Math. Inf. Sci. hum. (26^{e} année, n°101, *1988*, p.35-47)

MEASUREMENT REPRESENTATIONS OF ORDERED, RELATIONAL STRUCTURES WITH ARCHIMEDEAN ORDERED TRANSLATIONS

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During the past 12 years, research into the conditions under which measurement is feasible - that is, the characterization of those qualitative, ordered, relational systems that admit numerical representations - has progressed from axiomatizations of highly specific systems (see, for example, Krantz, Luce, Suppes & Tversky, 1971, and Roberts, 1979) to a far more complete understanding of the entire range of possibilities. At least this is true for structures that are highly symmetric - technically, homogeneous. The notes that follow, which are only a minor modification of the handout used at the talk, summarize the key concepts, definitions, major results, and selected references. The style is terse with little comment and no proofs. For more a discursive discussion, see Luce and Narens (1987) and Narens and Luce (1986); for more technical surveys see Luce, Krantz, Suppes, and Tversky (in press) and Narens (1985).

There is, of course, a closely related mathematical literature on ordered algebraic systems, in particular ordered groups and semigroups. For an early summary see Fuchs (1963), and for a more recent one, covering the numerous developments after 1960 through the late 1970s, see Glass (1981). To the best of my knowledge, the results reported here do not duplicate exactly those in the mathematical literature. To some extent this may be due to the motivation and guidance of our work issues by representing empirical information numerically - by measurement structures.

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This work was supported by National Science Foundation grant IST-8602765 to Harvard University. Final preparation was completed while the author was a Fellow at the Center for Advanced Study in the Behavioral Sciences with financial support provided by National Science Foundation grant BNS-8700864 and by the Alfred P. Sloan Foundation.

ORDERED RELATIONAL STRUCTURES

Ordered Relation Structure : $\mathcal{A} = \langle A, \succsim, S_j \rangle_{j \in J}$, where

A is a non-empty set (of empirical entities or numbers),

$\succsim$ is a total (or, sometimes, weak) order,

J is a non-empty set (usually integers) called an index set, and for each j in J , S_j is a relation of finite order on A .

Ordered Numerical Structure : $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$,

where R is a subset of the real numbers Re .

Isomorphism : Suppose $\mathcal{A}$ and $\mathcal{A}'$ are relational structures with

(i) the same index set J ,

(ii) for each j in J , order (S_j) = order $(S_{j'})$.

is a one-to-one mapping from A onto A' such that for all $a,b \in A$,

$$a \succsim b \text{ iff } \varphi(a) \geq \varphi(b) ;$$

and for all $j \in J$ and all $a_1, a_2, \ldots, a_{n(j)} \in A$,

$$(a_1, a_2, \ldots, a_{n(j)}) \in S_j \text{ iff } (\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{n(j)})) \in S_{j'} .$$

Numerical Representation : An isomporphism of $\mathcal{A}$ into (onto) a numerical structure $\mathcal{R}$.

AUTOMORPHISMS OF ORDERED RELATIONAL STRUCTURES

Automorphism : An isomorphism of a structure $\mathcal{A}$ onto itself. Let $\mathcal{J}$ denote the set of automorphisms, which is a group under function composition.

Translation : Any automorphism that either has no fixed point or is the identity, ι .

Dilation : Any automorphism with a fixed point.

Asymptotic Order : $\succsim'$ on $\mathcal{J}$ defined by : for $\alpha, \beta \in \mathcal{J}$, $\alpha \succsim' \beta$ iff for some $a \in A$ and all $b \in A$ such that $b \succ a$, $\alpha(b) \succsim \beta(b)$.

It is easy to see that $\succsim'$ is transitive and antisymmetric; however it may not be connected. Often, connectedness will derive from the assumptions made. Relative to function composition, denoted $\cdot$, it is monotonic : for all $\alpha, \beta, \gamma \in \mathcal{J}$, $\alpha \succsim' \beta$ iff $\gamma \cdot \alpha \succsim \gamma \cdot \beta$ iff $\alpha \cdot \gamma \succsim' \beta \cdot \gamma$.

