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Measurement Structures with Archimedean Ordered Translation Groups

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Abstract. The paper focuses on three problems of generalizing properties of concatenation structures (ordered structures with a monotonic operation) to ordered structures lacking any operation. (1) What is the natural generalization of the idea of Archimedeaness, of commensurability between large and small? (2) What is the natural generalization of the concept of a unit concatenation structure in which the translations (automorphisms with no fixed point) can be represented by multiplication by a constant? (3) What is the natural generalization of a ratio scale concatenation structure being distributive in a conjoint one, which has been shown to force a multiplicative representation of the latter and the product-of-powers representation of units found in physics? It is established (Theorems 5.1 and 5.2) that for homogeneous structures, the latter two questions are equivalent to it having the property that the set of all translations forms a homogeneous Archimedean ordered group. A sufficient condition for Archimedeaness of the translations is that they form a group, which is equivalent to their being 1-point unique, and the structure be Dedekind complete and order dense (Theorems 2.1 and 2.2). It is suggested that Archimedean order of the translations is, indeed, also the answer to the first question. As a lead into that conclusion, a number of results are reported in Section 3 on Archimedeaness in concatenation structures, including for positive structures sufficient conditions for several different notions of Archimedeaness to be equivalent. The results about idempotent structures are fragmentary.

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1. Introduction

This paper explores three broad questions about general weakly ordered relational structures:

(1) Is there some natural concept of Archimedeaness for such structures?
(2) When is such a structure isomorphic to a substructure of the positive real numbers that exhibits ratio scale ($x \to \rho x, \rho > 0$) uniqueness?
(3) Suppose a conjoint structure (i.e., a monotonically ordered Cartesian product) has a relational structure on the first component, ordered by the naturally induced order of the conjoint structure. When does this complex have a numerical representation analogous to the classical structures of

physics, namely, a ratio scale isomorphism of the relational structure and a mapping of the second component that multiplicatively represent the conjoint structure?

I do not know much about these questions in a completely general context, but if we restrict our attention to relational structures with a high degree of symmetry – technically, homogeneous ones – then answers will be given. And in fact, they are all very closely related.

The remainder of this introduction discusses the background of each question. If that is already familiar, the reader should skip to Section 2.

1.1. QUESTION OF GENERALIZING ARCHIMEDEANESS

Recent research in the general area of abstract measurement theory has led to a vastly greater understanding of ordered algebraic structures of two types [for a nontechnical survey, see Narens and Luce (1986), and for a highly technical one, see Narens (1985)]. The one is general concatenation structures that consist of a set X of objects, a weak order $\succsim$ on X, and a partial, locally-defined, binary operation O on X, such that the order and the operation are interconnected by the property of monotonicity. These structures generalize the additive and averaging ones of classical physics. The other is the general class of binary conjoint structures $\langle X \times P, \succsim \rangle$, where X and P are sets and $\sim$ is a weak ordering of $X \times P$ that satisfies the property of independence (i.e., monotonicity in each factor) over the product structure. In each axiomatization, at least one concept of Archimedeaness is invoked in an essential way to establish numerical representations (Cohen and Narens, 1979; Luce, 1986; Luce and Cohen, 1983; Luce and Narens, 1985; Narens and Luce, 1976).

This raises the general question: What might one mean by Archimedeaness in a general, (weakly) *ordered, relational structure*? Such a structure is of the form

$$\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J},$$

where $\succsim$ is a weak order on X, J is either a finite or countable sequence of successive integers, and for each $j \in J$, S_j is a relation of finite order $n(j)$ on X.

This issue of Archimedeaness is taken up at length in Luce and Narens (1987) for structures on a continuum; it draws heavily on the present results.

1.2. QUESTION OF GENERALIZED UNIT STRUCTURES

Cohen and Narens (1979) showed that a subclass of concatenation structures called homogeneous PCSs have a particularly simple representation into the positive real numbers that they called a *real unit structure*. By a PCS is meant a structure fulfilling all of the axioms of extensive measurement save associativity.

By *homogeneity* is meant the property that for each $x, y \in X$, there is an automorphism α of the structure that maps x into y, i.e., $\alpha(x) = y$. The real unit representation is of the form: there is function f that is strictly increasing, $f(r)/r$ is strictly decreasing, and the numerical operation $\otimes$ is of the form

$$r \otimes s = sf(r/s).$$

One nice property of real unit structures, which generalizes from extensive measurement, is that its automorphisms are very simple, namely, multiplication by any positive constant. This is called a *ratio scale* representation in the measurement literature.

A closely related result of Alper (1984, 1985, 1987) that will be described more fully later, begins with a fairly general class of relational structures on the real numbers and provides plausible general conditions on the automorphism group that result in its having a very simple structure, namely, conjugate to a subgroup of the affine transformations ($r \to \rho r + \sigma$, $\rho > 0$) that includes all of the translations ($r \to x + \sigma$, σ any real number). The latter property will prove important here.

So the second general question is: For a general ordered relational structure, under what conditions is there a representation that is analogous to the real unit structures for the case of PCSs?

1.3. QUESTION OF GENERALIZED DISTRIBUTION

The third question centers on the observation, which began with Narens (1976) and has been proved under increasingly weakened conditions in Narens and Luce (1976), Narens (1981a), and Luce and Narens (1985), that when a conjoint structure has a concatenation structure defined on one of its components and these two structures are interlocked in a certain way, then they behave just like interlocked physical dimensions, with the unit of the conjoint structure expressed as the products of powers of the units of the component measures. In particular, suppose a conjoint structure $\langle X \times P, \succsim \rangle$ has a closed operation O on X meeting two conditions. First, if $\succsim_X$ denotes the weak ordering induced by $\succsim$ on X, then $\mathscr{X} = \langle X, \succsim_X, O \rangle$ is a concatenation structure of some type – in the sequence of increasingly general results, $\mathscr{X}$ was first assumed to be extensive, next a PCS, and ultimately any concatenation structure with a ratio scale representation. Second, O is assumed to distribute in the conjoint structure in the following sense:

$$\text{if } (x, p) \sim (x', q) \quad \text{and} \quad (y, p) \sim (y', q), \quad \text{then } (xOy, p) \sim (x'Oy', q).$$

The upshot of these theorems is that the conjoint structure is forced to satisfy the Thomsen condition and so has a multiplicative representation. Indeed, if a distributive operation also exists on P and has a ratio scale representation, then the conjoint structure can be represented as a product of powers of the two ratio-scale representations of the operations.

These results are of interest mainly because this representation of conjoint-concatenation triples underlies the structure of physical quantities and the familiar product of powers of units (Krantz, *et al.*, 1971, Chapter 10; Luce, 1978). Thus, these theorems tell us about the kinds of generalizations of extensive structures that could conceivably be incorporated into the traditional system of units. This could be of importance to the biological, behavioral, and social sciences where it is far from clear that concatenation measurement structures with additive representations will be found.

So the third general question is: Is there a natural way to generalize the concept of distribution so that it applies not just to operations, but to general ordered relational structures and, at the same time, continues to lead to the same type of representation of the conjoint structure as products of powers of the representations of the structures on the components?

This paper establishes (Theorems 5.1 and 5.2) that for homogeneous structures there is a plausible answer to each question, and it is the same answer.

As a lead into that, I first take up two related matters. The first is to show that under weak restrictions, the important class of Dedekind complete structures have Archimedean ordered groups of translations (automorphisms with no fixed point). And the second is to explore within the context of concatenation structures, both positive and idempotent, the several concepts of Archimedeaness that can be found in the literature.

2. Archimedean Ordered Translation Groups in Dedekind Complete Structures

A clue to the key to the three problems posed above is found in two theorems in the literature. The first, Theorem 2.4 of Cohen and Narens (1979), established that the automorphism group of any PCS is an Archimedean ordered group under function composition. The ordering derives from that on the structure in the following way. They show that if α and β are any two automorphisms, then for all $x, y \in X$, $\alpha(x) \succ \beta(x)$ iff $\alpha(y) \succ \beta(y)$ and $\alpha(x) \sim \beta(x)$ iff $\alpha(y) \sim \beta(y)$. In this case, we say α and β *do not cross*, and it is natural to define $\succsim'$ on the automorphism group, $\mathscr{A}$, by: for $\alpha, \beta \in \mathscr{A}$,

$$\alpha \succsim' \beta \text{ iff for some } x, \text{ and hence all } x, \alpha(x) \succsim \beta(x).$$

Note further, these automorphisms have the property that none, save the identity, has a fixed point. Such automorphisms are called *translations*. It is also convenient to call the identity a (degenerate) translation as well. The set of all translations is denoted $\mathscr{T}$.

