.

**Theorem 1.** If \( A \) and \( C \) are, respectively, the Condorcet winner and loser, then \( A \) is the strict (no ties) Borda winner if and only if

\[
2P(A, B) + P(A, C) > P(B, C).
\]

If the inequality is reversed, the Borda ranking is \( B > A > C \).

For \( B \) to be a Borda winner, she must overcome a huge obstacle. She lost to \( A \) (because \( A \) is the Condorcet winner), so, according to Eq. 7, not only must her victory margin over the Condorcet loser \( C \) be greater than \( A \)'s victory over \( C \), but it must be greater than this victory plus twice \( A \)'s margin over \( B \)! As this is difficult to do, it is reasonable to expect, in general, that the Condorcet and Borda winners agree.

These observations also explain why it is much easier to find results for the Borda Count than for other positional methods: Borda tallies depend strictly on \( P(X, Y) \) values. In contrast, non-Borda positional methods are affected by \( P(X, Y) \) and reversal components; this extra dependency complicates finding relationships between pairwise and these positional rankings. Fortunately, with our geometric approach, all conclusions can be found with simple algebra. The following is a sample of results from [16] when Eq. 4 is satisfied.

- If \( A \) is a strict Condorcet winner (i.e., both \( P(A, B) \), \( P(A, C) \) are positive), at least one profile has \( A \) as the sole plurality winner. There may be other supporting profiles, however, with different plurality winners.
- So, being the Condorcet winner does not suffice to ensure \( A \) is the plurality winner. Instead, \( A \) needs to have substantial pairwise victories to ensure she is the plurality winner for all supporting profiles. The condition with \( n \) voters is that if \( P(A, Y) \) has the largest value (so \( A \) beats \( Y \) more soundly than any other paired election and \( X \) is the other candidate), then a necessary and sufficient condition for \( A \) to be the sole plurality winner for all profiles is \( 2P(A, X) + P(A, Y) > n \). If the inequality is reversed, some profiles elect \( X \) as the plurality winner.

This condition requires \( A \) to beat \( Y \) with at least two-thirds of the vote! To illustrate with the initial example where \( n = 100 \), because \( P(A, B) = 40 \) is the largest victory margin, \( Y = B \) and \( X = C \). This example fails to satisfy the inequality because \( 2P(A, C) + P(A, B) = 2(20) + 40 < 100 \). Thus, some supporting profiles elect \( X \) (which here is \( C \)) as the plurality winner.

- In fact, just to limit the number of possible plurality winners, paired victories must be surprisingly dominant. If \( P(X, Y) \) is the largest pairwise victory, and if \( 3P(X, Y) < n - 4 \), then each candidate is the plurality winner with some supporting profile. With a hundred voters, then, if the largest margin is less than 32 (that is, if the winner for each pair receives less than 66 votes), anyone can be the plurality winner! If \( 3P(X, Y) > n - 4 \), at most two candidates can be plurality winners. With our initial example, \( 3P(A, B) = 120 > 100 - 4 \), which means that no more than two candidates can be plurality winners. As \( A \) and \( C \) can, \( B \) cannot.
• Similar results hold for any positional method. To illustrate with the antiplurality rule ($s = 1$), if $P(A, Y)$ is the largest pairwise victory, then $A$ is the only antiplurality winner if and only if $2P(A, Y) > n + P(B, C)$. This coupling of the victory of $B$ over $C$ with the number of voters, makes the condition more difficult to be satisfied; it requires $A$ to beat someone by receiving over 75% of the vote!

The above samples results from [16]; many more are waiting to be derived. Notice how the farther a positional method is from the Borda Count (that is, the larger the $|s - \frac{1}{2}|$ value), the weaker the connections between the positional and pairwise outcomes. Compatible outcomes for these methods require shockingly large, unrealistic pairwise victories!

3. Power of a cube

Once the $P(X, Y)$ values are specified, all supporting profiles can be found. So, it remains is to find all possible $P(A, B), P(B, C), P(A, C)$ values. This objective, of course, is equivalent to finding all possible pairwise tallies that could ever occur with $n$ voters.

The answer (from Saari [8]) comes from the geometry of a cube; as it will be shown, a specific part of this cube identifies all possible pairwise tallies and a subregion identifies all possible cyclic behavior. (That is, pairwise outcomes of the $A \succ B, B \succ C, C \succ A$ type.) To illustrate how other results follow from this geometry, I will use the cube geometry to explain and extend the somewhat puzzling discursive paradox, and to provide new interpretations for Arrow’s seminal theorem [1].

