Classification of real measurement representations by scale type

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The scale type (M, N) of an ordered relational structure is defined in terms of two properties, called M-point homogeneity and N-point uniqueness, of the automorphism group of the structure. For real structures on an open interval, scale types (1, 1) and (2, 2) correspond to ratio and interval representations, respectively. Accepting certain key properties, such as transitivity of the ordering relation and, in the case of a binary operation, monotonicity, and assuming that a real representation exists, then for each scale type whose real transformation group is known, the possible forms for the representation can be derived. For structures with a monotonic, binary operation, this is done completely for the ratio and interval cases, and incompletely in what is shown to be the only other interesting case exhibiting substantial symmetry, (1, 2). These results are then used to gain a better understanding of the psychological theory of utility of gambles and the possible generalisations of multiplicative conjoint structures, which are of importance in dimensional analysis.

1 Introduction

Axiomatic theories of measurement lead to theorems of three general types. The first, known as a *representation theorem*, formulates, in terms of the primitives of a qualitative (and, potentially, empirical) relational structure, axioms that are sufficient (and occasionally necessary as well) for the existence of a homomorphism into a numerical relational structure.

The second, known as the corresponding *uniqueness theorem*, formulates the entire class of possible homomorphisms into (and sometimes on to) the same numerical structure. This is usually stated in terms of the group of real transformations that relate the several homomorphisms. In many cases it can be formulated in terms of the automorphisms of the structure itself.

The third, known as a *meaningfulness theorem*, formulates the class of statements that can be meaningfully asserted, either using the primitives of the structure itself or using the terms of its numerical representations. The concept of meaningfulness, which is to some degree controversial, has in many important cases to do with the invariance of statements under alternative numerical representations. In some cases of interest, this is equivalent to invariance of qualitative relations under the automorphisms of the qualitative structure. For physical scientists, the best known example of such a result is Buckingham's π-theorem of dimensional analysis (see Buckingham, 1914, and Chapter 10 of Krantz *et al*, 1971).

The paper also describes a fourth type of theorem which has only recently begun to be developed. The main goal of this fourth type is to accept a few of the most important axioms of a structure, and then to classify all possible numerical structures exhibiting those properties. This provides a listing of the possible representations for such structures. The principle used to classify structures is based upon a classification of the automorphism groups of these structures The key to arriving at these results is a well articulated concept of what is meant by the scale type of a structure. It is believed that results of this fourth type, of which a number are stated below, will aid significantly in *completing the traditional work of axiomatic measurement* theory. They make crystal clear which axiom systems need to be worked on; however, they do not seem to be very much help in actually finding the axioms needed to achieve the various representations.

As we shall see, the current results pertain only to structures that exhibit a considerable degree of symmetry, as is true of the familiar ratio and interval cases. There is not, at present, any useful classification of non-symmetric structures such as finite ones or ones with an intrinsic zero in the sense of an element invariant under all automorphisms of the qualitative relational structure. It is doubtful whether anything of use along these lines will arise for the finite ones or for any others that exhibit little symmetry, but there is more optimism about generalising concepts to cover ones that have substantial substructures that do exhibit symmetry, as is true of the conjoint structures discussed in section 6.

The paper is organised as follows: section 2 presents the general definition of scale type; section 3 describes some very general results about real structures, defined on infinite real intervals, when their scale type is known; and section 4 applies these results to the case of (non-associative) concatenation operations. Section 5 then applies the results of section 4 to one version of the subjective expected utility problem; and finally, section 6 generalises the concept of a scale type so as to apply the results of section 4 to conjoint structures (ordering of the Cartesian product of two sets) that are well endowed with factorisable automorphisms.