The second result pertains to general relational structures defined on an interval of real numbers (or, equivalently, Dedekind complete, order dense structures). Consider those meeting the following two conditions: homogeneity, which was defined earlier, and for N an integer, *N-point uniqueness* which is defined to mean that if any two automorphism that agree at N nonequivalent points, then they are necessarily identical. For the set of translations, it is not difficult to show that $N = 1$ iff they form a group (Luce, 1986, Theorem 2.1). Alper (1984, 1985, 1987) has shown that any such real structure meeting these two conditions is isomorphic to one whose automorphism group is a subgroup of the affine group (i.e., a subgroup of the mappings of the form $r \to \rho r + \sigma$, where $\rho > 0$) that includes all of the numerical translations (all mappings of the form $r \to r + \sigma$).

A central aspect of the result is the fact that under these assumptions the set of translations is, as in the case of the PCS, an Archimedean ordered group (see Luce, 1986). The ordering of the entire automorphism group is, however, somewhat more subtle than in the PCS case. Narens (1981b) introduced the idea of an asymptotic ordering in a unique structure, namely:

$$\alpha \succsim' \beta \text{ iff there is some } y \in X \text{ such that for all } x \in X \text{ with } x \succsim y, \text{ then } \alpha(x) \succsim \beta(x).$$

Actually, once one begins to focus on the Archimedeaness of the translations, it becomes clear that homogeneity is playing a somewhat indirect role, as shown in the first pair of theorems. Proofs are in Section 6.

THEOREM 2.1. *Suppose $\mathscr{X}$ is an ordered relational structure that is Dedekind complete and whose set $\mathscr{T}$ of translations is 1-point unique (equivalently, a group). If the elements of $\mathscr{T}$ do not cross, then $\mathscr{T}$ is Archimedean..*

A natural question, then, is what properties insure that the elements of $\mathscr{T}$ do not cross. The next result states two sufficient conditions, the first of which concerns $\mathscr{T}$ itself and the second concerns the structure.

THEOREM 2.2. *Suppose $\mathscr{X}$ is an ordered relational structure with the set of translations $\mathscr{T}$. If either*

(i) *$\mathscr{T}$ is a homogeneous and commutative group,* *or*
(ii) *$\mathscr{X}$ is order dense and Dedekind complete,*

then the elements of $\mathscr{T}$ do not cross.

These observations make one wonder exactly what significance there is to the translations being an Archimedean ordered group. To get at that, we first explore Archimedeaness more carefully within the context of concatenation structures, where we know how to define the concept directly in several plausible, but different ways. Basically, we work out conditions under which the several concepts of Archimedeaness agree.

3. Archimedeaness in Concatenation Structures

As Luce and Narens (1985) described in some detail, there are two major, very different classes of homogeneous concatenation structures: positive and idempotent. (There are also weakly positive structures in which $xOx \succ x$ holds, but positively fails.) The former is typified by extensive structures and their nonassociative generalization, PCSs, and the latter is illustrated by models of averaging. In the former, Archimedeaness is usually formulated in terms of bounded standard sequences being finite, whereas in the latter, there being no nontrivial standard sequences, the concept has to be captured in some other way, such as bounded difference sequences being finite. (For the formal definitions, see Definition 3.1 below.) This notion, however, applies to positive structures equally well, and so we now have two distinct notions of Archimedeaness in positive structure. Worse still, there are two other concepts which I shall refer to as Archimedeaness in generalized standard sequences and in standard differences. All four of these concepts are defined for the positive case, and so there is an issue of understanding how they relate. Ideally, within contexts of interest, they should be equivalent. The purpose of this section is to isolate those contexts. For the idempotent case, there is a second concept of Archimedeaness involving generalized standard sequences, but I do not know how it relates to the notion defined in terms of difference sequences.

3.1. GENERAL DEFINITIONS

We begin with a formal definition of the several concepts of sequences and Archimedeaness.

DEFINITION 3.1. Suppose that $\mathscr{X} = \langle X, \succsim, O \rangle$ is a totally ordered concatenation structure with O a closed operation. For $x, y \in X$ and n a positive integer, define the sequence $\theta(x, y, n)$ inductively as:

$$\theta(x, y, n) = \begin{cases} x, & \text{if } n = 1, \\ \theta(x, y, n-1)Oy, & \text{if } n > 1. \end{cases}$$

Let $\{x_n\}$ be a sequence of elements in X.

1. $\{x_n\}$ is said to be a *generalized standard sequence* iff for some $x, y \in X$, with $y \neq x$, and all positive integers n for which x_n is defined, $x_n = \theta(y, x, n)$. The case where $x = y$ is referred to simply as a *standard sequence* (*based on* y).
2. $\{x_n\}$ is said to be a *difference sequence* (*based on* p, q) iff $p, q \in X$, $p \neq q$, and for all $n, n+1$ for which x_n and x_{n-1} are defined, $x_n Op = x_{n-1} \mathrm{O} q$.
3. (a) If $\mathscr{X}$ is positive, $\mathscr{X}$ is said to be *Archimedean in (generalized) standard sequences* iff each bounded (generalized) standard sequence $\{x_n\}$ is finite.

(b) If $\mathscr{X}$ is positive, $\mathscr{X}$ is said to be *Archimedean in standard differences* iff for every $x, y, p, q \in X$ with $x \succ y$, there is some integer n such that $x_n O p \succ y_n O q$, where x_n and y_n are the nth terms of the standard sequences generated by x and y, respectively.

(c) If $\mathscr{X}$ is idempotent, $\mathscr{X}$ is said to be *Archimedean in generalized standard sequences* iff for each $x, y, z \in X$ with $x \prec y \prec z$, there exists an integer n such that $\theta(x, z, n) \succ y$.

4. $\mathscr{X}$ is said to be *Archimedean in difference sequences* iff each bounded difference sequence is finite.

The concept of a standard sequence and of Archimedeaness in terms of it is classical, dating back at least to Hölder (1901). For such sequences, the term x_n is often denoted nx, but I will not do so here. The notion of a generalized standard sequence seems to have been first introduced by Fuchs (1963, p. 182) and was studied by Luce (1986). The concept of Archimedeaness in generalized standard sequences for idempotent structures is due to Fuchs. The concept of Archimedeaness in difference sequences derives naturally from that used in conjoint measurement (Luce and Tukey, 1964), and that of Archimedeaness in standard differences (although not called such) was introduced by Roberts and Luce (1968) in order to state necessary and sufficient conditions for the numerical representation of a closed extensive structure. Observe that those concepts formulated in terms of standard sequences make sense only for positive structures. The difference sequence idea applies to all concatenation structures. And, of course, none of these concepts applies beyond concatenation structures.

In understanding their connections, three distinct ideas of solvability, as well as another concept, play an important role. The first two of these have been in a literature for some time, and the last two seem to be new.

DEFINITION 3.2. Suppose that $\mathscr{X} = \langle X, \succsim, O \rangle$ is a totally ordered concatenation structure and O is a closed operation.

1. $\mathscr{X}$ is *right (left) solvable* iff for each $x, y \in X$, with the restriction $x \prec y$ in the positive case, there exist z $(z') \in X$ such that $xOz = y$ $(z'Ox = y)$. These are referred to, respectively, as *right* and *left solutions*. When both exist, $\mathscr{X}$ is said to be *solvable*.
2. In the positive case, $\mathscr{X}$ is *restrictedly right solvable* iff for each $x, y \in X$ with $x \prec y$ there exist $z \in X$ such that $xOz \precsim y$. *Restrictedly left solvable* is defined similarly.
3. In the positive case, $\mathscr{X}$ is *uniformly, restrictedly right solvable* iff for each $x, y, r, s \in X$, $x \prec y$, $r \prec s$, there exists $z = z(x, y, r, s) \in X$ such that for all $u \in X$ for which $r \precsim u \precsim s$, $(uOx)Oz \precsim uOy$. *Uniformly, restrictedly left solvable* is defined similarly.

4. In the idempotent case, $\mathscr{X}$ is said to satisfy *left intern density* iff for each $x, y \in X$ with $x \prec y$, there exists $z \in X$ such that $z \prec x$ and $zOy \succ x$.

3.2. POSITIVE, ASSOCIATIVE CASE

We first examine the simplest, classical case, namely, when the operation O is positive and associative. Here the three concepts of standard, generalized standard, and difference sequences are very closely related, all four notions of Archimedeaness are defined and, in the restrictedly solvable case, are equivalent, and any restrictedly solvable structure is uniformly so. I state the relations precisely.