Archimedean Ordered Subgroup of Automorphisms : $\langle \mathcal{T}, \succsim', \cdot \rangle$ where

(i) $\mathcal{T}$ is a subgroup of $\mathcal{J}$,

(ii) $\succsim'$ is connected, and so a total order ,

(iii) $\cdot$ denotes function composition, and

(iv) for each $\alpha, \beta \in \mathcal{J}$ such that $\alpha \succ' \iota$, there is some integer n such that $\alpha^n \succ' \beta$.

Stevens (1946,1951) suggested a classification of measurement systems according to the behavior of the automorphism group of a numerical representation; it is summarized in Table 1. Although widely cited and used in the behavioral and social science literature, this classification was never very thoroughly investigated until the work to be described.

HOMOGENEITY, UNIQUENESS, AND SCALE TYPE

The following concept of M-point homogeneity, which has been extensively studied in the literature on ordered permutation groups (see Glass, 1981, for a summary of results), was independently introduced by Narens (1981a,b) in arriving at a formal definition of scale type. The results uncovered appear to be distinct from those reviewed by Glass.

A subset of $\mathcal{B}$ automorphisms is :

M-Point Homogeneous : Given any two strictly increasing sequences of M elements, there is an automorphism in $\mathcal{B}$ that takes one into the other.

Degree of Homogeneity : The largest M for which M-point homogeneity holds.

Homogeneous : Degree of homogeneity ≥ 1 .

N-Point Uniqueness : Whenever two automorphisms of $\mathcal{B}$ agree at N distinct points, they are identical.

Finite Uniqueness : Degree of uniqueness $< \infty$.

Scale Type (M,N) :

M = the degree of homogeneity, and

N = the degree of uniqueness of group of automorphisms.

Homogeneity means, among other things, that no element is distinguishable from others by properties formulated in terms of the primitives. Thus, any system is excluded that has a singular element, e.g., a zero or a bound.

CLASSICAL MODELS OF MEASUREMENT BASED ON AN OPERATION

Consider an ordered relational structure in which either one of the relations is a binary operation or there is an operation that along with the order is equivalent to the relational structure (as is true for some difference and conjoint structures).

The operation is assumed to be positive and Archimedean.

In the associative case, Hölder's theorem (1901) is used to show :

(i) the existence of a representation onto $\langle Re^+, \geq, + \rangle$,

(ii) the representation is unique up to similarities $x \to rx$, $r > 0$, i.e., the structure and its representation are of scale type (1,1).

Behavorial and social scientists speak of such a representation that is unique up to the similarities as a *ratio scale* representation.

In the non-associative case, a modification of Hölder's proof (Cohen & Narens, 1979; Luce et al., in press; Narens & Luce, 1976) is used to show

(i) the existence of a (non-additive) numerical representation, such that

(ii) the automorphisms form a subgroup of the similarity group, and so the scale type is (0,1) or (1,1) .

A series of results (Krantz et al.,1971;Luce et al., in press; Luce 1978; Luce & Narens, 1985; Narens, 1976; Narens & Luce, 1976) show how such operations enter into the structure of physical dimensions. Consider :

Conjoint Structure : $C = \langle A \times P, \succsim \rangle$, where for each $a,b \in A$ and $p,q \in P$, the following three conditions are satisfied :

1. **Weak Ordering** : $\succsim$ is a non-trivial weak ordering.

2. **Independence** (or monotonocity) :

   (i) $(a,p) \succsim (b,p)$ iff $(a,q) \succsim (b,q)$,

   (ii) $(a,p) \succsim (a,q)$ iff $(b,p) \succsim (b,p)$.

Observe that independence permits one to define induced weak orders on A and P , which are denoted $\succsim_A$ and $\succsim_p$, respectively .

3. $\succsim_A$ and $\succsim_p$ are total orders.

Suppose further :

4. O_A is a positive operation on A and that $a = \langle A, \succsim_A, O_A \rangle$ has a ratio scale representation, φ_A , onto the positive reals;

5. a *distributes* in $\mathcal{C} = \langle A \times P, \succsim \rangle$ in the sense that for all $a,b,c,d \in A$ and $p,q \in P$,

$(a,p) \sim (c,q)$ and $(b,p) \sim (d,q)$ imply $(a O_A b, p) \sim (c O_A d, q)$.