2 Intrinsic definition of scale type

According to traditional usage, scale type refers to the uniqueness that is established in the uniqueness theorem. Usually this is formulated in terms of the group of transformations that map one representation into another in the same numerical structure. The three most common groups are the similarity, affine, and monotonic increasing functions, corresponding to what are called ratio, interval, and ordinal scales. This classification is not complete, however, since there are structures whose uniquenesses are different from these three groups. This has been noted in various systems, including the unfolding structures of Coombs (1964) and semiorders of Luce (1956), but the best understood instance arose in the work of Narens and Luce (1976) on positive concatenation structures (see Def 4.1 and the following discussion). They were able to show that, when two homomorphisms of such a structure agree at a point, they are identical, but they did not fully classify the groups involved. Later, Cohen and Narens (1979) showed that the automorphism group of such a structure is isomorphic to a subgroup of the additive real numbers. They also noted that, of these subgroups, only certain ones have the following property called *homogeneity*: viz, given any two points in the structure, there is an automorphism that takes one into the other. (In some literatures, this concept is called transitivity (Glass, 1981).) In Narens (1981a, b), the author generalised these two concepts, the one having to do with the richness of the automorphism group and the other to do with the uniqueness of representations. We present these concepts here.

We do not give formal definitions of relational structure, of homomorphism of one structure into another, of isomorphism between two structures, or of endomorphism and automorphism within a structure, as they are the standard concepts. When one of the relations in a relational structure is a weak order, we speak of the structure as ordered, and the notation $\chi = \langle X, \succsim, S_1, S_2, \ldots \rangle$ is used for ordered qualitative structures and $R = \langle R, \geqslant, R_1, R_2, \ldots \rangle$ for numerical ones with the natural ordering of the real numbers. In what follows, R is usually the positive real numbers, Re^+, and occasionally the reals, Re.

Definition 2.1

Suppose χ is a totally ordered relational structure and Π is its group of automorphisms. Let M and N be non-negative integers. χ satisfies *M-point homogeneity* iff for all $x_1, \ldots, x_M$, $y_1, \ldots, y_M$ such that $x_i \succ x_{i+1}$ and $y_i \succ y_{i+1}$, $i = 1, \ldots, M-1$, there is an automorphism α in Π such that $\alpha(x_i) = y_i$, $i = 1, \ldots, M$. χ satisfies *N-point uniqueness* iff whenever two automorphisms α and β agree at N distinct points, then $\alpha = \beta$. The structure is said to be of *scale type (M, N)* iff M is the largest value of homogeneity and N is the smallest value of uniqueness for the structure. If the structure is M-point homogeneous for every positive integer M, then it is said to be of scale type (∞, ∞).

It is obvious that if χ is of scale type (M, N), then $M \leqslant N$. Furthermore, it is easy to show that if a structure has a ratio or interval or ordinal scale representation on to an open real interval, then it is of scale type (1, 1) or (2, 2) or (∞, ∞), respectively.

This observation leads to three question. First, is the converse of this statement true: does a structure on an open real interval that is of scale type (1, 1) necessarily have a ratio scale representation? Does one of type (2, 2) necessarily have an interval scale representation; And does one of type (∞, ∞) necessarily have an ordinal scale representation? The answer to the last is 'No', and the other two are discussed below.

Second, are there structures of scale type (M, N), $M < N$, and if so, what can be said about them? And, third, does the general nature of the structure impose any limits on the values of M and N?

In a sense, the rest of the paper provides some partial answers to these questions. Aside from the answer about the converse, none of these results is as complete as we would like.

3 General results for numerical structures

This section and the remainder of the paper will be largely concerned with structures that are isomorphic to a real structure defined on an open real interval. Rather than state this formally, the paper simply talks about real structures.

The first result, due to F. Roberts and proved in Luce and Narens (1983a) establishes one clear link between the structure and the degree of homogeneity.

Theorem 3.1

Suppose R is a real ordered structure defined on the real numbers, Re, ordered by $\geqslant$, and that it is M-point homogeneous. If the order of each defining relation is $\leqslant M$, then it is of scale type (∞, ∞) and it is an ordinal scale.