THEOREM 3.1. *Suppose $\mathscr{X} = \langle X, \succsim, O\rangle$ is a totally ordered concatenation structure for which O is closed, positive, and associative.*

(i) *Any generalized standard sequence is the concatenation of a fixed element with a standard sequence, and so Archimedeaness in standard sequences is equivalent to Archimedeaness in generalized standard sequences.*

(ii) (a) *Any standard sequence is a difference sequence, and so Archimedeaness in difference sequences implies Archimedeaness in standard sequences.*
(b) *If $\mathscr{X}$ is restrictedly left solvable, then Archimedeaness in standard sequences implies Archimedeaness in difference sequences.*

(iii) (a) *Archimedeaness in standard differences implies Archimedeaness in standard sequences.*
(b) *If O is restrictedly right solvable, then Archimedeaness in standard sequences implies Archimedeaness in standard differences.*

(iv) *If $\mathscr{X}$ is restrictedly right (left) solvable, then it is uniformly, restrictedly right (left) solvable.*

COROLLARY. *If O is closed, positive, associative, and restrictedly right and left solvable, then the four concepts of Archimedeaness are equivalent.*

3.3. ARCHIMEDEANESS IN POSITIVE CONCATENATION STRUCTURES

As the concept of uniform, restricted solvability will be seen (Theorem 3.3) to be important in relating the first two kinds of Archimedeaness in more general contexts, it is essential that it be satisfied in the kinds of numerical representing structures that we are sure are important. For positive concatenation structures, this is almost certainly the (homogeneous) unit structures defined by Cohen and Narens (1979). See Luce and Narens (1985) for a slightly improved definition and for the notation used here.

In the following result about unit structures we use two limits, which exist by the monotonicity properties of f:

$$c = \lim_{x \to 0} f(x) \quad \text{and} \quad k = \lim_{x \to \infty} f(x)/x.$$

THEOREM 3.2. *Suppose $\mathscr{R}$ is a (homogeneous) real unit concatenation structure.*

(i) *If $\mathscr{R}$ is idempotent, then $\mathscr{R}$ is right (left) solvable iff $k = 0$ ($c = 0$).*
(ii) *If $\mathscr{R}$ is positive, then $\mathscr{R}$ is right (left) solvable iff $k = 1$ ($c = 1$).*
(iii) *If $\mathscr{R}$ is positive and restrictedly right (left) solvable, then $\mathscr{R}$ is right (left) solvable and is uniformly restrictedly right (left) solvable.*

For the somewhat weaker assumption of Dedekind completeness, Luce and Narens (1985, Theorem 2.1) have shown, first, that positivity and left solvability imply the structure is Archimedean in standard sequences and, second, that left solvability alone implies it is Archimedean in difference sequences.

Under still weaker assumptions, we are able to show how these two Archimedean concepts relate. In particular, the next result establishes that uniform, restricted right solvability is adequate to show that the standard sequence concept of Archimedeaness implies the difference one, whereas solvability is needed for the other direction.

In the next result we need the property of 'the n-copy operator preserving the operation', which simply means that if $\{x_n\}$, $\{y_n\}$, and $\{(xOy)_n\}$ are standard sequences based on x, y, and xOy respectively, then for each n, $(xOy)_n = x_n O y_n$. This property obtains in unit structures.

THEOREM 3.3. *Suppose $\mathscr{X}$ is a totally ordered concatenation structure with an operation that is closed and positive.*

(i) *Archimedeaness in standard sequences is equivalent to Archimedeaness in generalized standard sequences.*
(ii) (a) *If $\mathscr{X}$ is solvable, then Archimedeaness in difference sequences implies Archimedeaness in standard sequences.*
 (b) *If $\mathscr{X}$ is uniformly restrictedly right solvable, then Archimedeaness in standard sequences implies Archimedeaness in difference sequences.*
(iii) (a) *If $\mathscr{X}$ is restrictedly right solvable and n-copy operators preserve the operation, then Archimedeaness in standard sequences implies Archimedeaness in standard differences.*
 (b) *If $\mathscr{X}$ is uniformly right solvable, order dense, and n-copy operators preserve the operation, then Archimedeaness in standard differences implies Archimedeaness in standard sequences.*

COROLLARY. *If $\mathscr{X}$ is a unit structure in the sense of Cohen and Narens* (1979) *that is positive, then all four Archimedean conditions obtain.*

3.4. IDEMPOTENT STRUCTURES

For the idempotent case, two distinct concepts of Archimedeaness were stated (Definition 3.1): in difference sequences and in generalized standard sequences. I do not know of any relation between them and am, indeed, suspicious that they are quite different. The reason is that a difference sequence is a very global concept, whereas a generalized standard sequence in the idempotent case concerns behavior only in a bounded interval. The following gives some insight into the more local concept.

THEOREM 3.4. *Suppose $\mathscr{X}$ is a totally ordered, idempotent concatenation structure.*

(i) *If $\mathscr{X}$ is Dedekind complete and satisfies left intern density, then $\mathscr{X}$ has no minimal element and is Archimedean in generalized standard sequences.*
(ii) *If $\mathscr{X}$ has no minimal element and is Archimedean in generalized standard sequences, then left intern density holds.*
(iii) *If $\mathscr{X}$ is a real unit structure on* Re^+, *then left intern density holds, and so it is Archimedean in generalized standard sequences.*

3.5. ARCHIMEDEANESS IN STRUCTURES WITH ARCHIMEDEAN ORDERED TRANSLATIONS

The final result of the section shows that homogeneity and Archimedean order of the translation group are, indeed, sufficient to imply the first two forms of Archimedeaness and with restricted right solvability also the third.

THEOREM 3.5. *Suppose $\mathscr{X}$ is a totally ordered concatenation structure whose translations form an ordered group (under function composition and the asymptotic order) that is Archimedean ordered and is homogeneous. Then,*

(i) *$\mathscr{X}$ is Archimedean in difference sequences;*
(ii) *if $\mathscr{X}$ is positive, then $\mathscr{X}$ is Archimedean in standard sequences;*
(iii) *if $\mathscr{X}$ is positive and restrictedly right solvable, then $\mathscr{X}$ is Archimedean in standard differences;*
(iv) *if $\mathscr{X}$ is idempotent and Dedekind complete, then $\mathscr{X}$ is Archimedean in generalized standard sequences.*

COROLLARY. *If $\mathscr{X}$ is an idempotent unit structure, then it is Archimedean in generalized standard sequences and in difference sequences.*

The proof, given in Section 6.8, is postponed until after the proof of a later theorem, which it uses.

It would be desirable to prove some version of a converse to Theorem 3.5. The strongest form would be to show that any concatenation structure that is Archimedean in some sense has a translation group that is Archimedean ordered. This is true for PCSs (Theorem 2.4 of Cohen and Narens, 1979). In the case of idempotent structures, less is known (but see Theorem 5.1 of Luce and Narens, 1985). Since idempotent structures are, under weak conditions, intern, they are automatically order dense. If, in addition, such a structure is solvable and Dedekind complete, then it is isomorphic to a structure on the entire real numbers. So by Alper's (1986) theorem, if it is homogeneous and N-point unique for some finite N, then the translations form an Archimedean ordered group. The issue is what happens when we drop Dedekind completeness and/or homogeneity. We do know (Theorem 5.2 of Luce and Narens, 1985) that the translations of solvable, idempotent, concatenation structures induce total concatenation structures that are two PCSs, one above and the other below an identity, and that the translations of the idempotent structure appear as isomorphisms between induced total concatenation structures. But I have been unable to see how to bring that fact and the Archimedeaness of the structure to bear on the translation group.

4. Generalized Definitions of Unit Structure and Distribution

We turn now to the question of generalizing the concept of Archimedeaness beyond structures for which an operation is given or one is readily defined, as in the case of difference or conjoint structures. For general relational structures one does not know how to define Archimedeaness in structural terms, and yet we know that there must be some concept of commensurability if the structure is to have a numerical representation. Perhaps, the Archimedeaness of the translations is a suitable generalization. In any event, it is surely reasonable to ask: what class of structures have Archimedean ordered groups of translations? We do not yet know the answer in general, but we do have a rather neat answer when the translation group is homogeneous. As was indicated earlier, this also answers the second and third questions posed initially.

