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An Axiomatic Theory of Conjoint, Expected Risk[1]

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This paper presents an axiomatization of subjective risk judgments that leads to a representation of risk in terms of seven free parameters. This is shown to have considerable predictive ability for risk judgments made by 10 subjects. The risk function retains many of the features of the expectation models—e.g., a constant number of parameters independent of the number of outcomes—but it also allows for asymmetric effects of transformations on positive and negative outcomes. This arises by axiomatizing independently the behavior with respect to gambles having entirely positive outcomes and those with entirely negative outcomes. Complex gambles are decomposed into these, and zero outcomes, using the expected risk property. The resulting risk function compares favorable with other functions previously suggested. It is also demonstrated that preference judgments are distinct from risk judgments, and that the theory does not apply as well to the former.

INTRODUCTION

Ever since expected utility models were shown to be empirically inadequate, interest has attached to finding other explanatory variables. One of the most prominent is the familiar, but undefined, concept of risk. Theories of choice that incorporate risk as a central concept—the most prominent being Coombs' portfolio theory of preference—have not achieved wide acceptance because a descriptively adequate measure of risk has not been developed. Some of Coombs' (1969, 1975; Coombs & Huang, 1970a, 1970b) intuitions about subjective risk—e.g., that the riskiness of equal expected value gambles should increase monotonically with the amount to be lost—were incorporated by Pollatsek and Tversky (1970) as axioms of risk judgments in a risk system closely related to an extensive structure. This model yielded a scale of risk for a money gamble that is a linear combination of its mean and variance. Coombs and Bowen (1971a), however, showed that Pollatsek and Tversky's risk measure is not empirically adequate because, despite the fact that perceived risk is indeed affected by both the expectation and variance of a gamble, they alone are insufficient to determine risk. In particular, by using transformations that maintained expectation and variance unchanged, they found risk varied systematically with the skewness of a gamble.

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Subsequent to the failure of Pollatsek and Tversky's risk measures few new suggestions have been made. One approach was to look at the effect of certain transformations of a gamble of its perceived riskiness. The experiments of Coombs and Bowen (1971a, 1971b), Coombs and Huang (1970a, 1970b), and Coombs and Lehner (1981), for example, followed this paradigm. Luce (1980, 1981) took this approach one step further by deriving some risk measures equivalent to certain functional equations relating the perceived risk to transformations on the gambles. He considered, first, the effect of a change of scale and, in particular, studied the two simplest possibilities, additive and multiplicative. Second, he considered two ways in which the density function representing a gamble might be aggregated into a single risk value. The two simplest possibilities seemed to be, first, a form analogous to expected utility integration, which resulted in an expected risk function already suggested by Huang (1971a) and, second, the density undergoing a transformation before integration. From the combination of options considered at these two choice points, four distinct possible measures were derived. Luce left examination of their descriptive validity to empirical investigation. Two sets of investigators have undertaken that task.

Weber (1984a, 1984b) set out to test Luce's assumptions. The first choice point, an additive or multiplicative effect of a change of scale on risk, was found to be indeterminate in the absence of sign dependence between the risk of the original gamble and the effect of the change of scale (Weber, 1984a). Turning to the second choice point, the second assumption considered by Luce, namely, that of density undergoing a transformation before integration, leads to risk functions that are insensitive to a change of origin of the random variable representing the gamble. This simple fact ruled out all measures of this type because it had been shown empirically that the subjective risk of a gamble is significantly affected by a change of origin (Coombs & Bowen, 1971a; Weber, 1984a). Of the expected risk functions, the additive one led to expectations of the logarithm of the random variable and the multiplicative one to expectations of power functions. The former function could be ruled out on a priori grounds because of its unreasonable behavior close to zero, namely, that the scale approaches negative infinity. The variables of the expected logarithmic risk function, parenthetically, provided a poor fit of subjective risk ratings (Weber, 1984b), primarily because of the insensitivity of moments, in this case $E(\log|X|)$, to the direction of transformations. This property which has led to the rejection of moments as useful variables in preference models (Coombs & Lehner, 1984; Payne, 1973) also seems to limit their role in models of risk (Coombs & Lehner, 1981). The risky choice literature gives growing support to the assumption that people treat positive and negative outcomes differently (Coombs & Lehner, 1984). As discussed by Lopes (1984), a separate consideration of gains and losses is often found in applied work (Fishburn, 1977; Holthausen, 1981).

Independently, Keller, Sarin, and Weber (1985) also investigated Luce's four cases. They observed that two of the four have the property that risk is unaffected by adding a constant to all outcomes, and they too disconfirmed this property. Of the two expectation forms, they also rejected the logarithmic form, concluding as did Weber (1984a, 1984b) that the main option is some variant on the expectation of a power of the outcomes of the gamble. This, of course, rests upon the assumption that risk satisfies the property that it can be computed from the risk of component gambles using an expected risk calculation. They studied this directly and showed what amounts to a risk version of the Allais paradox. Since we will invoke the expected risk property as part of the present axiomatization, their data make clear that it is not correct in detail. It may be possible to generate a subjective expected risk version, but we have not done so.

A variant of the most viable of Luce's (1980, 1981) risk functions, i.e., integration after a power transformation of positive and negative outcomes separately, is axiomatized in this article. Axioms 1 and 3 and parts of 2 and 4 are structural and thus are empirically less interesting properties. Indeed, Axioms 1 and 3 could be omitted altogether if we simply assumed as the domain of discourse all possible money gambles. Axiom 2 allows the risk function to retain the benefits of expectation models, namely, a constant number of parameters regardless of the number of outcomes, a property not shared by risk dimension models of the kind suggested by Payne (1973). The assumptions made to that end seem at least plausible. The crucial assumption that gambles are split into positive, negative, and zero components in the determination of risk is incorporated in Axiom 4. We do this by supposing, in effect, that changes of scale among gambles all of whose outcomes are positive result in a risk ordering that can be represented additively, and that the same is true for gambles all of whose outcomes are negative. It appears that the most problematic aspect of this assumption is the Archimedean property.