So, for example, a structure $\langle Re, \geqslant, o \rangle$, where o is a binary operation and therefore is a relation of order 3, can be at most 2-point homogeneous if it is not to degenerate into an ordinal scalable structure. In section 4 we will see a further limitation on the scale type of such structures with a binary operation.

The next theorem combines major results due to Narens (1981a, b).

Theorem 3.2

Suppose R is a real ordered relational structure with domain an open interval of Re. Then,

1 R is of scale type (1, 1) iff it has a real ratio scale representation.
2 R is of scale type (2, 2) iff it has a real interval scale representation.
3 There are no real structures of type (M, M) for $M > 2$.

As will be seen, this result is very useful in characterising the possible classes of structures of scale types (1, 1) and (2, 2). What are missing, and what have been found elusive, are results about scale type (M, N) when $M < N$. Such groups exist. For example the real mappings $x \rightarrow \sigma x^{\rho}$, where $\sigma > 0$ and ρ is generated from fixed $k > 0$ and all integers n by $\rho = k^n$, is a group of type (1, 2). This is referred to as the *discrete affine group*. Although this is a subgroup of the affine group and it includes the similarity group, it is not known, in general, if a group of type (M, N) is a subgroup of type (N, N) or if it includes a type (M, M) subgroup. The following specific result of Luce and Narens (1983a) is, however, of some use in studying specific classes of structures.

Theorem 3.3

Suppose R is a real, ordered relational structure on Re^+ and its automorphism group Π is of type (1, 2) and it contians a (1, 1) subgroup $\aleph$. Then Π is isomorphic to a subgroup of the affine group and $\aleph$ is isomorphic to all of the translations of the affine group.

4 Real representations of homogeneous concatenation structures

The concept of a concatenation structure is a natural generalisation of the model used in physics for those quantitites, such as mass, length, and time, that have an internal operation which is represented by addition. In such structures, which are called *extensive*, the operation of combination, which is called a *concatenation operation*, preserves the attribute being measured. Of the usual axioms for extensive measurement, which theoretically capture the essential properties of physical concatenation, four are abandoned and three maintained. Those abandoned are: associativity, positivity, solvability, and Archimedean. Those maintained are: local definability and monotonicity of the partial operation, and weak ordering. In addition, it is assumed that the ordering is non-trivial and that the partial operation is, in fact, an operation. To stress the latter assumption, such operations are referred to as *closed*. There is no loss in assuming the ordering to be a total one, which amounts to working with the equivalence classes of the weak ordering.

Definition 4.1

Let X be a non-empty set, $\succsim$ a binary relation on X, and o a closed-binary operation on X. Then $\chi = \langle X, \succsim, o\rangle$ is a *closed concatenation structure* iff for all x, y, u, v in X,

1 $\succsim$ is a total ordering;
2 for some x, y in $X, x \succ y$;
3 monotonicity: $x \succsim y$ iff $xou \succsim you$ iff $vox \succsim voy$.

The structure is said to be *idempotent, weakly positive*, or *weakly negative* iff for all x in X, $xox =, \succ$, or $\prec x$, respectively. It is *dense* iff for $xoa \succ you$, there exists v such that $xou \succ yov \succ you$. It is *solvable* if given any three of x, y, u, v in X, the fourth exists such that $xoy = uov$. A solvable structure is *Archimedean* iff any sequence $\{x_i\}$ for which $x_i ou = x_{i-1} ov$ holds for some u, v in X, $u \neq v$, and what is bounded has finitely many terms.

Theorem 4.1

Suppose χ is a closed concatenation structure that is at least 1-point homogeneous. Then it is either idempotent, weakly positive, or weakly negative. If for some positive integer N it is also N-point unique, then either $N = 1$ or χ is idempotent and $N = 2$.

Thus, for closed concatenation structures with 1-point homogeneity and finite uniqueness, there are only three possible scale types: (1, 1), (1, 2), and (2, 2). Concatenation structures that are not unique for any finite N appear to be unlikely, although it is not clear how to rule them out without further constraints. The following is sufficent to do so.