4.1. REAL UNIT STRUCTURES

Our first task is to generalize the concept of a unit structure, a type of PCS, to a more general context. For a PCS, the gist of the concept is the property from which the form of the operation $\otimes$ is derived, namely, that multiplication by a positive constant is an automorphism – actually, translation – and so for positive r, s, t the following functional equation holds:

$$tr \otimes ts = t(r \otimes s).$$

Basically, our task is to generalize this to structures of a general type with translations that are multiplications by constants.

DEFINITION 4.1. A real relational structure $\mathscr{R} = \langle R, \geqslant, R_j \rangle_{j \in J}$ is said to be a *real unit structure* iff $R \subseteq \mathrm{Re}^+$ and there is some $T \subseteq \mathrm{Re}^+$ such that

1. T is a group under multiplication.
2. T maps R into R, i.e., for each $r \in R$ and $t \in T$, $tr \in R$,
3. T restricted to R is the set of translations of $\mathscr{R}$.

It follows immediately from properties (1) and (2) and the fact $T \subseteq \mathrm{Re}^+$ that the translations of a real unit structure form an Archimedean ordered group. Further, the assumption that each $t \in T$ is an automorphism implies that for each R_j, $j \in J$, $r_i \in R$, $i = 1, \ldots, n(j)$, and $t \in T$, if $(r_1, \ldots, r_{n(j)}) \in R_j$, then $(tr_1, \ldots, tr_{n(j)}) \in R_j$.

This definition, which in a slightly different form was introduced by Luce (1986), generalizes that of Cohen and Narens (1979) for real, homogeneous, PCSs since in their case, $R = T = \mathrm{Re}^+$, $J = \{1\}$, and $R_1 =$ a binary operation. I continue to use the same term, real unit structure, for the generalization, but not restricted to homogeneous relational structures. In those cases, I use the more explicit phrase 'homogeneous, real unit structure'.

An important question about any class of structures is whether it is sufficient to study their Dedekind complete versions. In particular, can a structure be embedded in a Dedekind complete one in such a way that important properties are preserved in the completion. This has been studied for PCSs. The next result describes the situation for general unit structures.

THEOREM 4.1. *Suppose* $\mathscr{R} = \langle R, \geqslant, R_j \rangle_{j \in J}$ *is a real unit structure with* $T \subseteq \mathrm{Re}^+$ *its group of translations.*

(i) Then $\mathscr{R}$ *can be densely imbedded in a Dedekind complete unit structure,* $\mathscr{R}^*$.
(ii) *If* R *is order dense in* Re^+, *then each automorphism of* $\mathscr{R}$ *extends to an automorphism of* $\mathscr{R}^*$.
(iii) *If* T *is homogeneous and* R *is order dense in* Re^+, *then* $\mathscr{R}^*$ *of Part* (i) *is on* Re^+ *and* T^* *is homogeneous.*

I do not know the answers to the following two questions about unit structures. First, is the converse of part (ii) also true, i.e., is the restriction of an automorphism of $\mathscr{R}^*$ an automorphism of $\mathscr{R}$? And second, what can we say about uniqueness? Are unit structures, in general, N-point unique for some finite N? If not, does the assumption of finite uniqueness for $\mathscr{R}$ imply finite uniqueness for $\mathscr{R}^*$?

4.2. DISTRIBUTIVE RELATIONS

The next pair of definitions generalize, to structures far more general than concatenation ones, the idea of the structure distributing in a conjoint structure. Recall that an operation O on X is defined to be distributive in $\mathscr{C} = \langle X \times P, \succsim \rangle$ if whenever $(x, p) \sim (x', q)$ and $(y, p) \sim (y', q)$, then $(z, p) \sim (z', q)$ where $z = xOy$

and $z' = x'Oy'$. An alternative way of saying the same thing, one that generalizes naturally, is this: if the first three equations all hold and if $z = xOy$, then $z' = x'Oy'$. This I generalize as follows:

DEFINITION 4.2. Suppose $\mathscr{C} = \langle X \times P, \succsim \rangle$ is a conjoint structure.

1. The ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, $x_i, y_i \in X, i = 1, \ldots, n$, are said to be *similar* iff there exist $p, q \in P$ such that for each $i = 1, \ldots, n$, $(x_i, p) \sim (y_i, q)$.
2. Suppose S is a relation of order n on X. S is said to *distribute in* $\mathscr{C}$ iff for each x_i, $y_i \in X$, $i = 1, \ldots, n$, if $(x_1, \ldots, x_n) \in S$ and $(y_1, \ldots, y_n)$ is similar to it, then $(y_1, \ldots, y_n) \in S$. A relational structure is said to distribute in $\mathscr{C}$ iff each of its defining relations distributes in $\mathscr{C}$.

5. The Main Results

5.1. EQUIVALENCE OF CONCEPTS

In stating the major result of the paper, recall that the set of translations consists of the identity together with all automorphisms that have no fixed points. For the exact definitions of an Archimedean, solvable conjoint structure, see Definition 4.1 of Luce and Narens (1985).

THEOREM 5.1. *Suppose $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ is a relational structure, $\mathscr{T}$ is its set of translations, and $\succsim'$ is the asymptotic ordering of the automorphisms. Let $*$ denote function composition. Then the following are equivalent:*

(i) *$\mathscr{X}$ is isomorphic to a real unit structure with a homogeneous group of translations.*
(ii) *$\langle \mathscr{T}, \succsim', * \rangle$ is a homogeneous, Archimedean ordered group.*
(iii) *$\mathscr{T}$ is 1-point unique and there exists an Archimedean, solvable conjoint structure $\mathscr{C}$ with a relational structure $\mathscr{X}'$ on the first component such that $\mathscr{X}'$ is isomorphic to $\mathscr{X}$ and $\mathscr{X}'$ distributes in $\mathscr{C}$.*

COROLLARY 1. *The conjoint structure of part* (iii) *satisfies the Thomsen condition.*

COROLLARY 2. *Suppose, in addition, $\mathscr{X}$ is order dense and property* (ii) *holds. Then, the group of automorphisms of $\mathscr{X}$ is conjugate to a subgroup of the positive affine group restricted to X, and so $\mathscr{X}$ is 2-point unique.*

The significance of this result is considerable. It gives us a reasonably clear sense of just how far the dimensional structure of physics can be generalized. A structure with translations that are homogeneous and form an Archimedean ordered group has a representation as a unit structure, and so translations are represented as multiplication by positive numbers, and the structure can be imbedded distributively in a multiplicative conjoint structure. As we show in the next subsection, this is sufficient to lead to a representation of the conjoint structure as products of powers, as in dimensional analysis.

What these two results make clear is that the usual representation used in dimensional analysis, which deeply involves units of measurement that are related among themselves as products of powers, in no way depends upon extensive measurement or even on having an operation in the qualitative structure $\mathscr{X}$. The key to the representation is for the structure lying on one component of an Archimedean conjoint structure to have translations that form a homogeneous, Archimedean ordered group. Thus, if one is confronted with a particular axiom system for a relational structure, the first thing to do is to investigate its translations. If they form a homogeneous, Archimedean ordered group, then one knows from Theorem 5.1 that the structure can be represented numerically as a real unit structure with a homogeneous translation group.

One would like to know what happens when $\mathscr{T}$ is an uncrossed Archimedean ordered group, but not a homogeneous one. The proof of Theorem 5.1 involves mapping the relational structure onto the Archimedean ordered group of translations, which in turn is mapped into the reals using Hölder's representation of an Archimedean ordered group. Since the mapping of the structure into the translations depends upon homogeneity, this strategy cannot be followed in the nonhomogeneous case. Perhaps something is possible when $\mathscr{T}$ is nonhomogeneous provided it is not cyclic (see Levine, 1972), but I do not know of any result.

Finally, one would like to know if homogeneity of a real unit structure is equivalent to homogeneity of its translations.

5.2. REPRESENTATIONS OF DEDEKIND COMPLETE, ORDER DENSE, DISTRIBUTIVE TRIPLES

For purposes of dimensional analysis, it is useful to consider part (iii) of Theorem 5.1 in a more restrictive context, comparable to what is usually assumed in physics.

THEOREM 5.2. *Suppose* $\mathscr{C} = \langle X \times P, \succsim \rangle$ *is a conjoint structure that is solvable and Archimedean. Suppose, further, that* $\mathscr{X} = \langle X, \succsim_X, S_j \rangle_{j \in J}$ *is a relational structure whose translations form an Archimedean ordered group.*

(i) *If* $\mathscr{X}$ *distributes in* $\mathscr{C}$*, then* $\mathscr{X}$ *is* 1*-point homogeneous and* $\mathscr{C}$ *satisfies the Thomsen condition.*

(ii) *If, in addition,* $\mathscr{X}$ *is Dedekind complete and order dense, then under some mapping* φ *from* X *onto* Re^+ $\mathscr{X}$ *has a homogeneous unit representation and there exists a mapping* ψ *from* P *into* Re^+ *such that* $\varphi\psi$ *is a representation of* $\mathscr{C}$.