Then there exists a function ψ on P such that $\varphi_A \psi$ represents $\mathcal{C}$. If there is also an operation O_P on P with a ratio scale representation then for some real ρ , $\varphi_A \psi_P^{\rho}$ represents $\mathcal{C}$.

This is the typical product of powers exhibited by physical units. It is potentially important to other sciences to know the degree to which the conditions leading to the product of powers representation can be generalized. So we turn to that.

QUESTION : HOW DOES ALL OF THIS GENERALIZE WHEN NO OPERATION IS DEFINED ?

In particular :

1. Is there some natural concept of Archimedeanness in the absence of an operation ?

2. When does an ordered relational structure have a real, ratio scale representation ?

3. How can the idea of an operation distributing in a conjoint structure be generalized so one continues to arrive at the product of powers representation of physical dimensions ?

These questions are of special interest in the social sciences since they place mathematical limits on the possibility of adding new dimensions to those currently known.

ANSWERS IN THE HOMOGENEOUS CASE

Answers are not known in general, but assuming homogeneity the situation is clear and moderately simple.

Real Unit Structure : $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$, where R is a subset of Re^+ and there is some subset T of Re^+ such that

1. T is a group under multiplication,
2. T maps R into R , i.e., for each $R \in R$ and $t \in T$, then $tr \in R$,
3. T restricted to R is the set of translation of $\mathcal{R}$.

Dedekind Complete : Every bounded subset of elements has a least upper bound.

Order Dense : $\langle A, \succsim \rangle$ is such that if $a, b \in A$ and $a \succ b$, then there exists $c \in A$ such that $a \succ c \succ b$.

THEOREM (Luce, 1986) Suppose $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$ is a real unit structure with T its group of translations.

(i) Then $\mathcal{R}$ can be densely imbedded in Dedekind complete unit structure, $\mathcal{R}^*$.

(ii) If R is order dense in Re^+, then each automorphism of $\mathcal{R}$ extends to an autoporphism of $\mathcal{R}^*$.

(iii) If T is homogeneous and R is order dense in Re^+, then $\mathcal{R}^*$ of Part (i) is on Re^+ and T^* is homogeneous.

THEOREM (Luce, 1986) Suppose $\mathcal{A} = \langle A, \succsim, S_j \rangle_{j \in J}$ is a relational structure, $\mathcal{T}$ is its set of translations, and $\succsim'$ is the asymptotic ordering of the automorphisms. Then the following are equivalent :

(i) $\mathcal{A}$ is isomorphic to a real unit structure with a homogeneous group of translations.

(ii) $\langle \mathcal{T}, \succsim', \cdot \rangle$ is a homogeneous, Archimedean ordered group.

COROLLARY. If, in addition, $\langle A, \succsim \rangle$ is order dense, then the automorphism group of the real unit structure is a subgroup of the power group (see Table 1) restricted to its domain.

This theorem answers question 2 - when does a relational structure have a ratio scale representation ? - and suggests that condition (ii) is an answer to question 1 - is there a natural concept of Archimedeanness in the absence of an operator ? The latter point is discussed more fully in Luce and Narens(in press). Moreover, it provides a routine way to proceed : given an axiom system, investigate whether the translations are :

(i) homogeneous,

(ii) closed under function composition, and so a group,

(iii) Archimedean ordered.

The following is a sufficient condition for translations that form a group to be Archimedean ordered.

THEOREM (Luce, 1987) Suppose $\mathcal{A}$ is a relational structure that is Dedekind complete and order dense. If its set $\mathcal{T}$ of translations is 1-point unique (equivalently, a group), then $\mathcal{T}$ is Archimedean.

This means that for order dense, Dedekind complete cases the issue is to show that the translations form a group. That is the difficult part in proving the following sufficient condition :

THEOREM (Alper, 1985, 1987, Narens, 1981 a,b). Suppose that $\mathcal{R} = \langle Re^{+}, \geq, R_j \rangle_{j \in J}$ is a numerical relational structure that is homogeneous and finitely unique. Then the following are true.