The present model differs from the earlier ones of Luce in two ways. First, it accepts the expected risk property [although that has been called into question by Keller *et al.* (1985)]. Second, and most important, it modifies the assumption about the impact of change in scale to take into account the fact that subjects seem to deal with the positive and negative outcomes rather differently. Otherwise, it is in the same spirit as the earlier work. The representation is somewhat more complex, as is the proof.

Another, related, avenue has been pursued by Fishburn (1982, 1984). From a relatively complicated set of relatively elementary axioms, he derived a risk function that is multiplicative in two factors, one based on the distribution of gains and the other an expectation of a transformation of the distribution of losses. Even his most specific result involves four free functions, and so it is not possible to compare his result directly with either our risk function or the others that have been proposed, all of which involve only free constants. In general spirit, Fishburn's axiomatization and ours are similar, differing in two major ways. We definitely postulate that risk is an expectation, and he does not although one of his factors is, and we make far more use of rescaling random variables, i.e., multiplication of the outcomes by a constant factor, than he did. The net effect is that our axioms are somewhat easier to state, our result is far more specific, and it arrives at a measure that is additive over gains and losses rather than multiplicative.

AXIOMS

In this section we formulate axioms for a qualitative theory of risk orderings and in the next we derive from them the possible numerical risk measures that can arise as representations. Of the two types of measures that are mathematically admissible, we show that one has a property which is clearly incorrect empirically, and so we are left with a single family of measures.

Our basic domain will be taken to be a set $\mathscr{G}$ of gambles of money, which we may therefore interpret as a set of random variables where the real numbers are identified with amounts of money. And we assume that the decision maker involved has an ordering according to risk, $\gtrsim$, of the pairs of random variables in $\mathscr{G}$. Thus, $\gtrsim$ is a binary relation over $\mathscr{G}$.

AXIOM 1. *For each* $\mathbf{X}$ *and* $\mathbf{Y}$ in $\mathscr{G}$, *a in* Re^+, *c in* $\mathrm{Re} - \{0\}$, *and p in* $[0, 1]$, *each of the following random variables is also in* $\mathscr{G}$:

(i) $a\mathbf{X}$ *where* $\Pr(a\mathbf{X} \leqslant x) = \Pr(\mathbf{X} \leqslant x/a)$.

(ii) $\mathbf{X}o_p\mathbf{Y}$ *where* $\Pr(\mathbf{X}o_p\mathbf{Y} \leqslant z) = p\Pr(\mathbf{X} \leqslant z) + (1-p)\Pr(\mathbf{Y} \leqslant z)$.

(iii) $\mathbf{c}$ *where*

$$\begin{aligned} \Pr(\mathbf{c} \leqslant x) &= 0, \quad &\text{if} \quad x < c, \\ &= 1, \quad &\text{if} \quad x \geqslant c. \end{aligned}$$

(iv) $\mathbf{X}_c$ *where for* $c > 0$,

$$\begin{aligned} \Pr(\mathbf{X}_c \leqslant x) &= 0, \quad & x < 0, \\ &= x/c, \quad & 0 \leqslant x \leqslant c, \\ &= 1, \quad & c < x. \end{aligned}$$

and for $c < 0$,

$$\begin{aligned} \Pr(\mathbf{X}_c \leqslant x) &= 0, \quad & x < -c, \\ &= 1 + x/c, \quad & -c \leqslant x \leqslant 0, \\ &= 1, \quad & 0 < x. \end{aligned}$$

It is clear that we could simplify Axiom 1 by simply assuming that $\mathscr{G}$ contains all random variables; the present version makes clear, however, exactly what is needed to prove the result. It is also clear that we could treat the properties of Axiom 1 as simply a definition of the domain of discourse rather than as an axiom. It is a minor matter of taste how it is handled.

LEMMA 1. *If Axiom* 1(i) *holds, a, b* > 0, *and* $\mathbf{X}$ *is in* $\mathscr{G}$, *then* $a(b\mathbf{X}) = (ab)\mathbf{X}$ *is also in* $\mathscr{G}$.

Proof.

$$\begin{aligned}\Pr[a(b\mathbf{X}) \leqslant x] &= \Pr[b\mathbf{X} \leqslant x/a]\\ &= \Pr(\mathbf{X} \leqslant x/ab)\\ &= \Pr[(ab)\,\mathbf{X} \leqslant x],\end{aligned}$$

so $a(b\mathbf{X}) = (ab)\mathbf{X}$. It is obviously in $\mathscr{G}$ by Axiom 1(i). ∎

LEMMA 2. *If Axioms* 1(i) *and* (ii) *hold,* $a > 0$, p *is in* $[0, 1]$, *and* $\mathbf{X}$, $\mathbf{Y}$ *are in* $\mathscr{G}$, *then* $a(\mathbf{X}o_p\mathbf{Y}) = (a\mathbf{X})\, o_p(a\mathbf{Y})$ *is also in* $\mathscr{G}$.