(iii) *If, further, there is a Dedekind complete relational structure on P that, under some ψ from P onto Re^+, has a homogeneous unit representation, then for some real constant ρ, $\varphi\psi^\rho$ is a representation of $\mathscr{C}$.*

This result extends considerably the structures for which the classical product of powers representation of physics holds [see Section 10.7 of Krantz *et al.* (1971)]. Two things are clear. First, one need not postulate that $\mathscr{C}$ has a product representation since that follows from the uniqueness of the translations of $\mathscr{X}$ and the assumption that $\mathscr{X}$ is distributive in $\mathscr{C}$. Second, the representation is not restricted to extensive structures or even to concatenation ones; it holds for any structure that is isomorphic to a real unit structure. Theorem 5.2 can thus be invoked in the formulation of dimensional analysis, much as was done in Chapter 10 of Krantz *et al.* (1971) and somewhat better in Luce (1978).

It is important to recognize that the solvability assumption is crucial to this result and that we do not know what can happen when it (partially) fails. An example (Luce and Narens, 1985, p. 74) exists of a (1,1) structure that is distributive in a conjoint structure, fails to satisfy the Thomsen condition, and, of course, is not solvable.

6. Proofs

6.1. THEOREMS 2.1 AND 2.2

Suppose $\mathscr{T}$ is not Archimedean, i.e., there exist $\sigma, \tau \in \mathscr{T}$ such that $\sigma \succ' \iota$ and for all integers n, $\tau \succsim' \sigma^n$. Since the translations are assumed to be uncrossed, for each $x \in X$, $\tau(x) \succsim \sigma^n(x)$. By Dedekind completeness, let

$$u(x) = \text{l.u.b.}\{\sigma^n(x) \mid n \text{ an integer}\}$$

Clearly $\sigma[u(x)] \succsim u(x)$. If $\sigma[u(x)] \succ u(x) \succsim \sigma^n(x)$ for all n, then $u(x) \succ \sigma^{-1}[u(x)] \succsim \sigma^{n-1}(x)$, which means $\sigma^{-1}[u(x)]$ is a smaller upper bound than $u(x)$, contrary to choice. So $u(x)$ is a fixed point of σ, which is impossible since σ is a nontrivial translation. Thus, $\mathscr{T}$ is Archimedean. □

Turning to Theorem 2.2i, suppose α and β are crossed, i.e., $\sigma = \alpha^{-1}\beta$ and ι are crossed. So for distinct $x, y \in X$, $\sigma(x) \succ x$ and $\sigma(y) \prec y$. By the homogeneity of $\mathscr{T}$, there exists $\tau \in \mathscr{T}$ such that $\tau(x) = y$. Using commutativity of $\mathscr{T}$,

$$\tau(x) = y \succ \sigma(y) = \sigma\tau(x) = \tau\sigma(x) \succ \tau(x),$$

which is impossible.

Part (ii) is shown in Theorem 2.4 of Luce (1986). □

6.2. THEOREM 3.1

(i) Observe that in the associative case, $\theta(x, y, n) = xO\theta(y, y, n-1)$. Thus, a standard sequence $\theta(y, y, n)$ is bounded iff the generalized standard sequence $\theta(x, y, n)$ is bounded.

(iia) Suppose $\{x_n\}$ is a standard sequence, i.e., $x_n = x_{n-1}Ox$. Using monotonicity and associativity,

$$x_n Op = [x_{n-1}Ox]Op = x_{n-1}O(xOp),$$

proving $\{x_n\}$ is a difference sequence.

(iib) Suppose $\{x_n\}$ is a difference sequence, i.e., for some $p \prec q$, $x_n Op = x_{n-1}Oq$. Select u such that $q \succsim uOp$, and so

$$x_n Op = x_{n-1}Oq \succsim x_{n-1}O(uOp) = (x_{n-1}Ou)Op,$$

Thus, $x_n \sim x_{n-1}Ou$, and so by induction $x_n \succsim x_1 Ou_{n-1} \succ u_{n-1}$, where $\{u_n\}$ is a standard sequence. Since, by hypothesis, $\{u_n\}$ is unbounded, so is $\{x_n\}$, proving the assertion.

(iiia) By Roberts and Luce (1968) (see Theorem 3.1 of Krantz *et al.*, 1971), such a structure has an additive representation, and so Archimedeaness in standard sequences obtains.

(iiib) By Narens (1985, Theorem 8.1, p. 79), such a structure has an additive representation, and so Archimedeaness in standard differences obtains.

(iv) Suppose $p \prec q$, then by restricted right solvability there exists u such that $pOu \precsim q$. So for all x, $(xOp)Ou = xO(pOu) \precsim xOq$. □

6.3. THEOREM 3.2

(i) Consider the equation $xOz = y$ which is equivalent in the unit structure to $zf(x/z) = y$. This may be rewritten $f(x/z)/(x/z) = y/x$. Since y/x can assume any positive value and f is onto Re^+, we see the solution exists iff $k = 0$. The left solution is similar.

(ii) Suppose $\mathscr{R}$ is positive and right solvable, then by Theorem 3.8.(5) of Luce and Narens (1985) $k = 1$. Conversely, suppose $k = 1$, then for any $y > x$, a right solution exists since the structure is on Re^+. The left case is similar.

(iii) Suppose $\mathscr{R}$ is positive and restrictedly right solvable. Suppose $x, y, r, s \in \mathrm{Re}^+$ are such that $x < y$ and $r < s$. Select ε such that $0 < \varepsilon < f(r)$. We first show that for $u > r$, $[f(u) - \varepsilon]/u$ is strictly decreasing. Suppose, on the contrary, there exist u, v, $r \leqslant u < v$, such that

$$[f(u) - \varepsilon]/u \leqslant [f(v) - \varepsilon]/v,$$

i.e.,

$$vf(u) - uf(v) \leqslant \varepsilon(v - u).$$

Since f is a unit representation, f is strictly increasing, so $f(v) > f(u)$, whence $f(u) \leqslant \varepsilon$. But $\varepsilon < f(r) \leqslant f(u)$, a contradiction.

Now, consider

$$\begin{aligned} zOy - zOx &= yf(z/y) - xf(z/x) \\ &= y[f(z/y) - \varepsilon] - x[f(z/x) - \varepsilon] + (y - x)\varepsilon \\ &= z\{[f(z/y) - \varepsilon]/(z/y) - [f(z/x) - \varepsilon]/(z/x)\} + (y - x)\varepsilon. \end{aligned}$$

Since $(z/y) < (z/x)$, the first term is positive, and so

$$zOy - zOx > (y - x)\varepsilon. \tag{1}$$

Let $g(x) = f(x)/x$. Since $\mathscr{X}$ is positive and restrictedly solvable, we know by Theorem 3.8 of Luce and Narens (1985) that g is strictly decreasing to 1 as $x \to \infty$. Let δ be such that

$$0 < \delta < \min_{r \leqslant z \leqslant s} \{(y - x)\varepsilon/(sOy), \max[g(z) - 1]\},$$

and let v be the solution to $g(v) = 1 + \delta$. We show that $u = (rOp)/v$ exhibits the desired property. Consider z such that $r \leqslant z \leqslant s$. Using the monotonicity of O and the fact g is decreasing, we see that

$$(zOy)\, g[(zOx)/u] \leqslant (sOy)\, g[(rOx)/u]. \tag{2}$$

Consider

$$\begin{aligned} zOy - (zOx)Ou &= zOy - (zOx)\, g[(zOx)/u] \\ &> zOy - [zOy - (y - x)\varepsilon]\, g[(zOx)/u] \quad \text{(Eq. 1)} \\ &= -(zOy)\{g[(zOx)/u] - 1\} + (y - x)\varepsilon g[(zOx)/u] \\ &> -(sOy)\{g[(rOx)/u] - 1\} + (y - x)\varepsilon \quad \text{(Eq. 2, } g > 1) \\ &= -(sOy)[g(v) - 1] + (y - x)\varepsilon \\ &= -(sOy)\delta + (y - x)\varepsilon \\ &> 0 \quad [\text{since } \delta < (y - x)\varepsilon/(sOy)]. \end{aligned}$$

□

6.4. THEOREM 3.3

(i) Suppose $\mathscr{X}$ is Archimedean in standard sequences. For any $x, y \in X$, by positivity $xOy \succ y$, and so by induction $\theta(x, y, n) \succ \theta(y, y, n-1)$. Thus, if $\theta(x, y, n)$ is bounded, so is $\theta(y, y, n)$, which by assumption is impossible. The converse is trivial.