Theorem 4.2;

If $R = \langle Re^+, \geqslant, o\rangle$ is a closed concatenation structure such that o is on to Re^+ and continuous, then the structure is 2-point unique.

If we now consider those concatenation structures that have a representation on to Re^+ (*see* Theorem 4.4 for a set of sufficient conditions), then, by Theorem 3.2, we know that for the (1, 1) and (2, 2) cases there exist representations that are, respectively, ratio and interval scales, ie, the automorphism groups are the similarity and affine groups. This fact permits us to use the following device to characterise these representations: the numerical concatenation operation o can be thought of as a real function F on two variables, where $F(x, y) = z$ iff $z = xoy$. Then F must satisfy the following functional equation: for all automorphisms α and x, y in X,

$$\alpha[F(x, y)] = F[\alpha(x), \alpha(y)] \qquad \ldots(1)$$

Since we know the groups corresponding to (1, 1) and (2, 2), we need only solve this functional equation for these groups. For the (1, 2) case we have not been able to characterise all of the possible groups, but the discrete affine group is one example. The next result summarises the solutions for these groups.

Definition 4.2

A real closed concatenation structure $\langle Re^+, \geqslant, o\rangle$ is said to be a *unit structure* iff there exists f: $Re^+ \rightarrow Re^+$ such that

(i) f is strictly increasing,
(ii) f/ι, where ι is the identity function, is strictly decreasing,
(iii) for all x, y in Re^+,

$$xoy = yf(x/y) \qquad \ldots(2)$$

Theorem 4.3

Suppose $R = \langle Re^+, \geqslant, o\rangle$ is a real closed concatenation structure. Then,

1 R is of scale type (1, 1) iff R is isomorphic to a unit structure with the property that if for all $x > 0$ and some $\rho > 0$,

$$f(x^\rho) = f(x)^\rho, \qquad \ldots(3)$$

then $\rho = 1$.

2 R is of scale type (1, 2) with a (1, 1) subgroup iff R is isomorphic to a unit structure with the property that there is a unique $k > 0$ such that Eqn (3) holds iff $\rho = k^n$, where n ranges over the integers.

3 R is of scale type (2, 2) iff R is isomorphic to a unit structure with the property that Eqn (3) holds for all $\rho > 0$.

Statement (3) may be rewritten in two equivalent ways. First, there exist constants a, b, where $0 < a$, $b < 1$, such that

$$f(x) = \begin{cases} x^a, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ x^b, & \text{if } x < 1. \end{cases} \qquad \ldots(4)$$

And second, by taking logarithms, the representation can be placed on Re, which is customary in economics and psychology, in which case o is represented as: for u, v in Re,

$$uov = \begin{cases} au + (1-a)v, & \text{if } u > v, \\ u, & \text{if } u = v, \\ bu + (1-b)v, & \text{if } u < v. \end{cases} \qquad \ldots(5)$$

There are examples of all three types of structures; however, the (1, 2) one is too lengthy to present here.

When the remaining (1, 2) groups, if any, are discovered, the same technique can be used to achieve corresponding theorems about all possible numerical representations.

Given that we understand the possible representations on to a real interval, the next question is the traditional one: under what conditions does a qualitative structure have such a representation? Representations of concatenation structures that are positive in the sense that for all x, y, $xoy \succ x, y$ were axiomatised in Narens and Luce (1976) and studied further in Cohen and Narens (1979), and both $M = 0$ and $M = 1$ cases were found. The latter was shown to be equivalent to the structural condition that the n-copy operators defined by $\theta_n(x) = x$ for $n = 1$ and $= \theta_{n-1}(x)ox$ for $n > 1$ are automorphisms of the structure. In Narens and Luce (1976) idempotent structures were defined that are very closely related to positive concatenation ones, but in Luce and Narens (1983a) these were shown to be extremely special (see Luce and Narens (1983b) for a summary of the results). Axiomatisations of bisymmetric structures, such as that in section 6.9 of Krantz *et al* (1971), lead to representations as in part 3 of Theorem 4.3 but with the added restriction that $a = b$.