Proof. Using Axioms 1(i) and (ii) freely,

$$\begin{aligned}\Pr[a(\mathbf{X}o_p\mathbf{Y}) \leqslant z] &= \Pr(\mathbf{X}o_p\mathbf{Y} \leqslant z/a)\\ &= p\Pr(\mathbf{X} \leqslant z/a) + (1-p)\Pr(\mathbf{Y} \leqslant z/a)\\ &= p\Pr(a\mathbf{X} \leqslant z) + (1-p)\Pr(a\mathbf{Y} \leqslant z)\\ &= \Pr[(a\mathbf{X})\, o_p(a\mathbf{Y}) \leqslant z],\end{aligned}$$

which proves the equality, and it is in $\mathscr{G}$ by Axiom 1(i). ∎

AXIOM 2. *The family* $\langle \mathscr{G}, \gtrsim, o_p \rangle_{p \in [0,1]}$ *satisfies an axiom system for a mixture space so that there is a real representation* R *such that* R *is order preserving and for all* $\mathbf{X}$, $\mathbf{Y}$ *in* $\mathscr{G}$ *and* p *in* $[0, 1]$

$$R(\mathbf{X}o_p\mathbf{Y}) = pR(\mathbf{X}) + (1-p)\, R(\mathbf{Y}).$$

This axiom, which was first postulated for risk by Huang (1971a), bears some comment. We assume the reader is familiar with the expected utility literature and, in particular, one or another of the axiom systems that have been given for a mixture space (see, for example, Fishburn, 1970). Usually these axioms are taken to be postulates for a concept of utility of gambles, but as is well known this has encountered empirical difficulties. Indeed, that is one of the reasons that risk, together with other properties of the gamble such as expected value, has come to be examined as a possibly relevant concept. The key assumptions of a mixture space that have been some source of difficulty for utility theory are the assumption that $\gtrsim$ is a weak order, and so is transitive, and, possibly, the monotonicity property that if $\mathbf{X} \gtrsim \mathbf{Y}$, then for every p in $(0, 1)$ and $\mathbf{Z}$ in $\mathscr{G}$,

$$\mathbf{X}o_p\mathbf{Z} \gtrsim \mathbf{Y}o_p\mathbf{Z}.$$

(This property is often referred to in the utility literature as strong independence or as the substitution principle; in the axiomatic measurement literature, it is called monotonicity.) Care must be taken in concluding whether existing data show violations of monotonicity, per se. Some examples, such as the Allais paradox

(Fishburn, 1970, p. 109), which violate expected utility theory only implicate monotonicity indirectly. Specifically, the paradox is a violation of the property:

$$\mathbf{X}o_p\mathbf{Z} \gtrsim \mathbf{Y}o_r\mathbf{Z} \quad \text{iff} \quad \mathbf{X}o_{ps}\mathbf{Z} \gtrsim \mathbf{Y}o_{rs}\mathbf{Z}.$$

This can be decomposed into two separate statements. The first is monotonicity:

$$\mathbf{X}o_p\mathbf{Z} \gtrsim \mathbf{Y}o_r\mathbf{Z} \quad \text{iff} \quad (\mathbf{X}o_p\mathbf{Z})\, o_s\mathbf{Z} \gtrsim (\mathbf{Y}o_r\mathbf{Z})\, o_s\mathbf{Z},$$

and the second may be described as a probability accounting equation:

$$(\mathbf{X}o_p\mathbf{Z})\, o_s\mathbf{Z} \sim \mathbf{X}o_{ps}\mathbf{Z}.$$

What the experiments demonstrate is that monotonicity together with a correct probability analysis of compound gambles does not hold. The move from expected utility to subjective expected utility preserves monotonicity but abandons the probability accounting. For a much more detailed study of these matters, see Section 7 of Luce and Narens (1985).

If one contemplates the riskiness of gambles, it seems plausible that risk should exhibit both transitivity and monotonicity. It is, of course, another matter whether in fact people's judgments of risk do. A number of empirical studies have reported data in which the expected risk hypothesis is sustained (Aschenbrenner, 1978; Coombs, 1975; Coombs & Bowen, 1971b; Coombs & Meyer, 1969; Huang, 1971b). In contrast, Lehner (1980) made an assumption about risk that is not necessarily consistent with expected risk. For a gamble in which the outcomes are $k+x$, k, and $k-x$ with probabilities p, $1-2p$, and p, respectively, he assumed the risk to be a monotonic function of p and x. Under expected risk, it is easy to see that the risk of the gamble is $[R(k+x)+R(k-x)-2R(k)]\,p+R(k)$, which is clearly monotonic in p but need not be in x.

As was noted above, Keller *et al.* (1985) showed that the expected risk property is not a fully satisfactory assumption. We do not yet know if monotonicity itself fails for risk or if a subjective expected risk hypothesis is viable. If so, it remains to develop a theory like the present one for that assumption.

DEFINITION 1. Let $\mathscr{G}^+$ denote the subset of random variables in $\mathscr{G}$ for which all realizable outcomes are positive and $\mathscr{G}^-$ the subset for which all realizable outcomes are negative. The restrictions of $\gtrsim$ to $\mathscr{G}^+$ and to $\mathscr{G}^-$ are denoted, respectively, $\gtrsim^+$ and $\gtrsim^-$.

Note that if $a > 0$ and $\mathbf{X}$ is in $\mathscr{G}^+$ and $\mathbf{Y}$ is in $\mathscr{G}^-$, then $a\mathbf{X}$ is in $\mathscr{G}^+$ and $a\mathbf{Y}$ is in $\mathscr{G}^-$.