(iia) Suppose $\{x_n\}$ is a bounded standard sequence. Select any $p \in X$ with $p \prec x_1$, and using right solvability let q_{n-1} solve $x_n Op = x_{n-1} O q_{n-1}$. We show that $\{q_n\}$ has a lower bound $q \succ p$. If not, then for some n sufficiently large, $q_{n-1} \prec x_1$, and so

$$x_n Op = x_{n-1} O q_{n-1} \prec x_{n-1} O x_1 = x_n,$$

which violates positivity. So, such a q exists. Thus, $x_n Op \succsim x_{n-1} Oq$. Using left solvability, there is a sequence $\{y_n\}$ satisfying $y_1 = x_1$ and $y_n Op = y_{n-1} Oq$. By induction, $y_n \precsim x_n$, since

$$y_nOp = y_{n-1}Oq \precsim x_{n-1}Oq \precsim x_nOp,$$

whence by monotonicity, $y_n \precsim x_n$. Therefore, $\{y_n\}$ is a bounded difference sequence, and so it is finite, whence $\{x_n\}$ is also finite.

(iib) Let $\{x_n\}$ be an increasing difference sequence that is bounded from above by z. So there are $p, q \in X$, $p \prec q$, such that for all n, $x_nOp = x_{n-1}Oq$. Let $u = u(p, q, x_1, z)$ be the element insured by right uniform solvability. Let $v = \min(u, x_1Op)$. Define $y_1 = v$, $y_n = y_{n-1}Ov$. Clearly $\{y_n\}$ is a standard sequence. We show by induction $x_nOp \succsim y_n$. It is true for $n = 1$ by definition of v. Suppose the statement is true for $n-1$, then

$$y_n = y_{n-1}Ov \precsim (x_{n-1}Op)Ou \precsim x_{n-1}Oq = x_nOp.$$

Because zOp bounds $\{y_n\}$ and the structure is Archimedean in standard sequences, the sequence can have only finitely many terms. Thus, it is also Archimedean in difference sequences.

(iiia) Suppose $x \succ y$ and $p \prec q$. By restricted right solvability, there exists u such that $x \succsim yOu$. By the hypothesis that n-copy operators preserve O, $x_n \succsim (yOu)_n = y_nOu_n$. By Archimedeaness in standard sequences, select n such that $u_n \succ q$. Then,

$$x_nOp \succ x_n \succsim y_nOu_n \succ y_nOq,$$

proving $\mathscr{X}$ is Archimedean in standard differences.

(iiib) Suppose $\{x_n\}$ is a standard sequence bounded by u. By density there is some v with $x_1 \prec v \prec x_1Ox_1$. Since the n-copy operator preserves O, $v_n \prec (x_1Ox_1)_n = x_nOx_n$. Select any $q \in X$. Since $x_1 \prec x_n \prec u$, by uniform, restricted right solvability there exists $p \in X$ such that for all n, $(x_nOu)Op \precsim x_nOq$. But

$$(x_nOu)Op \succsim (x_nOx_n)Op \succsim v_nOp,$$

so $x_nOq \succsim v_nOp$, which violates the Archimedeaness of standard differences. □

Proof of Corollary. Two of the defining properties of a positive unit structure are Archimedeaness in standard sequences and restricted right solvability, and it is easy to verify restricted left solvability. By Theorem 3.2iii, uniform restricted right and left solvability hold, and so by Part (iib), Archimedeaness in differences follows. As is readily verified, the n-copy operator is operation preserving (Cohen and Narens, 1979), so by part (iiia) it is also Archimedean in standard differences. □

6.5. THEOREM 3.4

(i) It is immediate that left intern density implies there is no minimal element. Suppose it is false that $\mathscr{X}$ is Archimedean in generalized standard sequences, i.e., there exist $x \prec y \prec z$ such that for all n, $\theta(x, z, n) \precsim y$. Since the structure is assumed to be Dedekind complete, this generalized standard sequence has a l.u.b. v. By left intern density, there exists $u \prec v$ such that $v \prec uOz$. By the l.u.b. property, there exists an n such that $u \prec \theta(x, z, n)$, and so

$$v \prec uOz \prec \theta(x, z, n)Oz = \theta(x, z, n+1),$$

which contradiction proves the result.

(ii) Suppose $x \prec y$. Since there is no minimal element, there exists $u \prec x$. Since $\mathscr{X}$ is Archimedean in generalized standard sequences, there exists n such that $\theta(u, y, n) \precsim x \prec \theta(u, y, n+1) = \theta(u, y, n)Oy$. So $z = \theta(u, y, n)$ fulfills the left intern density property.

(iii) Suppose f is the unit representation of $\mathscr{X}$, and suppose $x < y$. Since $yf(x/y) = xOy > x$, by the continuity of f we may select $\varepsilon > 0$ such that $(x-\varepsilon)Oy = yf[(x-\varepsilon)/y] > x$. Thus, $x - \varepsilon < x$ fulfills the condition for left intern density. □

6.6. THEOREM 4.1

(i) Let $\langle R^*, \geqslant \rangle$ be the Dedekind completion of $\langle R, \geqslant \rangle$ and $\langle T^*, \geqslant \rangle$ that of $\langle T, \geqslant \rangle$, excluding 0 and ∞. Define extensions of R_j as follows:

$$R_j^* = \{(r_1^*, \ldots, r_n^*) \mid \text{there exists finite or countable sequences } \{r_{ik}\},\ i = 1, \ldots, n, \text{ such that } r_i^* = \text{l.u.b.}_k(r_{ik}) \text{ and } (r_{1k}, \ldots, r_{nk}) \in R_j\}.$$

We verify the three properties of a generalized unit structure.

1. T^* under multiplication is a group. Since it is obviously associative, closed, and has an identity, it suffices to show that each element has an inverse. Suppose $t^* \in T^*$. Since $1/t^*$ is a lower bound of the set $\{1/t \mid t \in T \text{ and } t \leqslant t^*\}$, it has a g.l.b., say u^*. Suppose $u^* > 1/t^*$, then $t^* > 1/u^*$ and so there exists $t \in T$ and $t^* > t > 1/u^*$, when $u^* > 1/t$, contrary to choice. So $u^* = 1/t^*$.

2. T^* maps R^* into R^*. Suppose $r^* \in R^*$ and $t^* \in T^*$, then we show $r^*t^* \in R^*$. For each $r, t \in R$ with $r \leqslant r^*$ and $t \leqslant t^*$, $tr \in R$ and the l.u.b. of such tr, u^*, is $\leqslant t^*r^*$. Suppose $u^* < t^*r^*$, then by definition of t^* there exists $t \in T$ with $t^* > t > u^*/r^*$, and so $u^* < tr^*$. Select $r \in R$ sufficiently close to r^* so $u^* < tr$, which is a contradiction. So $u^* = t^*r^*$.

3. We show T^* are translations of the extension. Suppose $t^* \in T^*$, and select $\{t_k\}$ such that $t_k \in T$ and $t^* = \text{l.u.b.}\{t_k\}$. If the sequence is finite, make it countable by adding 1's. Observe that by hypothesis, if $(r_{1k}, \ldots, r_{nk}) \in R_j$, then $(t_k r_{1k}, \ldots, t_k r_{nk}) \in R_j$ and that $t^*r_i^* = \text{l.u.b.}_k\{t_k r_{ik}\}$. Thus, if $(r_i^*, \ldots, r_n^*) \in R_j^*$, it follows by definition and what we have just shown that $(t^*r_1^*, \ldots, t^*r_n^*) \in R_j^*$.