The following representation theorem, although quite general, is distinct from the one for positive concatenation structures in that it has a stronger solvability condition and a different Archimedean one. In addition, it would be desirable to have specific axiomatisations for the three unit structures of Theorem 4.3.

Theorem 4.4

If χ is a closed, dense, solvable, Archimedean concatenation structure, then χ is isomorphic to a real structure.

The proofs of the results in this section are in Luce and Narens (1983a).

5 Application to choice under uncertainty

Following the classic work of von Neumann and Morgenstern (1947, 1953), a sizeable literature has developed concerning choices among gambles, in which the outcome depends upon chance events whose probabilities may or may not be known to the decision maker. The major representation theorem has been that of (subjective) expected utility. According to that representation, (i) preferences are ordered according to a utility function having the property that the utility of each gamble is given by the expected value of the utility of its components, where the expectation is taken relative to a (subjective) probability measure over the events; and (ii) the utility representation forms an interval scale. A substantial critical and empirical literature has cast considerable doubt upon the descriptive and perhaps even the normative accuracy of this model. To a great extent, the difficulties have been attributed to a particular type of monotonicity principle, which in this context is often called *substitutability* or the *extended sure-thing principle.* A useful survey is Fishburn (1982).

It is desirable to question this conclusion by examining the class of representations that are compatible with such monotonicity principles and, of course, transitivity of the preference ordering. To this end, we consider a structure having the following primitives. Let X be a non-empty set, $\succsim$ a binary relation on X, ξ a non-empty set of subsets of a non-empty set, and for each A in ξ let o_A be a closed binary operation. We interpret xo_Ay to be the gamble in which x is the outcome if event A occurs and y if it fails to occur. In a gamble whose components are also gambles, we interpret each gamble as an independent 'experiment' in much the same sense as one means independence in a random sample.

Theorem 5.1

Suppose $\langle X, \succsim, \xi, \{o_A\} \rangle_A$ in ξ satisfies the following properties: for each A in ξ,

1 $\langle X, \succsim, o_A \rangle$ is a closed, idempotent concatenation structure;
2 these concatenation structures have a common interval scale representation U on to the real numbers.

Then, there exist functions P and Q from ξ into (0, 1) such that for all x, y in X and A in ξ,

$$U(xo_Ay) = \begin{cases} U(x)P(A) + U(y)[1-P(A)], & \text{if } U(x) \geq U(y) \\ U(x)Q(A) + U(y)[1-Q(A)], & \text{if } U(x) < U(y) \end{cases} \quad \ldots(6)$$

This obviously derives from part 3 of Theorem 4.3 (see Eqn (5)).

Corollary 1

Under the conditions of the theorem, for all x, y and A, B in ξ,

$$(xo_Ay)o_By = (xo_By)o_Ay. \quad \ldots(7)$$

Corollary 2

Under the conditions of the theorem and assuming that ξ is closed under complementation, for all A in ξ and all x, y in X,

$$xo_Ay = yo_{\sim A}x \quad \ldots(8)$$

iff

$$P(A) + Q(\sim A) = 1. \quad \ldots(9)$$

Corollary 3

Under the conditons of the theorem, bisymmetry, ie, for all x, y, u, v in X and A in ξ,

$$(xo_Ay)o_A(uo_Av) = (xo_Au)o_A(yo_Av), \quad \ldots(10)$$

iff

$$P = Q \quad \ldots(11)$$

The main significance of these results is that there exist representations that have the following three properties: they are consistent with both transitivity and monotonicity principles; to a considerable degree they exhibit the usual subjective utility property; and they are not inconsistent with the classic counter-examples of Allais (1951) and Ellsberg (1961). The main difference is that, in general, P is not the same as Q and these functions do not necessarily satisfy all of the usual properties of probability. It is really only under the conditions of Corollary 3 that the model exhibits the properties that fail to hold in these counter-examples. Furthermore, the model can accommodate all of the phenomena discussed by Kahneman and Tversky (1979) in motivating their prospect theory, and that theory (for at least the gambles it discusses) is a special case of Eqn (6).