![Figure 3. Pairwise tallies](image)

3.1. All pairwise outcomes. With $n$ voters, normalize the pairwise tallies to be of the form $\frac{P(A, B)}{n}$, which means, as illustrated in Fig. 3a, the values range from $-1$ (where $B$ receives a unanimous vote) to $+1$ (where $A$ receives a unanimous vote), and 0 corresponds to a tie. (This differs from Fig. 1a.) Thus the tallies for all three pairs are given by a point $(x, y, z) = \left( \frac{P(A, B)}{n}, \frac{P(B, C)}{n}, \frac{P(C, A)}{n} \right)$ in the Fig. 3b cube $[-1, 1]^3$. 
The cube has eight vertices; six of them correspond to an unanimous vote for a particular transitive ranking. As listed on the cube, they are

<table>
<thead>
<tr>
<th>Name</th>
<th>Ranking</th>
<th>Name</th>
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<th>Ranking</th>
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<tr>
<td>1</td>
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<td>2</td>
<td>A ≻ C ≻ B</td>
<td>3</td>
<td>C ≻ A ≻ B</td>
</tr>
<tr>
<td>4</td>
<td>C ≻ B ≻ A</td>
<td>5</td>
<td>B ≻ C ≻ A</td>
<td>6</td>
<td>B ≻ A ≻ C</td>
</tr>
</tbody>
</table>

For convenience, these names are located in the Fig. 3c triangle (from Sect. 2.1). The remaining two cube vertices represent an unanimous vote for a cyclic ranking; vertex 7 corresponds to \(A ≻ B, B ≻ C, C ≻ A\), while vertex 8, which is diametrically opposite 7 and hidden in the back, corresponds to the opposite cycle \(B ≻ A, C ≻ B, A ≻ C\).

Vertices 7 and 8 demonstrate that some points in this cube cannot be represented by the votes of voters with complete transitive outcomes. So, the next task is to find all points that can be the outcome of voters with these preferences. To do so, notice that if two-thirds of the voters have preference 1 while the last one-third have preference 2, then the election outcome is on the line connecting these two vertices; it is two-thirds of the way toward vertex 1. In other words, the election outcomes are convex combinations of the six vertices, so admissible outcomes are in the convex hull of the six identified transitive vertices. In particular, the tetrahedron region defined by vertices 1, 3, 5, and 7 that is above the plane defined by the first three points must be excised because points in this region can never be attained by rational voters. A similar region involving the four vertices identified with even integers must also be cut away. I call what remains the representation cube; see [8, p. 100] for a pattern to make a physical copy.

If a point in this representation cube has fractions as coordinates, it corresponds to a pairwise election outcome. But the parity of \(P(A, B), P(B, C), P(A, C)\) and \(n\) (the number of voters) must agree. That is, either all are odd integers, or all are even integers. So, all possible pairwise outcomes with \(n\) voters are given by all \((x, y, z)\) points that are in the representation cube where integers \(x, y\) and \(z\) have the same parity as \(n\). Because any point with fractions can be rewritten in a form where the common denominator and the numerators are even integers (multiply the numerator and denominator by an appropriate even integer), all rational points represent election outcomes.

With my choice of coordinates, all points in the positive and negative orthants represent cycles. The positive orthant in Fig. 3b is given by the dotted lines that define the tetrahedron with vertices \((0,0,0)\) (a complete tie), \((1,0,0)\) (for \(A ≻ B, B ≻ C, A ≻ C\)), \((0,1,0)\) (for \(A ∼ B, B ≻ C, C ≻ A\)), and \((0,0,1)\) (for \(A ∼ B, B ∼ C, C ≻ A\)). A similar cyclic region lies diametrically opposite in the negative orthant.

As a direct computation shows, these cyclic regions consist of \(\frac{1}{16}\) of the total volume, which suggests that, with large number of voters, cycles occur with probability \(\frac{1}{16} = 0.0625\): This estimate agrees with probability values based on a particular assumption about the probability distribution of voter preferences (Gherlein [4]). This agreement is

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1This is because \(P(X,Y)\) is \(X\)'s tally minus \(Y\)'s tally, and \(n\) is the sum of these tallies. So if \(n\) is odd, then one of these tallies is odd and the other even, which means that \(P(X,Y)\) is an odd integer. If \(n\) is even, then either both both tallies are odd, or both are even, which makes \(P(X,Y)\) even.
not a coincidence; McIntee and I proved (extending ideas developed in [16]) that this assertion holds for a wide class of probability distributions.