DEFINITION 2. If $\mathbf{X}$ is any random variable for which $\Pr(\mathbf{X} > 0) > 0$, then its positive projection $\mathbf{X}^+$ is defined by:

$$\Pr(\mathbf{X}^+ \leqslant x) = \Pr(0 < \mathbf{X} \leqslant x)/\Pr(\mathbf{X} > 0), \quad \text{if} \quad x > 0,$$
$$= 0, \quad \text{if} \quad x \leqslant 0.$$

If $\Pr(\mathbf{X} < 0) < 0$, then its negative projection $\mathbf{X}^-$ is defined by:

$$\Pr(\mathbf{X}^- \leqslant x) = 1, \quad \text{if} \quad x \geqslant 0,$$
$$= \Pr(\mathbf{X} \leqslant x)/\Pr(\mathbf{X} < 0), \quad \text{if} \quad x < 0.$$

LEMMA 3. *Suppose* $\mathbf{X}$ *is a random variable with* $\mathbf{X}^+$ *and* $\mathbf{X}^-$ *its positive and negative projections, if they exist. Let* $q = \Pr(\mathbf{X} > 0) + \Pr(\mathbf{X} < 0)$, *and if* $q > 0$, *let* $p = \Pr(\mathbf{X} > 0)/q$. *Then,*

$$\mathbf{X} = (\mathbf{X}^+ o_p \mathbf{X}^-)\, o_q \mathbf{0}.$$

Proof. Observe that $\Pr(\mathbf{X} < 0) = q(1-p)$ and $\Pr(\mathbf{X} \leqslant 0) = q(1-p) + (1-q) = 1 - pq$. Let $\mathbf{X}' = (\mathbf{X}^+ o_p \mathbf{X}^-)\, o_q \mathbf{0}$, then we show that $\mathbf{X} = \mathbf{X}'$ simply by calculating

$$\begin{aligned}
\Pr(\mathbf{X}' \leqslant x) &= pq \Pr(\mathbf{X}^+ \leqslant x) + 1 - pq, && \text{if} \quad x > 0,\\
&= 1 - pq, && \text{if} \quad x = 0,\\
&= (1-p)\, q \Pr(\mathbf{X}^- \leqslant x), && \text{if} \quad x < 0,\\
&= \Pr(0 < \mathbf{X} \leqslant x) + \Pr(\mathbf{X} \leqslant 0), && \text{if} \quad x > 0,\\
&= \Pr(\mathbf{X} \leqslant 0), && \text{if} \quad x = 0,\\
&= \Pr(\mathbf{X} \leqslant x), && \text{if} \quad x < 0,\\
&= \Pr(\mathbf{X} \leqslant x). \quad \blacksquare
\end{aligned}$$

This result motivates the following assumption.

AXIOM 3. *For each* $\mathbf{X}$ *in* $\mathscr{G}$, *either it has no positive projection* $\mathbf{X}^+$ *or* $\mathbf{X}^+$ *is in* $\mathscr{G}^+$, *and either it has no negative projection* $\mathbf{X}^-$ *or* $\mathbf{X}^-$ *is in* $\mathscr{G}^-$.

Axiom 3, as Axiom 1, would be unnecessary if we assumed $\mathscr{G}$ to consist of all random variables. One referee suggested that Axiom 3 can be derived from the preceding axioms, but we have not seen how to do it.

AXIOM 4. *The structures* $\langle \mathrm{Re}^+ \times \mathscr{G}^+, \gtrsim^+ \rangle$ *and* $\langle \mathrm{Re}^+ \times \mathscr{G}^-, \gtrsim^- \rangle$ *are each conjoint structures satisfying the following conditions, where* $\mathbf{X}$, $\mathbf{Y}$ *are both in* $\mathscr{G}^+$ *or both in* $\mathscr{G}^-$ *and* a, b, b', b'' *in* Re^+:

(i) *Independence*: $\mathbf{X} \gtrsim \mathbf{Y}$ *iff* $a\mathbf{X} \gtrsim a\mathbf{Y}$, *and* $a\mathbf{X} \gtrsim b\mathbf{X}$ *iff* $a\mathbf{Y} \gtrsim b\mathbf{Y}$.

(ii) *The ordering induced by independence on* Re^+ *is* $\geqslant$, *i.e.*, $a\mathbf{X} \gtrsim b\mathbf{X}$ *iff* $a \geqslant b$.

(iii) *Restricted solvability*: *if* $b'\mathbf{Y} \gtrsim a\mathbf{X} \gtrsim b''\mathbf{Y}$, *then there exists* b *in* Re^+ *such that* $b\mathbf{Y} \sim a\mathbf{X}$.

(iv) *Archimedean*: *if* $1\mathbf{X} > 1\mathbf{Y}$, *then there exists* a *in* Re^+ *such that* $a\mathbf{Y} \gtrsim 1\mathbf{X}$.

The major assumptions here, aside from the richness of the space of gambles forced by restricted solvability, are independence and the Archimedean property. The first part of independence says that the relative risk ordering of two gambles that are entirely positive or entirely negative, is unchanged by a shift in the units of play. So, for example, if the 50–50 gamble between 50 cents and 10 cents is viewed as more risky than a 75–25 gamble between 30 cents and 20 cents, then the same risk ordering will hold if the cents are changed to dollars, i.e., $a = 100$. The second part says that if, for a particular gamble all of whose outcomes are positive (negative), one choice of unit is seen as more risky than another unit, then that ordering will be true for any other gamble all of whose outcomes are positive (negative). Although these properties seem plausible to us, they require empirical verification. Assuming independence, the second condition says that riskiness for gambles that are entirely positive or entirely negative is an increasing function of the scale value. This, too, is plausible.

The solvability condition, which is asserted only for the continuum representing the possible scale transformations, is a way of saying that risk is a continuous function of scale changes. For if solvability did not hold, there would have to be a gap in the measure of risk, destroying its continuity. Recall that Axiom 2 also involves a solvability condition, namely, that if $\mathbf{Z} \gtrsim \mathbf{Y} \gtrsim \mathbf{X}$, then there is p in $[0, 1]$ such that $\mathbf{Z}o_p\mathbf{X} \sim \mathbf{X}$. We do not know if, in the presence of the other axioms, these two solvability conditions are independent axioms.