If we assume that a representation into a unit structure (Def 4.2), then the conditions formulated in Eqns (7) and (8) imply that f satisfies the following functional equation: for all $r > 0$,

$$f[rf(1/r)] = f(r)f[1/f(r)] \quad \ldots (12)$$

This is satisfied by Eqn (4), but we do not know if there are other solutions.

If we examine various relations, of which Eqn (7) is a simple example, in which logically the two sides are equivalent, it turns out that idempotence, Eqn (7), and Eqn (8), are the only ones that involve at most two outcomes, at most two events, and at most two successive gambles. We have shown – in some cases, the argument is lengthy – that if one either increases the number of outcomes, events, or stages beyond two, the bisymmetric case $P = Q$ is forced. So, in a very real sense, imposing just Eqns (7) and (8) constitutes the unique model of 'bounded rationality' within the framework of binary concatenations of gambles, and Eqn (6) is the only interval scale generalisation of the traditional theory.

The results in this section are proved in Luce and Narens (1983a).

6 Real representations of homogeneous conjoint structures

A conjoint structure is simply an ordering of a Cartesian product of two (or more) sets, in which case the objects under consideration have factors each of which affects the attribute being measured. Although examples of this abound in physics – as is reflected in the fact that some units of measurement are products of powers of other units – such structures were only axiomatised in the late 1950s (for a general survey as of 1970 see chapters 6 and 10 of Krantz *et al* (1971)). Again, we shall search for possible numerical conjoint representations that satisfy what appear to be the most important properties.

Definition 6.1

Suppose A and P are sets, a_0 in A, p_0 in P, and $\succsim$ a binary relation on $A \times P$. $\tau = \langle A \times P, \succsim\rangle$ is a conjoint structure that is A-solvable relative to a_0p_0 iff for all a, b in A and p, q in P,

1 $\succsim$ is a weak ordering (transitive and connected);
2 $\succsim$ satisfies *independence (monotonicity)*:
   $ap \succsim bp$ iff $aq \succsim bq$ and $ap \succsim aq$ iff $bp \succsim bq$;
3 *solvability*: there exists $\xi(a,p)$ in A and $\pi(a)$ in P such that:
   $\xi(a,p)p_0 \sim ap$ and $a_0\pi(a) \sim ap_0$;
4 *density*: if $ap_0 \succ bp_0$, then there exist p, q in P such that:
   $ap_0 \succ bp \succ bp_0$ and $ap_0 \succ aq \succ bp_0$.

Obviously, a comparable definition of being P-solvable can be given. The key role of the solvability assumption is to define an operation $*_A$ on A: for all a, b in A, let

$$a *_A b = \xi[a, \pi(b)] \quad \ldots (13)$$

This was first introduced in Holman (1871) to study additive conjoint structures, but it is also useful in the non-additive case. Without being completely formal about it, the following remarks summarise the relations between conjoint and concatenation structures for the case where a_0p_0 is a minimal element (see Theorems 2, 3, and 5 of Luce and Cohen (1983). The induced operation (Eqn (13)) of an A-solvable conjoint structure in which a_0p_0 is minimal, together with the induced ordering arising from independence, from a positive concatenation structure (PCS) on the set A. A PCS is, in essence, the type of structure one obtains by abandoning associativity in the usual extensive model. This is the best possible result since for each PCS there is a conjoint structure for which the induced PCS is isomorphic to the given one. It can also be shown that a conjoint structure satisfies the Thomsen condition (ie, for all a, b, f in A and p, q, x in P, if $ax \sim fq$ and $fp \sim bx$, then $ap \sim bq$) iff the induced operation $*_A$ is associative. These results generalise nicely when a_0p_0 is not minimal, but to state them requires an additional concept, called a *total concatenation structure*, whose parts above and below a_0p_0 are related PCSs (see Luce and Cohen (1983)).