The geometry also indicates how to prevent having an outcome of any specified type, say a cyclic outcome. This is because a point in my representation cube must be a convex combination of appropriately selected Eq. 8 vertices. So, with experimentation, it follows that it is impossible for a point to be in the positive orthant without using vertices 1, 3, and 5; leave one out (that is, no voters can have that particular preference ranking) and the outcome cannot be cyclic in the positive orthant. As a similar statement holds for cycles in the negative orthant (with vertices 2, 4, 6), a host of conclusions follow immediately. The first is that:

> If a profile has no voters with preferences from at least one of the three odd vertices, and no voters with preferences from at least one of the three even integers, then it is impossible for the profile to define a cycle.

Using the Sect. 2.1 profile representation, this means that if a profile has no voters in at least one of the shaded and one of the unshaded Fig. 3c regions, then a cycle is impossible. This simple statement includes as special cases Black’s well known single-peaked condition [2] and Ward’s generalization [19] asserting that a cycle cannot occur should any one of the following three conditions be satisfied:

1. Some candidate never is top-ranked by anyone.
2. Some candidate never is middle-ranked by anyone.
3. Some candidate never is bottom-ranked by anyone (which is Black’s condition).

To see how my statement includes Ward’s conditions with candidate A, if she is never top-ranked, then the Fig. 3c representation has no voters in regions 1 and 2, which satisfies my condition. If she is never second-ranked, then the profile have no voters in regions 3 and 6, which satisfies the condition. If she is never bottom-ranked, then the profile has no voters in regions 4 and 5, which also satisfies the condition.

To include the Sect. 2 results into this discussion, each point in the cube represents an essential profile. To see this, for a given \( p \) in this cube, draw a line from the origin through \( p \) and continue until it hits a face of the representation cube. Each of the four possible faces represents one form of the essential profile. On each face, the point has a unique representation, which defines the essential profile. The distance of \( p \) to the point on the face corresponds to the number of reversal profiles that can be added to define all possible supporting profiles. In this way, new assertions about the likelihood of cycles and other behaviors can be computed.

3.2. Aggregation losing logic. This cube geometry explains a large number of other mysteries, including how majority vote outcomes can violate principles of logic. To introduce the basic ideas in terms of the discursive dilemma, which has been extensively studied by several including Christian List [5] and his coauthors, suppose a group of three is evaluating a candidate for tenure. The rules are that the candidate must do a good job in teaching and in research; the decision is based on a majority vote. The poor candidate
was not promoted because, as shown in the following (the last column), two of the three reviewers – a majority – had a negative assessment.

<table>
<thead>
<tr>
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<th>Teaching</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
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<td>No</td>
<td>No</td>
</tr>
<tr>
<td>2</td>
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<td>No</td>
</tr>
<tr>
<td>3</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
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</tbody>
</table>

But had the decision been made by taking a majority vote over each issue, the candidate would have succeeded. (The majority vote over each of the first two columns has a positive outcome. Receiving a favorable response on both traits, the candidate would have succeeded.) This conflict is manifested by the totals of the three columns with the illogical combination of Yes on research, Yes on teaching, but No on tenure. Daniel Eckert and Christian Klamler [3] used my geometry of the cube, as follows, to explain this disagreement.

The inadmissible outcomes are

\[ (11) \quad 2 - (Yes, No; Yes), \quad 6 - (No, Yes; Yes), \quad 7 - (Yes, Yes; No), \quad 8 - (No, No; Yes). \]

The inadmissible outcomes are

\[ (10) \quad 1 - (Yes, Yes; Yes), \quad 3 - (Yes, No; No), \quad 4 - (No, No; No), \quad 5 - (No, Yes; No). \]

All majority vote aggregated outcomes are in the convex hull defined by the admissible vertices 1, 3, 4 and 5. Notice, this hull includes a portion of the first orthant (defined by the dotted lines in Fig. 4a), which agrees with a cyclic region of Fig. 3b. \textit{All points in this region inherit the ranking of vertex 7, which is the inadmissible (Yes, Yes; No).} Thus, the earlier cyclic region for voting (non-transitive outcomes) becomes a region of illogical assertions as captured by the outcomes of the three columns of Eq. 9. (The Eq. 9 example is in the convex sum of vertices 1, 3, 5; the outcome is the center point of the Fig. 4a equilateral triangle with dashed lines.)
More generally,

*a necessary and sufficient condition for a setting to allow a series of majority votes to have undesired outcomes is for the convex hull of the vertices corresponding to admissible principles to include a region where its vertex represents an undesired outcome.*

Notice how I used this property above to derive Ward’s and Black’s condition ensuring that cycles cannot occur: Namely, the admissible profiles (vertices) were selected to ensure that the convex hull they define misses the cyclic regions.