(i) $\mathcal{R}$ is of scale type (1,1), (1,2), or (2,2).

(ii) $\mathcal{R}$ is of scale type (1,1) iff $\mathcal{R}$ is isomorphic to a real structure whose automorphisms are the similarity group.

(iii) $\mathcal{R}$ is of scale type (2,2) iff $\mathcal{R}$ is isomorphic to a real structure whose automorphisms are the power group (see Table 1).

(iv) $\mathcal{R}$ is of scale type (1,2) iff $\mathcal{R}$ is isomorphic to a real structure whose automorphisms are a proper subgroup of the power group that properly includes the similarity group.

One should not assume this result generalizes. In unpublished work, Cameron (1987) has shown that structures on the rationals of scale type (M,N) can be found for any integers M and N for which $1 \leq M < N < \infty$. To my knowledge, the case M = N is not understood.

THEOREM (Luce & Narens, 1985). If $\mathcal{A}$ is a homogeneous concatenation structure, then :

(i) Either $\mathcal{A}$ is weakly positive $[(\forall a) a \mathrm{O} a \succ a]$ or idempotent $[(\forall) a \mathrm{O} a \sim a]$ or weakly negative $[(\forall) a \mathrm{O} a \prec a]$

(ii) If $\mathcal{A}$ is also finitely unique, then it is of scale type (1,1), (1,2) or (2,2).

THEOREM (Cohen & Narens, 1979; Luce & Narens, 1985) $\mathcal{A} = \langle Re, \geq 0, \rangle$ is a homogeneous and finitely unique concatenation structure iff $\mathcal{A}$ is isomorphic to a real unit structure $\mathcal{R} = \langle Re^{+}, \geq, \circledast \rangle$, in which case $\circledast$ can be characterized as follows : there is a function $f : Re^{+} \rightarrow$ (onto) Re^{+} such that :

(i) f is strictly increasing,

(ii) f/ι is strictly decreasing, where ι is the identity,

(iii) for all $x,y \in Re^{+}$,

$$x \circledast y = y f(x/y) .$$

COROLLARY : Consider

(1) $$f(x^{\rho}) = f(x)^{\rho}\ , \quad x \in Re^{+}\ .$$

Then $\mathcal{R}$ is of scale type :

(1,1) iff Eq.(1) is satisfied only for $\rho = 1$.

(1,2) iff Eq. (1) is satisfied for $\rho = k^{n}$, where $k > 0$ is fixed and n is any integer.

(2,2) iff Eq. (1) holds for all $\rho > 0$.

Applications of homogeneity ideas, using both concatenation and conjoint structures, are given by Falmagne (1985) to psychophysical problems and by Luce and Narens (1985) to the study of preferences among uncertain alternatives.

DISTRIBUTION IN A CONJOINT STRUCTURE

Suppose $\mathcal{C} = \langle A \times P, \succsim\rangle$ is a conjoint structure (defined above) :

Similar n-Tuples : The n-tuples $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ from A are such that for some $p,q \in P$ and for each $i = 1,\ldots,n$,

$$(a_i,p) \sim (b_i,q)\ .$$

Distributive Relation in $\mathcal{C}$: S is a relation of order n on A such that if $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ are similar and one is in S , then so is the other.

Note : the induced order $\succsim_A$ is always distributive in $\mathcal{C}$.

Distributive Structure in $\mathcal{C}$: Any relational structure on a component of $\mathcal{C}$ each of whose relations distributes in $\mathcal{C}$.

The following is the culmination of a series of increasinly more general results by Narens and Luce.

THEOREM (Luce, 1987). Suppose $\mathcal{A} = \langle A, \succsim, S_j\rangle_{j\in J}$ is an ordered relational structure with translations $\mathcal{T}$ and asumptotic ordering $\succsim'$. Then the following are equivalent :

(i) $\langle \mathcal{T}, \succsim', \rangle$ is a homogeneous, Archimedean ordered group.

(ii) $\mathcal{T}$ is 1-point unique and there exists an Archimedean, solvable conjoint structure $\mathcal{C}$ with a relational structure $\mathcal{A}'$ on the first component such that $\mathcal{A}'$ is isomorphic to $\mathcal{A}$ and $\mathcal{A}'$ distributes in $\mathcal{C}$.