The Archimedean property is also stated just for the scale dimension, and it is of the classical form: given any two gambles with one riskier than the other, it asserts that for a sufficiently large change of scale, the less risky one can be transformed into one that is more risky than the other gamble. So, for example, suppose $\mathbf{X}$ denotes the gamble of winning \$1 with probability 1/2 or winning \$1000 otherwise and $\mathbf{Y}$ denotes the sure outcome, which is a gamble, of winning \$1, so in dollar units $\mathbf{Y} = 1$. Assuming $\mathbf{X} > \mathbf{1}$, which is plausible, then the axiom says that for some sure outcome \$$a$, $\mathbf{a} = a\mathbf{1} \gtrsim \mathbf{X}$. Many feel that this is in error. They argue that any sure win has no risk, i.e., $R(\mathbf{a}) = 0$ for $a > 0$. But if that is so, then by induction on Axiom 2 any gamble in $\mathscr{G}^+$ has risk 0 and so the structure $\mathrm{Re}^+ \times \mathscr{G}^+, \gtrsim^+ \rangle$ is trivial. This would reflect itself in the following representation theorem as $A(+) = B(+) = 0$. The data presented later (see Table 2) do not support this prediction at all. So either the intuition about the Archimedean axiom is incorrect or the rest of the structure is in error.

In proving the main theorem, we will establish that both restricted solvability and the conjoint Archimedean property hold separately on $\mathscr{G}^+$ and on $\mathscr{G}^-$.

It is, perhaps, surprising that we do not impose the Thomsen condition, which is necessary for there to be an additive representation of a conjoint structure, but that property turns out to be a consequence of our other assumptions.

The final axiom simply postulates that the risk function of Axiom 2 is well behaved near 0.

AXIOM 5. *For each real c, $R(\mathbf{c})$ is bounded as c approaches* 0.

We do not know of any way to restate this property in terms of the primitives of the system.

REPRESENTATION THEOREM

THEOREM. *Suppose $\mathscr{G}$ is a set of random variables, $\gtrsim$ a binary relation on $\mathscr{G}$, and for each p in* $[0, 1]$ *o_p is a binary operation on $\mathscr{G}$ such that $\langle \mathscr{G}, \gtrsim, o_p \rangle_{p \in [0,1]}$ satisfies Axioms 1–5. Then the real representation R of Axiom 2 is of the form: for each $\mathbf{X}$ in $\mathscr{G}$,*

$$R(\mathbf{X}) = A(0)\Pr(\mathbf{X}=0) + A(+)\Pr(\mathbf{X}>0) + A(-)\Pr(\mathbf{X}<0)$$
$$+ B(+)\,E[\mathbf{X}^{k(+)} \mid \mathbf{X}>0]\Pr(\mathbf{X}>0) + B(-)\,E[|\mathbf{X}|^{k(-)} \mid \mathbf{X}<0]\Pr(\mathbf{X}<0),$$

where $A(0)$, $A(+)$, $A(-)$, $B(+)$, $B(-)$, $k(+)$, and $k(-)$ are constants with $k(+)>0$, and $k(-)>0$.

Proof. We first show that $\langle \mathrm{Re}^+ \times \mathscr{G}^+, \gtrsim^+ \rangle$ satisfies the Thomsen condition. We omit the superscript $^+$ on $\gtrsim$. Suppose $a\mathbf{Y} \sim b\mathbf{Z}$ and $b\mathbf{X} \sim c\mathbf{Y}$, then by independence [Axiom 4(i)] and Lemma 1,

$$(ca)\,\mathbf{Y} = c(a\mathbf{Y}) \sim c(b\mathbf{Z}) = (cb)\,\mathbf{Z}$$
$$(ac)\,\mathbf{Y} = a(c\mathbf{Y}) \sim a(b\mathbf{X}) = (ab)\,\mathbf{X}.$$

Since $ac = ca$,

$$b(c\mathbf{Z}) = (bc)\,\mathbf{Z} = (cb)\,\mathbf{Z} \sim (ca)\,\mathbf{Y} = (ac)\,\mathbf{Y} \sim (ab)\,\mathbf{X} = (ba)\,\mathbf{X} = b(a\mathbf{X}),$$

whence by independence $a\mathbf{X} \sim c\mathbf{Z}$.

Next we show that restricted solvability holds on the $\mathscr{G}^+$ component (it is assumed on the Re^+ component). Suppose $b\mathbf{Y}' \gtrsim a\mathbf{X} \gtrsim b\mathbf{Y}''$. Then by the axioms of a mixture space and Lemma 2, there exists a p such that

$$a\mathbf{X} \sim b\mathbf{Y}' o_p b\mathbf{Y}'' \sim b(\mathbf{Y}' o_p \mathbf{Y}'').$$

So $\mathbf{Y} = \mathbf{Y}' o_p \mathbf{Y}''$ suffices.

To show the conjoint Archimedean property, consider first a standard sequence $\{a_n\}$ on Re^+. By definition, for some $\mathbf{Y} > \mathbf{X}$, $a_{n+1}\mathbf{X} \sim a_n\mathbf{Y}$. We show, by induction, that $a_n = a_1(a_1/a_0)^{n-1}$. The assertion is clearly true for $n=1$. Observe that by the defining relations and Lemma 1,

$$a_0 a_{n+1}\mathbf{X} \sim a_0 a_n \mathbf{Y} = a_n a_0 \mathbf{Y} \sim a_n a_1 \mathbf{X}.$$

Thus, by Axioms 4(i) and (ii), $a_0 a_{n+1} = a_n a_1$, and the result follows from the induction hypothesis. Since $\mathbf{Y} > \mathbf{X}$, it follows from Axiom 4(ii) that $a_1/a_0 > 1$. So, by the Archimedean property of numbers, any bounded standard sequence is finite. For the other component, a standard sequence $\{\mathbf{X}_n\}$ satisfies $a\mathbf{X}_{n+1} \sim b\mathbf{X}_n$, $b > a$. Thus, by Lemma 1 and independence, $\mathbf{X}_{n+1} \sim (b/a)\,\mathbf{X}_n$, and so by induction $\mathbf{X}_{n+1} \sim (b/a)^n\,\mathbf{X}_0$. Suppose that for all integers n, $\mathbf{Z} > \mathbf{X}_n$, then in particular $\mathbf{Z} > \mathbf{X}_0$ and so by the Archimedean assumption [Axiom 4(iv)] there exists real c such that $c\mathbf{X}_0 \gtrsim \mathbf{Z} > (b/a)^n\,\mathbf{X}_0$. So, by independence and Axiom 4ii, $c > (b/a)^n$, which can be true only for finitely many n, proving that the bounded standard sequence is finite.