(ii) Let α be an automorphism of $\mathscr{R}$, and extend it to $\mathscr{R}^*$ by: for $r^* \in \mathrm{Re}^+$,

$$\alpha^*(r^*) = \text{l.u.b.}\{\alpha(r) \mid r \in R \,\&\, r \leqslant r^*\}.$$

First, we show that it is invariant under the defining relations R_j^*. Let r_{ik} be sequences in R such that for each k, $(r_{ik}, \ldots, r_{nk}) \in R_j$ and $r_i^* = \text{l.u.b.}_k\{r_{ik}\}$. Thus, by definition, $(r_1^*, \ldots, r_n^*) \in R_j^*$. Since α is an automorphism of $\mathscr{R}$, $(\alpha(r_{1k}), \ldots, \alpha(r_{nk})) \in R_j$. But since α is strictly increasing, it follows from the definition of $\mathscr{R}^*$ that $(\alpha^*)r_1^*), \ldots, \alpha^*(r_n^*)) \in R_j^*$. Next we show that α^* is 1 : 1. Suppose for $r^*, s^* \in R^*$, $r^* < s^*$, $\alpha^*(r^*) = \alpha^*(s^*)$. By the order density of R in Re^+ and the definition of l.u.b., we may select s_j, s_{j+1} such that $r^* \leqslant s_j < s_{j+1} < s^*$. Since α^* is monotonic on Re^+ and strictly monotonic on R, we have a contradiction. Finally, we observe that α^* is onto since R is order dense in Re^+.

(iii) Assuming T is homogeneous and R is order dense, we wish to show that T^* is homogeneous. By order density, $R^* = \mathrm{Re}^+$, and so it suffices to show $T^* = \mathrm{Re}^+$. This follows if T is order dense in Re^+, which we now show. Suppose $u, v \in \mathrm{Re}^+$ and $u < v$. Choose any $r \in R$, and by the density of R there eixsts $s \in R$ such that $ru < s < rv$, in which case by homogeneity $s/r \in T$ and $u < (s/r) < v$, proving the density of T. □

6.7. THEOREM 5.1

(i) is equivalent to (ii). Theorem 3.4 of Luce (1986).

To establish that (ii) is equivalent to (iii), we need the following sub-result, which generalizes Theorem 4.3 of Luce and Narens (1985). In that paper the concept of a right translation of an operation $*$ is introduced: $x \to x * y$ (Definition 4.4).

LEMMA. *Suppose $\mathscr{C} = \langle X \times P, \succsim \rangle$ is a conjoint structure that is solvable relative to $(x_0, p_0) \in X \times P$, that $*$ is the induced Holman operation on A, and that $\mathscr{T}(*)$ is the set of right translations of $*$. Suppose, further, that S is a relation of order n on X, that $\mathscr{E}$ is the set of endomorphisms, and $\mathscr{A}$ the set of automorphisms of $\langle X, \succsim_X, S \rangle$. Then, the following are true:*

1. *If S is distributive in $\mathscr{C}$, then $\mathscr{T}(*) \subseteq \mathscr{E}$.*
2. *$\mathscr{T}(*) \subseteq \mathscr{A}$ iff S is distributive in $\mathscr{C}$ and $\mathscr{C}$ is unrestrictedly X-solvable.*
3. *$\mathscr{T}(*) = \mathscr{A}$ iff $\mathscr{T}(*) \subseteq \mathscr{A}$ and $\mathscr{A}$ is 1-point unique.*

Proof. For $\tau \in \mathscr{T}(*)$, we know $\tau(x) = x * y$ for some y. So,

$$(\tau(x), p_0) \sim (x * y, p_0) \sim (x, \pi(y)).$$

Thus, $(\tau(x_1), \ldots, \tau(x_n))$ is similar to $(x_1, \ldots, x_n)$. Therefore, if $(x_1, \ldots, x_n) \in S$, then since S is distributive, $(\tau(x_1), \ldots, \tau(x_n)) \in S$. So τ is an endomorphism.

2. Suppose S is distributive in $\mathscr{C}$ and $\mathscr{C}$ is unrestrictedly X-solvable. If $\tau \in \mathscr{T}(*)$, then by unrestricted solvability it is onto; since it is order preserving it is 1 : 1; and by part (1) it is an endomorphism. So $\mathscr{T}(*) \subseteq \mathscr{A}$.

Conversely, suppose $\mathscr{T}(*) \subseteq \mathscr{A}$. Let τ_w denote the right translation $\iota * w$. Observe that $(x, p) \sim (y, q)$ iff $\tau_{\pi^{-1}(p)}(x) = \tau_{\pi^{-1}(q)}(y)$. Now suppose $(x_1, \ldots, x_n) \in S$ and $(x_i, p) \sim (y_i, q)$, $i = 1, \ldots, n$, then using invariance under automorphisms, we see

$$\begin{aligned}(\tau_{\pi^{-1}(p)}(x_1), \ldots, \tau_{\pi^{-1}(p)}(x_n))\\ = (\tau_{\pi^{-1}(q)}(y_1), \ldots, \tau_{\pi^{-1}(q)}(y_n)) \in S,\end{aligned}$$

and so $(y_1, \ldots, y_n) \in S$, proving that S is distributive in $\mathscr{C}$.

3. The same proof as in Theorem 4.3 of Luce and Narens (1985). □

We now continue the proof of the theorem.

(iii) implies (ii). Fix $x \in X$, and let $*$ be one of the Holman operations for $\mathscr{C}$ relative to x. By the Lemma, the set of right translations of $*$, $\mathscr{T}(*)$, is a subset of the automorphisms of $\mathscr{X}$. We show $\mathscr{T}(*)$ is a subset of $\mathscr{T}$. Suppose not, then for some right translation, there is a fixed point z, i.e., $z * w = z$. Since $\langle X, \succsim_X, *, x\rangle$ is a total concatenation structure [Theorem 2 of Luce and Cohen (1983)] $w \succsim x$ iff $z = z * w \succsim z * x = z$. Thus, $w = x$, and so the right translation is the identity. Next we show that $\mathscr{T}(*) = \mathscr{T}$. Suppose $\tau \in \mathscr{T}$, and let $y = \tau(x)$. Then $\tau(x) = y = x * y$, whence by 1-point uniqueness $\tau = \iota * y \in \mathscr{T}(*)$.

To show $\mathscr{T}$ is homogeneous, suppose $y, z \in X$. Observe that the right translations of $*$, and so translations of $\mathscr{X}$, $\tau_w = \iota * w$ have the property $\tau_w(x) = x * w = w$. Since the translations are a group, $\tau_y \tau_z^{-1}(z) = \tau_y(x) = y$, proving $\mathscr{T}$ is homogeneous.

To show that $\mathscr{T}$ is Archimedean, we first observe that $*$ is associative. Suppose $y, z, w \in X$. By the closure of $\mathscr{T}(*)$, there exists some u such that $\tau_w \tau_z = \tau_u$ and, indeed, $u = z * w$ since

$$u = x * u = \tau_u(x) = \tau_w \tau_z(x) = (x * z) * w = z * w.$$

Thus,

$$y * (z * w) = y * u = \tau_u(y) = \tau_w \tau_z(y) = (y * z) * w.$$

So, by induction, $(\tau_y)^n = \iota * y_n$. Since by Theorem 2 of Luce and Cohen (1983), $\langle X, \succsim_X, *, x\rangle$ is a total concatenation structure, we know that for $y, z \succ_X x$, there is some n such that $y_n \succsim_X z$, whence $\tau_y^n \succsim' \tau_z$, proving that $\mathscr{T}$ is Archimedean.

(ii) implies (iii). We first imbed $\mathscr{T}$ in $\mathscr{X}$. For $x \in X$, define $*_x$ as follows: for each $y, z \in X$, by 1-point homogeneity and by 1-point uniqueness (which follows from the fact $\mathscr{T}$ is a group) there exist unique $\tau, \sigma \in \mathscr{T}$ such that $\tau(x) = y$ and $\sigma(x) = z$. Let $y *_x z = \sigma\tau(x)$. It is easy to show that $\langle X, \succsim, *_x\rangle$ is isomorphic to $\langle T, \succsim', *\rangle$, where $*$ denotes function composition. Note that this means it is (trivially) a total concatenation structure that is commutative and associative. By Theorem 3 of Luce and Cohen (1983) we may construct a conjoint structure $\mathscr{C}$ on $X \times A$ such that the induced Holman operation relative to (x, x) is $*_x$. Map the defining relations of $\mathscr{X}$ onto $\langle X, \succsim_X, *_x\rangle$ under the isomorphism used to construct $\mathscr{C}$. By the Lemma, these relations are distributive provided that $\mathscr{T}(*_x)$ is a subset of $\mathscr{T}$. For $\tau \in \mathscr{T}(*_x)$, there is some w such that for all $y \in X$, $\tau(y) = y *_x w$. By monotonicity, this can have a fixed point if and only if $w = x$, in which case $\tau = \iota$. So τ is a translation.