COROLLARY. The conjoint structure of Part (ii) satisfies the *Thomsen condition* : for $a,b,c \in A$ and $p,q,u \in P$,

$$(a,u) \sim (c,q) \quad \text{and} \quad (c,p) \sim (b,u) \quad \text{imply} \quad (a,p) \sim (b,q) .$$

Solvability : Given any three of $a,b \in A$, $p,q \in P$, the fourth exists such that $(a,p) \sim (b,q)$.

In solvable conjoint structures with an operation, the general concept of distribution implies the one introduced earlier.

Archimedean : If $\{a_i\}$ and $\{a_{i+1}\}$ are similar and bounded, then the sequence is finite. And a similar condition on the P-component.

THEOREM (Luce, 1987). Suppose $\mathcal{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure that is solvable and Archimedean. Suppose, further, that $\mathcal{A} = \langle A, \succsim_A, S_j \rangle_{j \in J}$ is a relational structure whose translations form an Archimedean ordered group.

(i) If $\mathcal{A}$ distributes in $\mathcal{C}$, then $\mathcal{A}$ is 1-point homogeneous and $\mathcal{C}$ satisfies the Thomsen condition.

(ii) If, in addition, $\mathcal{A}$ is Dedekind complete and order dense, then under some mapping φ_A from A onto Re^+ $\mathcal{A}$ has a homogeneous real unit representation and there exists a mapping ψ from P into Re^+ such that $\varphi_A \psi$ is a representation of $\mathcal{C}$.

(iii) If, further, there is a Dedekind complete relational structure on P that, under some φ_P from P onto Re^+ , has a homogeneous unit representation, then for some real constant ρ , $\varphi_A \varphi_P{}^{\rho}$ is a representation of $\mathcal{C}$.

The latter result is highly relevant to issues in the foundations of dimensional analysis; see Luce (1978). It says :

> Real unit structures that distribute in a conjoint structure are such that products of powers of their representations form a representation of the conjoint structure.

This is the typical pattern of physical dimensions, e.g., $E = (1/2)\,mv^2$ and is reflected in the pattern of physical units.

This result establishes that such a dimensional triple is not, as was for some time believed, restricted to extensive structures; it can be true for any real unit structure. The net effect is to increase considerable the opportunities for augmenting the system of physical units.

For a somewhat different, but related, approach to this problem of products of powers, see Falmagne and Narens (1983).

CONCLUSION

We now understand a good deal about homogeneous measurement structures; but we still do not have an equally satisfactory understanding of non-homogeneous ones. Some of these are important. In particular, we do not understand the interlock of conjoint structures and bounded structures on one of the factors. One example is relativistic velocity, which superficially looks like any other physical dimension except for being bounded; however, unlike the unbounded physical attributes, velocity does not distribute in the distance = velocity × time conjoint structure. Another example is probability, which has never been incorporated into the system of physical units although it has been, to an extent, into economic measures (e.g., subjective expected utility). Presumably, sensory variables, like loudness and brightness, are bounded. These structures, unlike many other non-homogeneous one, are richly endowed with automorphisms or partial automorphisms, and seem in many ways comparable to the homogeneous ones. So results about them may be anticipated.

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TABLE 1*

Stevens' Classification of Scale Types (Augmented)

Numerical Domain		Parameters
Re	Re^+	
Ordinal (homeomorphism group) $x \to f(x)$		countable
$f : Re \to (onto)Re$	$f : Re^+ \to (onto)Re^+$	
f strictly increasing		
Interval (positive affine group) $x \to rx+s$, $r > 0$	**Log-interval** (power group) $x \to tx^r$, $r > 0$, $t > 0$	2
translation if $r = i$		
dilation if $r \neq i$ or if the identity		
Difference (translation group) $x \to x+s$	**Ratio** (similarity group) $x \to tx$, $t > 0$	1
Ratio (similarity group) $x \to rx$, $r > 0$		1
Absolute (identity group) $x \to x$		0

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* This is Table 20.1 of Luce, Krantz, Suppes and Tversky, in press.