To show that the first component is essential, consider any $a > b$. By Axioms 1(i), 4(i), and (ii), for any $\mathbf{X}$ in $\mathscr{G}^+$, $a\mathbf{X} > b\mathbf{X}$. To show that the second component is essential, note that $\mathbf{Y} = a\mathbf{X} > b\mathbf{X} = \mathbf{Z}$. So, for any $c > 0$, $c\mathbf{Y} > c\mathbf{Z}$, showing that the second component is essential.

By Theorem 6.2 of Krantz *et al.* (1971), the conjoint structure $\langle \mathrm{Re}^+ \times \mathscr{G}^+, \gtrsim^+ \rangle$ has an additive representation $S^+ + T^+$. If we select the linear transformation so that $S^+(1) = 0$, then from the fact that $1\mathbf{X} = \mathbf{X}$, we see that $S^+ + T^+$ and T^+ agree on $\mathscr{G}^+$ in the sense that $[S^+ + T^+](1\mathbf{X}) = S^+(1) + T^+(\mathbf{X}) = T^+(\mathbf{X})$.

By Axiom 2, the entire mixture space has a representation R. Since both R and T^+ are order preserving over $\mathscr{G}^+$, then for that domain there is a strictly increasing function f such that $R = f(T^+)$. By Lemma 2 and properties of R and T^+, we see that

$$\begin{aligned}
f\{T^+[a(\mathbf{X}o_p\mathbf{Y})]\} &= R[a(\mathbf{X}o_p\mathbf{Y})] \\
&= R[(a\mathbf{X})\,o_p(a\mathbf{Y})] \\
&= pR(a\mathbf{X}) + (1-p)\,R(a\mathbf{Y}) \\
&= pf[T^+(a\mathbf{X})] + (1-p)f[T^+(a\mathbf{Y})] \\
&= pf[S^+(a) + T^+(\mathbf{X})] + (1-p)f[S^+(a) + T^+(\mathbf{Y})].
\end{aligned}$$

Equally well,

$$\begin{aligned}
f\{T^+[a(\mathbf{X}o_p\mathbf{Y})]\} &= f[S^+(a) + T^+(\mathbf{X}o_p\mathbf{Y})] \\
&= f\{S^+(a) + f^{-1}[R(\mathbf{X}o_p\mathbf{Y})]\} \\
&= f\{S^+(a) + f^{-1}[pR(\mathbf{X} + (1-p)\,R(\mathbf{Y})]\} \\
&= f(S^+(a) + f^{-1}\{pf[T^+(\mathbf{X})] + (1-p)f[T^+(\mathbf{Y})]\}).
\end{aligned}$$

Equating, taking f^{-1}, and setting $T^+(\mathbf{X}) = x$, $T^+(\mathbf{Y}) = y$, and $S^+(a) = z$, we see that f satisfies the functional equation

$$f^{-1}[pf(x+z) + (1-p)f(y+z)] = f^{-1}[pf(x) + (1-p)f(y)] + z.$$

Aczél (1966, pp. 152–153) dealt with this functional equation, showing that its possible solutions are:

$$f(x) = cx + d \qquad \text{or} \qquad f(x) = ce^{kx} + d, \qquad c > 0, k > 0.$$

By an exactly parallel argument using $\mathscr{G}^-$, one shows the existence of a representation T^- with the property $T^-(a\mathbf{X}) = S^-(a) + T^-(\mathbf{X})$, and R and T^- over $\mathscr{G}^-$ are either linearly or exponentially related.

Applying Axiom 2 to the decomposition stated in Axiom 3,

$$\begin{aligned} R(\mathbf{X}) &= pqR(\mathbf{X}^+) + (1-p)\,qR(\mathbf{X}^-) + (1-q)\,R(\mathbf{0}) \\ &= \Pr(\mathbf{X} > 0)\,R(\mathbf{X}^+) + \Pr(\mathbf{X} < 0)\,R(\mathbf{X}^-) + \Pr(\mathbf{X} = 0)\,R(\mathbf{0}). \end{aligned}$$

We consider the three terms separately. For the positive part, we know that R is related to T^+ either linearly or exponentially. Consider the former case first. For $a > 0$,

$$R(a\mathbf{X}^+) = cT^+(a\mathbf{X}^+) + d = c[S^+(a) + T^+(\mathbf{X}^+)] + d = cS^+(a) + R(\mathbf{X}^+).$$

So,

$$\begin{aligned} R[(ab)\,\mathbf{X}^+] &= cS^+(ab) + R(\mathbf{X}^+) \\ &= R[a(b\mathbf{X}^+)] \\ &= cS^+(a) + R(b\mathbf{X}^+) \\ &= cS^+(a) + cS^+(b) + R(\mathbf{X}^+), \end{aligned}$$

whence $S^+(ab) = S^+(a) + S^+(b)$. By Axiom 4(ii), S^+ is a strictly increasing function of a, and so (Aczél, 1966, p. 41) the solution is for some $C > 0$,

$$S^+(a) = C \log a.$$

Consider the random variable $\mathbf{X}_a$ defined in Axiom 1(iv), and note that since $a > 0$,

$$\begin{aligned} \Pr[(1/a)\,\mathbf{X}_a = x) = \Pr(\mathbf{X}_a = ax) &= 1, \qquad \text{if} \quad ax \text{ in } [0, a] \text{ iff } x \text{ in } [0, 1] \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