While this geometric principle holds for any dimensional cube, it is particularly easy to apply with three pairs where the vertices have three components. Here, we just need that each of three admissible vertices is attached by a leg to the same inadmissible vertex. (This statement echoes the discussion in [12, Chap. 2].) For the cyclic outcome, each of the Fig. 3b admissible vertices 1, 3, 5 is connected by an edge to inadmissible vertex 7. (Just change one appropriate ranking for each vertex.) The same is true for Fig. 4a; just change one appropriate response for each admissible vertex so it becomes (Yes, Yes; No).

An example using this geometric approach developed by Tommy Ratliff and me [7] has to do with the election of a diverse committee; e.g., the goal may be to elect a mixed gender committee where each of three divisions has a man and a woman candidate. If each voter votes according to this mixed gender principle, three of the six admissible vertices are

\[(\text{Man, Man}, \text{Woman}), \ (\text{Man}, \text{Woman}, \text{Man}), \ (\text{Woman, Man}, \text{Man})\]

But each vertex is connected by an edge to the inadmissible (Man, Man, Man) vertex! (For each vertex, define the edge connection by replacing “Woman” with “Man.”) Thus, *even though each voter votes for a diverse committee, it is possible to have a non-diverse outcome!* Using the above choices, for instance, if each is supported by one-third of the voters, an all male committee is elected.

What seems to have been overlooked is that this geometric property identifies other settings where the majority vote can violate principles of logic. To illustrate by replacing “and” in the Eq. 10 tenure example with “or,” the candidate can obtain tenure by excelling in Research or Teaching. To determine whether conflict can arise, just check whether each of three admissible vertices has a leg connection to the same inadmissible one. As shown in Eq. 12, this is the case: Changing an appropriate “Yes-No” response in an admissible outcome shows that each is but an edge away from the inadmissible (No, No, Yes). Thus, conflicting majority vote behavior is assured.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Reviewer} & \text{Research} & \text{Teaching} & \text{Outcome} & \text{Vertex} \\
\hline
1 & \text{Yes} & \text{No} & \text{Yes} & 2 \\
2 & \text{No} & \text{No} & \text{No} & 4 \\
3 & \text{No} & \text{Yes} & \text{Yes} & 6 \\
\text{No} & \text{No} & \text{Yes} & 8 \\
\hline
\end{array}
\]

Here the candidate receives tenure because reviewers 1 and 3 (vertices 2 and 6) believe the candidate performs satisfactorily in at least one category; reviewer 2 disagrees. Had the
procedure used a majority vote over issues, the candidate would have been unsuccessful. The conflict, captured by the sum of the columns, has the inadmissible outcome of vertex 8.

The above geometric characterization makes it easy to create a large number of examples from logic and other areas where majority votes over parts conflict with intended principles. Indeed, examples come from many “truth tables,” where voters can evaluate two or more inputs. In fact, the “only an edge away” special case explains a considerable portion of the literature concerning paired comparison difficulties (Saari [15]). To suggest how this happens with $N$ alternatives, start with the inadmissible cyclic outcome

$$A_1 \succ A_2, A_2 \succ A_3, \ldots, A_{N-1} \succ A_N, A_N \succ A_1.$$ 

Reversing any one ranking creates a transitive ranking. Thus this cyclic outcome is but an edge away from $N$ transitive rankings. These $N$ transitive rankings (called “ranking wheel configurations” in [15]) are what cause all sorts of difficulties.

3.3. “No voting system is fair”: Arrow’s theorem. As it should be expected from this discussion, the cube’s geometry also can be used to provide a simple proof of Arrow’s Theorem. Details are in Saari [10, pp. 217-227], so only an outline is given here but with the goal of providing added insights and a new interpretation of Arrow’s seminal assertion.