Our primary concern here, however, is to use the automorphisms of conjoint structures in order to arrive at the possible numerical representations. This is slightly more complex than might first seem because, as defined, the conjoint structure is not really in the form of a relational structure. If we treat $C = A \times P$ as a set, then we lose the fact that it is a Cartesian product. Rather, let $X = A \cup P$, where $A \cap P = 0$, and for each a, b, p, q in X define S to be the relation of order 4 on X such that

$$S(a, p, b, q) \quad \text{iff} \quad ap \succsim bq. \quad \ldots (14)$$

Then $\chi = \langle X, S, a_0, p_0\rangle$ is a relational structure and the concepts of weak order, independence, solvability, and density are all definable in terms of its primitives. Let α be an automorphism of χ, then for each a, b in A and p, q in P,

$$\begin{aligned} ap \succsim bq \quad &\text{iff} \quad S(a, p, b, q) \\ &\text{iff} \quad S[\alpha(a), \alpha(p), \alpha(b), \alpha(q)] \\ &\text{iff} \quad \alpha(a)\alpha(p) \succsim \alpha(b)\alpha(q). \end{aligned} \quad \ldots (15)$$

Thus, each automorphism of χ is an automorphism (order preserving mapping) of τ with the special property of being *factorisable* into distinct mappings $\langle \alpha_A, \alpha_P\rangle$ on the two components. We shall confine our attention to such factorisable automorphisms.

It is evident from Eqn (14) that it is only possible for χ to exhibit 0-point homogeneity because there is no mapping from A to P or from P to A. Thus, we need a more restricted version of the concept, one that treats the components separately. Such a definition can be found either in Luce and Narens (1983b) or Luce and Cohen (1983), but it and the exact theorems are too lengthy to summarise there in detail. Suffice it to say that two major cases arise. In the one, there is an element a_0p_0 that is an intrinsic zero in the sense that it maps into itself under all factorisable automorphisms. Assuming a real conjoint structure F on $Re \times Re$ that is continuous and strictly increasing in each variable, solvable relative to each pair in the structure, and has an intrinsic zero, then conditions called 1-*point component homogeneity* and 1-*point component uniqueness* are sufficient to show the existence of functions f_+ from Re on to $[1, \infty)$ and f_- from Re on to $(-\infty, -1]$ such that f_+ and $-f_-$ satisfy the conditions (i) and (ii) of Definition 4.2 and for all x, y in Re,

$$F(x, y) = \begin{cases} yf_+(x/y), & y > 0, \\ x, & y = 0, \\ |y|f_-(x/|y|), & y < 0. \end{cases} \quad \ldots (16)$$

For the case of no intrinsic zero, various combinations of homogeneity, uniqueness, and smoothness (formulated as differentiability assumptions about F) lead to the conclusion that F must be additive in the sense that there are functions f, f_1, and f_2 such that

$$F(x, y) = f[f_1(x) + f_2(y)]. \qquad \ldots(17)$$

The exact statement of these results can be found in Theorems 9.4 and 9.5 of Luce and Narens (1983b) or as Theorems 13, 17–19 of Luce and Cohen (1983); the proofs are in the second paper.

One of the major conclusions of this work is that there is really not much hope for a generalisation of dimensional analysis of non-additive conjoint structures. The reason is that dimensional analysis depends crucially on the existence of many factorisable automorphisms – that is how the condition of dimensional invariance is formulated – and, except when there is an intrinsic zero, their existence together with some smoothness forces one to the classical, additive case. Of course, there remains open the possibility that something interesting will arise when we understand concatenation structures of scale type (1, 2), but this is really the only possibility that offers a possible generalisation.

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