By Axiom 2, R and so T^+ is an expectation, hence the relation

$$T^+[(1/a)\,\mathbf{X}_a] = S^+(1/a) + T^+(\mathbf{X}_a)$$

coupled with the distribution function of $\mathbf{X}_a$ yields,

$$D = \int_0^1 T^+(x)\,dx = -C \log a + \int_0^a T^+(x)(1/a)\,dx,$$

where in a minor abuse of notation we write $T^+(a)$ for $T^+(\mathbf{a})$. By Axiom 1(iii), $T^+(a)$ is defined for each $a>0$. Differentiate this with respect to a:

$$0=-C/a+T^+(a)/a+(-1/a^2)\int_0^a T^+(x)\,dx.$$

Solving for the integral, substituting into the preceding equation, and solving for $T^+(a)$ yields,

$$T^+(a)=D+C+C\log a.$$

Using the expectation property of Axiom 2,

$$\begin{aligned}T^+(\mathbf{X}^+)&=\int_0^\infty T^+(x)\Pr(\mathbf{X}^+=x)\,dx\\&=D+C+CE(\log\mathbf{X}^+)\\&=D+C+CE(\log\mathbf{X}\mid\mathbf{X}>0).\end{aligned}$$

For $a>0$, $T^+(a)=D+C+C\log a$, so substituting $R=cT^++d$, we see that R is not bounded as a approaches 0. Thus, by Axiom 5, this case is impossible.

Turning to the exponential case, one carries out exactly the parallel argument to show, first,

$$\exp S^+(ab)=\exp S^+(a)\exp S^+(b),$$

whence for some $k(+)>0$,

$$\exp S^+(a)=a^{k(+)}.$$

Next, carrying out the parallel argument on the same special case as before, we find that

$$\exp T^+(a)=[1+k(+)]\,D'a^{k(+)},$$

where $D'=\int_0^1\exp T^+(x)\,dx$ and so

$$R(\mathbf{X}^+)=c\exp T^+(\mathbf{X}^+)+d=A(+)+B(+)\,E(\mathbf{X}^{k(+)}\mid\mathbf{X}>0),$$

where $A(+)=d$ and $B(+)=c[1+k(+)]\,D'$.

To obtain the results for $\mathbf{X}^-$, one follows exactly the same line of argument, of course taking into account the fact we are dealing with negative quantities and so absolute values must be used. ∎

Discussion

The plausibility of axioms underlying the Theorem make the risk function

$$R(\mathbf{X}) = A(0)\,\Pr(\mathbf{X}=0) + A(+)\,\Pr(\mathbf{X}>0) + A(-)\,\Pr(\mathbf{X}<0) + B(+)\,E[\mathbf{X}^{k(+)} \mid \mathbf{X}>0)]\,\Pr(\mathbf{X}>0) + B(-)\,E[|\mathbf{X}|^{k(-)} \mid \mathbf{X}<0]\,\Pr(\mathbf{X}<0)$$

a reasonable candidate. We refer to is as CER, standing for conjoint, expected risk.

Weber (1986) ran several experiments on subjective risk and obtained results which motivated the development of the present axiom system. One of these experiments, however, allows a test of the present model. The details of the experiment are described there. Suffice it to say here that subjects rated the risk of 30 gambles by marking locations on a line. Each gamble consisted of two independent repetitions of the same two outcome gambles, and these were all generated from a single basic gamble by three families of transformations. The first yielded five different levels of skewness, the second two shifts of the origin, and the third three values of scale. Ten student subjects, five of each sex, made the judgments on four separate occasions. For each subject, regression coefficients were selected by numerical search so as to maximize the proportion of variance accounted for. This was done for the variance expectation (V–E) measure of Pollatsek and Tversky (1970) and the risk dimensions (R–D) of Payne (1973) as well as for CER. The

TABLE 1

Values of R^2 for Three Risk Measures Obtained from Risk Judgments of 30 Gambles (Weber, 1985)

Subject	$R-D^a$ (26 *df*)	$V-E^b$ (28 *df*)	CER[c] (24 *df*)	Average Reliability
1	.30	.45	.56	.49
2	.47	.65	.77	.77
3	.24	.34	.45	.42
4	.50	.63	.71	.69
5	.39	.60	.82	.69
6	.55	.72	.80	.82
7	.46	.66	.75	.71
8	.40	.57	.64	.41
9	.39	.55	.65	.44
10	.43	.61	.70	.64

[a] $R-D$ is the risk dimensions of Payne (1973).

[b] $V-E$ is the measure $aV+bE$, where E is the expected value and V is variance of the gamble, of Pollatsek and Tversky (1970).

[c] Although the general CER measure involves seven parameters, the gambles all had $\Pr(\mathbf{X}=0)=0$, and so $A(0)$ was not relevant.

TABLE 2

Values of the Parameters in the CER Model of Risk

Subject	$A(+)$	$A(-)$	$B(+)$	$B(-)$	$k(+)$	$k(-)$
1	−13.0	128.7	−255.5	412.1	.10	.20
2	−261.5	73.8	−163.0	367.6	.20	.20
3	−197.8	49.4	−32.4	182.6	.30	.30
4	−44.7	219.3	−317.5	446.8	.25	.30
5	−408.2	44.9	−175.0	341.2	.10	.15
6	−4.9	259.9	−413.8	472.9	.20	.30
7	−30.8	250.2	−406.7	478.6	.25	.25
8	−258.6	42.5	−289.8	387.7	.25	.30
9	−221.7	88.4	−134.5	446.9	.20	.25
10	−304.2	19.4	−182.9	392.8	.20	.50

results, along with the average reliability of the data between replications, are shown in Table 1. Without exception, the R^2 values of the regression are ordered

$$R\text{–}D < V\text{–}E < \text{CER} \cong \text{Reliability}.$$

It should be noted that different numbers of free parameters are involved, being 4, 2, and 6, respectively; the latter being one less than in the theory because there were no zero outcomes. Given the reliability of these data, much improvement on CER is not to be anticipated. Additional experimentation is required to test it further.