But first, recall how Fig. 2a was used to identify transitivity conditions. In the same way, the ranking regions (the bullets) of the Fig. 4a cube identify all relationships associated with paired comparisons. To illustrate how this is done, the $A \succ B$ relationship forces attention to the cube’s front face while $A \succ C$ is given by the bottom face. Combining these conditions corresponds to the intersection of these faces, which is the bottom front edge. This edge has the three relationships (three bullets) given by vertex 1 (or $A \succ B \succ C$), vertex 2 ($A \succ C \succ B$), and the edge’s midpoint $(1, 0, -1)$ ($A \succ B \sim C$).

Similarly, combining $A \succ B$ (front face) with $C \succ A$ (top face) is the front top edge with the three rankings (bullets) of vertex 3 (or $C \succ A \succ B$), vertex 7 (the cycle $A \succ B, B \succ C, C \succ A$), and the midpoint $(1, 0, 1)$ ($A \succ B, B \sim C, C \succ A$). The last two are not transitive, which underscores the reality that transitivity is a constraint that drops certain paired combinations. (More generally, the $A \succ B$ front face has nine rankings (bullets); the three not described are on the horizontal line passing through $(1, -1, 0)$ (or $A \sim C \succ B$), the midpoint (point $(1, 0, 0)$) (or $A \succ B, B \sim C, A \sim C$) and $(1, 1, 0)$ (or $A \succ B, B \succ C, A \sim C$). Of the nine rankings represented on this face, five are transitive. Of the 27 bullets in the cube, 13 represent transitive rankings and 14 do not.

My version of Arrow’s Theorem ([8, pp. 83-99]) is more general than usual descriptions. While for simplicity of exposition only the three alternative case is considered, everything extends in an immediate manner to any finite number of alternatives. The goal is to design a decision rule that satisfies the following requirements.

(1) Each voter has a strict (i.e., no ties) complete transitive ranking of the three alternatives \{A, B, C\}.

(2) The decision rule is to yield a complete, transitive ranking of the alternatives, which is called the societal ranking.
(3) (Independence of Irrelevant Alternatives, or IIA) The societal ranking of each pair is based strictly on how each voter ranks that particular pair.

(4) (Involvement) Each pair admits at least two different societal rankings. That is, no pair can have a fixed ranking; instead, there are profiles that yield one ranking for this pair, while other profiles yield at least one other ranking.

The first three conditions are standard. To add a twist to the third condition, notice how it reflects the \textit{reductionist philosophy} whereby to handle a complex problem, reduce it to more tractable parts. Find the answer for each part, and reassemble them to have an answer for the whole.”

For Arrow, the complex problem is to find a complete transitive ranking for the three alternatives. The IIA condition is the reductionist step; it requires each pair’s societal ranking to be separately determined. How this is done is up to the designer to find a clever method. Assembling the answers for the pairs, then, answers the original objective.

The fourth condition, Involvement, includes, as a special case the usual Pareto condition where if everyone has the same ranking of a pair, that is the societal ranking. (Pareto requires each pair to have two strict rankings, so Pareto is a special case of Involvement.) Wilson’s [20] negative Pareto condition is where if everyone has the same ranking of a pair, then the opposite ranking is the societal ranking of the pair; again, this is a special case of Involvement. For another choice, if everyone prefers $A \succ B$, that is the societal ranking; but if everyone prefers $B \succ A$, the societal ranking is $A \sim B$. Another choice, which illustrates the generality of this condition, is to allow only the two possible societal rankings $A \succ B \succ C$ and $C \succ B \succ A$; this requirement satisfies Involvement. In fact, “Involvement” can be relaxed to require that this property is satisfied by at least two pairs.

To offer the designer added flexibility in carrying out this task, no monotonicity condition (where the outcome somehow resembles voter preferences) is imposed. Instead, the following extension of Arrow’s result shows that the reductionist philosophy as applied to finding complete transitive rankings fails if it is to include the preferences of more than one voter.

\textbf{Theorem 2.} If a decision rule can be created that satisfies the above conditions, then, a single voter can be identified with the following property: For all possible profiles, the rule’s outcome always depends just on the preferences of the identified single voter.

The goal is to show that if a method satisfying the four conditions exists, its outcome always is determined by the preferences of the same single voter: What other voters want is immaterial. To identify who has the power to make societal changes, start with a particular pair, say $\{A,B\}$. The fourth condition ensures there are two profiles, $p_1$ and $p_2$, with different $\{A,B\}$ rankings. One-by-one, change each voter’s ranking from what it is in $p_1$ to what it is in $p_2$. At some point in this process, the outcome must change. (The outcome changes, but to what we do not know, nor care.)