Corollary 1 follows from the fact that the Holman operation is associative and commutative (see Theorem 5 of Luce and Cohen, 1983).

We establish Corollary 2. Note that by the construction of R in Part (i), R is a subgroup of the additive reals. If α is an automorphism and τ a translation, then $\alpha\tau\alpha^{-1}$ is a translation. For if x is a fixed point, then $\alpha^{-1}(x)$ is a fixed point of τ, contrary to choice. Treating these automorphisms as in $\mathscr{R}$, we may therefore write them as $\tau(x) = x + t$ and $\alpha\tau\alpha^{-1}(x) = x + s$, for some s, $t \in T$. Letting $y = \alpha(x)$,

$$\begin{aligned}\alpha(x) + s &= y + s \\ &= \alpha\tau\alpha^{-1}(y) \\ &= \alpha\tau(x) \\ &= \alpha(x + t).\end{aligned}$$

Since R is a subgroup of the additive reals, $0 \in R$ and so $s = \alpha(t) - \alpha(0)$. Setting $h(x) = \alpha(x) - \alpha(0)$, we see

$$h(x + s) = h(x) + h(s). \tag{3}$$

Since $\mathscr{X}$ is order dense, so is $\mathscr{R}$. Since α is order preserving, so is h, and hence h can be extended to a real interval so that Eq. (3) holds. It is well known that the solution is $h(x) = rx$, and so $\alpha(x) = rx + \alpha(0)$, as asserted. Obviously, $\mathscr{X}$ is at most 2-point unique. □

6.8. THEOREM 3.5

By Theorem 5.1, the given concatenation structure $\mathscr{X}$ is isomorphic to a homogeneous, unit structure, $\mathscr{R} = \langle R, \geqslant, \otimes \rangle$ for which the translation group R is homogeneous and, of course, Archimedean. Fix $r_0 \in R$ and define f by:

$$f(r) = (r_0 r \otimes r_0)/r_0.$$

For any $s \in R$, the homogeneity of T implies there exists $t \in T$ such that $ts = r_0$. So, using the invariance property of automorphisms,

$$r_0(r \otimes s)/s = tr \otimes ts = r_0(r/s) \otimes r_0.$$

Thus,

$$r \otimes s = sf(r/s).$$

From the monotonicity of $\otimes$, it is easy to verify that f is strictly increasing and

f/ι is strictly decreasing. With no loss of generality, we work with this numerical structure.

(i) Suppose that $\{r_n\}$ is a nontrivial difference sequence of $\mathscr{R}$ relative to $u, v \in R$ that is bounded. With no loss of generality, suppose $r_n < r_{n+1}$ and the sequence is bounded from above. Let w be the least upper bound. Since $r_n \otimes u = uf(r_n/u)$, in the limit as $n \to \infty$, $r_n \otimes u$ approaches $uf(w/u)$. Similarly, $r_{n+1} \otimes v$ approaches $vf(w/v)$. Since for all n, $r_n \otimes u = r_{n+1} \otimes v$, we have $f(w/u)/(w/u) = f(w/v)/(w/v)$. By the strict monotonicity of f, we conclude $u = v$, which is contrary to choice. So a nontrivial difference sequence cannot be bounded.

(ii) Next, suppose $\mathscr{R}$ is positive. By Theorem 3.3, it suffices to show Archimedeaness in standard sequences. Let $\{r_n\}$ be a nontrivial, bounded standard sequence. So, by induction, $r_1 = rf(1)$, $r_n = rf^n(1)$, where $f^1(1) = f(1)$ and $f^n(1) = f[f^{n-1}(1)]$. By positivity, $f(1) > 1$. If s is the l.u.b. of the sequence, then $f^n(1) \leqslant s/r$. Since f is strictly increasing and $f(1) > 1$, $f^n(1)$ is a strictly increasing bounded sequence, and so it has a l.u.b. $u > 1$. Note that $f(u) = u$. If $u \in T$, then for $s \in R$, $r = us \in R$ and

$$r \otimes s = sf(r/s) = sf(u) = su = r,$$

which is impossible since the structure is positive. If $u \notin T$, then for $\varepsilon > 0$, there exists some n such that $r_n < ur \leqslant r_n + \varepsilon$. Since the structure is homogeneous, there exists $t \in T$ such that $r_n = tr$, so $tr < ur \leqslant tr + \varepsilon$. Observe,

$$tr \otimes r = rf(t) < rf(u) = ru \leqslant tr + \varepsilon.$$

Since as $\varepsilon \to 0$, $tr \to s$, we see $s \otimes r \leqslant s$, which is contrary to positivity. The negative case is similar.

(iii) Since for a standard sequence $\{x_n\}$, $x_n = xf^n(1)$, it is easily verified that the n-copy operators preserve $\otimes$. So, using Part (ii) of this Theorem and Theorem 3.3iii, Archimedeaness in standard differences follows.

(iv) We show that left intern density holds, in which case Theorem 3.4i implies the result. Suppose $x \prec y$. Then, by internness, $x \prec xO(xOy) \prec xOy \prec y$. By homogeneity, there exists a translation τ with $\tau(x) = xOy \succ x$. Let $z = \tau^{-1}[xO(xOy)]$. We show that $z \prec x$ and $zOy \succ x$. Suppose $z \succsim x$, then $\tau(z) = xO(xOy) \succsim \tau(x) = xOy$, which is impossible. Next, suppose $zOy \precsim x$, then $\tau(z)O\tau(y) = \tau(zOy) \precsim \tau(x) = xOy$. Since translations cannot be crossed when the translation group is Archimedean ordered, $\tau(y) \succ y$, and so $\tau(z)Oy \precsim \tau(z)O\tau(y) \precsim xOy$. Thus, by monotonicity, $\tau(z) \precsim x$, which is impossible since $\tau(z) = xO(xOy) \succ x$.

We establish the Corollary. Since the translations of a unit structure are multiplication by positive constants, they form an Archimedean ordered group. So by Part (i) of the Theorem, the unit structure is Archimedean in difference sequences. Theorem 3.4iii shows it to be Archimedean in generalized standard sequences. □

6.9. THEOREM 5.2

(i) Since the set $\mathscr{T}$ of translations are assumed to be a group, they are 1-point unique so, by Theorem 5.1, $\mathscr{T}$ is homogeneous and by Corollary 1 $\mathscr{C}$ satisfies the Thomsen condition.

(ii) By Theorem 6.2 of Krantz *et al.* (1971), $\mathscr{C}$ has a multiplicative representation, say $\varphi'\psi'$, and so φ' is a multiplicative representation of the total concatenation structure induced by $\mathscr{C}$ on A. By Part (i) and the proof of Theorem 5.1, we know that $\mathscr{T}(*) = \mathscr{T}$. Thus, for $\tau \in \mathscr{T}$, there exists some $x \in X$ such that $\tau(y) = y * x$. So.

$$\varphi'[\tau(y)] = \varphi'(y * x) = \varphi'(y)\varphi'(x).$$

Since φ is a unit representation of $\mathscr{X}$, for some real function r on $\mathscr{A}$

$$\varphi[\tau(y)] = r(\tau)\varphi(y).$$

Since φ and φ' both preserve the order $\succsim_X$, there is a strictly increasing function g such that $\varphi' = g(\varphi)$. Thus,

$$\begin{aligned} g[\varphi(y)]\, g[\varphi(x)] &= \varphi'(y)\varphi'(x) \\ &= \varphi'[\tau(y)] \\ &= g\{\varphi[\tau(y)]\} \\ &= g\{r(\tau)\varphi(y)\}. \end{aligned}$$

Since $\mathscr{X}$ is Dedekind complete and order dense, φ is onto Re^+. So for some y, $\varphi(y) = 1$, whence $g(1)\, g[\varphi(x)] = g[r(\tau)]$, and so

$$g[r(\tau)]\, g[\varphi(y)] = g[r(\tau)\varphi(y)]\, g(1).$$

Since $\mathscr{X}$ is homogeneous, r is onto Re^+, so by a well known result (Aczél, 1966, p. 41) for some $\beta > 0$,

$$g(x) = g(1)x^{\beta}.$$

Thus,

$$[\varphi'\psi']^{1/\beta} = [g(1)\varphi^{\beta}\psi']^{1/\beta} = \varphi\psi,$$

where $\psi = [g(1)\psi']^{1/\beta}$, is a representation of $\mathscr{C}$.

(iii) This follows immediately from a proof parallel to (ii) and the fact that multiplicative representations of $\mathscr{C}$ form a log-interval scale. □

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