Table 2 shows the values of the six parameters of the CER model for the risk judgments. The values of $k(+)$ and $k(-)$ are both rather small and nearly the same. In fact, requiring $k(+)=k(-)$ reduces R^2 by at most 0.01 for these data. Before either of these "facts" are taken seriously, experiments must be run in which the outcomes span a much larger monetary range than the −\$8.40 to +\$8.40 used in this experiment.

Some on hearing about this model are extremely skeptical about the positive part of the gamble having any bearing on its riskiness. They feel that all positive gambles are equally risky—or really, riskless. We examine Table 2 to see the degree to which subjects agree with this belief. The only clear pattern we have noticed is that subjects 1, 4, 6, and 7 all have values of $A(-)$ substantially larger than the $|A(+)|$, whereas for the other subjects $|A(+)|$ is both larger than $A(-)$ and larger than $|A(+)|$ of these four subjects. However, the weights $B(+)$ and $B(-)$ are both substantial with no clear differential pattern.

Slovic (1967), in a comparison of perceived risk and preference, found that the two judgments were determined by different aspects of the gamble. We may check this for these data since Weber (1986) also had the subjects judge preference for the same set of gambles. Table 3 shows the average correlations between their risk and preference judgments. Observe that all correlations are negative, indicating that

TABLE 3

Values of the Risk–Preference Correlations and of R^2 for the CER Fit to the Preference Judgments for the Same Subjects and Same Gambles as in Table 1 (Weber, 1986)

		CER Model	
Subject	Risk–preference correlation	Preference R^2	Risk R^2 – preference R^2
1	−.65	.41	.15
2	−.75	.61	.16
3	−.64	.36	.09
4	−.62	.54	.17
5	−.74	.67	.15
6	−.87	.71	.09
7	−.71	.59	.16
8	−.62	.48	.16
9	−.71	.57	.08
10	−.69	.58	.12

subjects generally preferred the lower risk gambles. In some cases, the squared correlation approaches the magnitude of the reliability coefficients for risk judgments, but for most subjects it is substantially lower. To test further the distinctiveness of risk judgments from preference ones, the CER measure was also fit to the preference ratings. These results are also shown in Table 3. The overall fit of CER is poorer for preference than it was for risk, with the differences in R^2 varying from .08 to .17.

The six parameters of the CER model for preferences are shown in Table 4. Two things are notable. First the subjects all seem moderately similar, with both the

TABLE 4

Values of the Parameters of the CER Model of Preferences

Subject	$A(+)$	$A(-)$	$B(+)$	$B(-)$	$k(+)$	$k(-)$
1	233.1	−82.8	64.0	−49.7	.90	.15
2	285.7	−136.9	60.1	−144.8	1.10	.20
3	147.9	−55.1	24.9	−43.3	1.40	.20
4	309.3	−157.8	53.4	−95.6	1.25	.15
5	352.0	−149.4	42.5	−151.5	.80	.25
6	316.2	−167.3	75.6	−127.3	.95	.30
7	352.7	−138.7	102.0	−149.2	1.50	.30
8	264.5	−131.7	44.2	−145.2	1.10	.20
9	225.4	−141.7	73.8	−214.5	1.10	.20
10	299.1	−104.6	59.9	−183.4	.90	.35

positive and negative parts of the gamble having comparable weights. The most striking difference from the risk model is that the ratio of $k(+)$ to $k(-)$ is substantially larger than for risk, which agrees with Slovic's results. The expected value of the positive part, which corresponds to $k(+)=1$, but not of the negative, seems to play an important role in these preference judgments.

The risk function axiomatized here can also be evaluated against previous empirical evidence regarding subjective risk judgments. Coombs and Bowen (1971b) reported that when two gambles are convolved, the risk of the resulting gamble is not an additive function of the risk of the two component gambles. This fact was an additional strike against Pollatsek and Tversky's (1970) risk function as well as eliminating Coombs and Huang's (1970a) polynomial model of perceived risk because both models predicted additivity. It is easy to see that the present risk function does not predict such additivity.

Another instance where CER seems to provide a superior prediction of empirical phenomena is the effect of a change in expected value on risk. One of Coombs' (1972) assumptions about risk was that relative order remains unaffected by changes in expected value. Payne, Laughhunn, and Crum (1980), on the other hand, claim to have brought about a reversal in relative risk by increasing the expected value of two gambles by the same amount. It is not difficult to construct examples where, with the right choice of parameters, CER predicts such reversals.

It should be noted that, for the parameter values estimated by Weber (1985), the present risk function can take on negative values when the positive outcome contributions outweigh the negative outcome contributions. This is in direct contrast to Axiom B4 of Fishburn (1982) which restricts risk functions to positive values and postulates that gambles without losses have zero risk. This axiom, which is part of all risk measures axiomatized by Fishburn (1982, 1984) in his comprehensive catalogue, in fact rules out all models that are additively separable in gains and losses. Instead, Fishburn considered multiplicatively separable representations which allow for an effect of gain on risk without changing his assumption that risk is zero when no loss is possible. This assumption, however, seems questionable. The role of aspiration level in preference has received considerable attention in recent years (e.g., Kahnemann & Tversky, 1979). If risk were related, for example, to the probability of not meeting one's aspiration level, then one should have to discriminate, with regard to risk, gambles with only positive payoffs whenever one's aspiration level exceeded the lower bound of the range of payoffs.

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