To avoid missing any special cases, carry out this exercise for all pairs, all possible profiles having different outcomes for the pairs, and all possible ways to change voter preferences from one ranking to the other. If it always turns out that the same voter, let’s say Heili,
is the one who changes the outcome for all possible pairs and scenarios, then the decision rule depends on her preferences, and only her preferences.

If, in some way, the rule depends on the preferences of more than one voter, then there are scenarios where Heili changes the outcome for one pair, say \( \{A, B\} \) and a scenario where someone else, say Tatjana, changes the outcome for another pair, say \( \{B, C\} \). The “scenarios” specify who must have what ranking of each pair to enable either Heili or Tatjana to change the societal outcome. For everyone other than Heili and Tatjana, assign them a transitive ranking with their assigned \( \{A, B\} \) and \( \{B, C\} \) ranking (from the scenarios).

To empower Heili to change \( \{A, B\} \) outcomes, Tatjana might need to have a specific \( \{A, B\} \) ranking. If it is \( A \succ B \) (the front face of Fig. 4a), let Tatjana’s preferences range between vertices \( \{1, 2\} \); if it is \( B \succ A \) (the back face), let them range between vertices \( \{4, 5\} \). In either case, Tatjana is free to change \( \{B, C\} \) rankings while keeping fixed her \( \{A, B\} \) and \( \{A, C\} \) preferences. (This is where the Sect. 2.1 indeterminacy plays a role.)

Similarly to empower Tatjana to change \( \{B, C\} \) preferences, Heili might need to have specific \( \{B, C\} \) preferences. If they are \( B \succ C \) (right-side face), let her preference vary between vertices \( \{1, 6\} \); if they are \( C \succ B \) (left-side face), let them vary between \( \{3, 4\} \). Again, Heili is free to vary her \( \{A, B\} \) preferences with fixed \( \{B, C\} \) and \( \{A, C\} \) preferences.

For simplicity, suppose Heili’s actions can change societal rankings between \( A \succ B \) and \( B \succ A \), while Tatjana can change the outcome for her pair between \( B \succ C \) and \( C \succ B \). By construction, they can do what they want independent of what the other person does. Thus one outcome can be \( A \succ B \) and \( B \succ C \), which is the Fig. 4a edge with vertices 1 and 7. To be transitive (condition 2), the outcome must be vertex 1 on the bottom \( \{A, C\} \) face. Another choice is the opposite \( B \succ A \) and \( C \succ B \), which is the hidden edge with vertices 4 and 8; here only 4 on the top face has a transitive ranking. Notice: No voter changes \( \{A, C\} \) ranking, yet the outcome jumps between the top and bottom faces to change the \( \{A, C\} \) outcome. This violates IIA! The contradiction proves the theorem.

I leave it to the reader to handle settings such as where Heili can change the outcome between a strict ranking, say \( A \succ B \) and indifference \( A \sim B \). Here the outcomes are on the front face or middle slice. If Tatjana’s choices make strict societal changes, then there are settings where the society outcome is on the front-right edge, or the midpoint of the edge connecting 3 and 4. Again, to be transitive, the \( \{A, C\} \) ranking must change. A similar story holds if both outcomes have an indifference societal ranking.

As this analysis proves, the real message of Arrow’s result (which differs from other descriptions) is that the reductionist approach fails with this project. Arrow’s theorem does not mean, as some have claimed, that “no voting system is fair.” It only asserts that never try to find the transitive societal ranking by using paired comparison methods because, no matter how clever the approach, there are settings where it will fail. In this way, Arrow’s result is an indictment against the majority vote and Condorcet’s approach.

The proof has an added message based on how the \( \{A, C\} \) ranking jumped depending on the societal ranking for other pairs. It follows that to determine the societal ranking, each pair’s societal ranking must depend upon information about how other pairs are ranked.
4. Summary

Simple geometry can be used to explain, understand, and extend many voting mysteries. Beyond the paired comparison results that are emphasized here, much more is possible. For instance, the survey [14] uses geometry to explain strategic action, why a voter can be rewarded by not voting, why a winning candidates can lose by receiving added support, etc., etc. In a different survey [13], geometry is used to explain issues with a game theoretic concept, called the “core,” that arise in spatial voting.

References