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FACTORIZABLE AUTOMORPHISMS IN SOLVABLE CONJOINT STRUCTURES I

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1. Introduction

For much measurement, both in the behavioral and physical sciences, conjoint structures play a significant role. A conjoint structure is a weak ordering $\succsim$ of a cartesian product $A \times P$ (it can be generalized to any product of finitely many factors) that satisfies the property of independence: for all a, b in A, p, q in P, $ap \succsim aq$ if and only if $bp \succsim bq$ and $ap \succsim bp$ if and only if $aq \succsim bq$. This immediately induces weak orders on each factor, which we denote $\succsim_A$ and $\succsim_P$. As usual, $\precsim$ denotes the converse of $\succsim$, and $\succ$ denotes $\succsim$ and not $\precsim$, and $\sim$ denotes $\succsim$ and $\precsim$.

In practice, the most studied conjoint structures have been the additive ones, those for which there are mappings ϕ_A from A into the positive reals (Re^+) and ϕ_P from P into Re^+ such that for all a, b in A and p, q in P,

$$ap \succsim bp \quad \text{iff} \quad \phi_A(a)\phi_P(p) \geq \phi_A(b)\phi_P(q).$$

Such a representation is referred to as additive because it becomes that if we take logarithms of it; we write it in multiplicative form because that is what is done in physics. The additive case is quite well understood; see Chs. 6 and 9 of Krantz, Luce, Suppes, and Tversky [5] and Ch. 5 of Fishburn [4].

It is natural to inquire whether there are important non-additive structures. Do some merit consideration as possible empirical measurement structures? Almost surely some constraints must be imposed on the nature of the non-additivity. The best we might hope for without any constraint is some sort of classification, but nothing along these lines has been reported. The literature so far includes two types of constraints. One limits the nature of the representation, e.g., to simple polynomials when there are three or more factors (Ch. 7 of [5]). The other concerns conjoint structures that have an operation on one of the factors and it is distributive

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over the conjoint structure. In essence, it has been shown that if the operation together with the induced ordering on that factor have a ratio scale representation and if the operation is qualitatively distributive, then the conjoint structure must be additive (Narens and Luce [9]; Narens [7]). This result is improved in Section 5.

Here we examine a third constraint, one suggested by the theory of dimensional analysis. In that theory, possible physical laws are numerical expressions that remain invariant under transformations called similarities. Luce [6], in studying the qualitative equivalence to such dimensionally invariant laws, showed that a similarity corresponds to an automorphism of the qualitative structure, and these automorphisms have the very nice property that they factor into automorphisms on the components of conjoint structures. In general, of course, an automorphism of $\langle A \times P, \succsim \rangle$, i.e., a $1:1$ mapping α from $A \times P$ onto $A \times P$ that preserves the ordering $\succsim$, need not be factorizable into a $1:1$ function θ from A onto A and a $1:1$ function η from P onto P such that

$$\alpha(a, p) = (\theta(a), \eta(p)).$$

But if there are structures with such factorizable automorphisms, then one could generalize the domain of dimensional analysis without altering either the concept of a dimensionally invariant law or the procedures of analysis. So our problem is to understand such conjoint structures.

The paper is structured as follows. Section 2 provides the full definition of a solvable conjoint structure, defines a natural operation on one of its components, and characterizes that operation. Section 3 establishes that associativity of the operation is equivalent to additivity of the conjoint structure and formulates an interconnection between pairs of induced operations. Little of this is new. Section 4 focuses on the concept of a factorizable automorphism, establishes how the two components of such an automorphism are related to one another via two induced operations, shows that all of the induced structures are isomorphic, and characterizes additivity in terms of properties of factorizable automorphisms. Section 5 deals with conjoint structures having an operation on one component. It relates a concept of distributivity to automorphisms of the induced structure and to factorizable automorphisms of the conjoint structure. Moreover, if the operation is representable on the positive reals, then distributivity forces the conjoint structures to be additive. Section 6 summarizes Narens' [7, 8] very general results about the existence of ratio scale (uniqueness up to similarity transformations) and interval scale (uniqueness of the logarithms of the scale up to affine transformations) representations of numerical relational structures when they have suitable automorphism groups. Section 7 introduces related concepts that are defined for factorizable automorphisms of conjoint structures that are richly endowed with factorizable automorphisms. The remaining sections analyze this classification more fully, although still incompletely, when the conjoint structure has a representation *onto* the real numbers and the induced structures on the components are also onto the reals. In essence, three general cases arise, two of which are studied in this paper and the remaining one in Cohen [1]. Here we assume that the factorizable automorphisms exhibit a good deal of independence on the components (formulated in Section 7). Under these assumptions, either the structure has an intrinsic zero, defined as an element that maps onto itself under all factorizable automorphisms, in which case a representation in terms of unit structures (Cohen and Narens [2]) exists, or the structure is rescalable into an additive representation. There is considerable tradeoff between smoothness assumptions about the representation and richness assumptions about the automorphisms. In [1], no assumptions are made about the independence of the factors of the automorphisms, but certain topological assumptions are added. In addition to the cases we obtain in this paper, several other quite simple structures arise.

2. Solvable conjoint structures and concatenation structures

Definition 1. Suppose $\succsim$ is a binary relation on $A \times P$, a_0 is in A, and p_0 is in P. $\langle A \times P, \succsim, a_0, p_0\rangle$ is said to be a *conjoint structure that is A-solvable relative to $a_0 p_0$* iff for all a, b in A and p, q in P:

1. *Weak ordering:* $\succsim$ is transitive and connected.
2. *Independence: $ap \succsim bp$* iff *$aq \succsim bq$*; *$ap \succsim aq$* iff *$bp \succsim bq$*.
3. *Solvability:* There exist *$\xi(a,p)$ in A and $\pi(a)$ in P such that*

$$\xi(a, p)p_0 \sim ap, \qquad a_0\pi(a) \sim ap_0.$$

4. *Density:* If $ap_0 \succ bp_0$, then there exists p in P such that $ap_0 \succ bp \succ bp_0$.
5. *Archimedean:* For any a define a *standard sequence na* inductively by: $1a = a$, $na = \xi[(n-1)a, \pi(a)]$. Then, for a, b in A such that $ap_0, bp_0 \succ a_0p_0$ and p in P such that $a_0p_0 \succsim a_0p$, the set $\{n \mid n$ an integer and $bp_0 \succsim na, p\}$ is finite, and for every a, b in A such that $a_0p_0 \succ ap_0$, bp_0 and p in P such that $a_0p \succsim a_0p_0$, the set $\{n \mid n$ an integer and $na, p \succsim bp_0\}$ is finite.

A symmetric definition of being P-solvable relative to a_0p_0 can be given; the only changes required are in Axioms 4 and 5. (Axiom 3 is already symmetric since $\xi(a_0, p)p_0 \sim a_0p$ and $ap \sim \xi(a, p)p_0 \sim a_0\pi\xi(a, p)$.)

The major purpose of the solvability conditions is to be able to induce concatenation operations either on one of the components or on the structure itself. This permits us to bring into play known results about such concatenation structures. We use $*$ as the generic symbol for such induced operations.

Definition 2. Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0, p_0\rangle$ is a conjoint structure that is A-solvable realtive to a_0p_0. Define operations $*_A$, $*_P$, and $*$ relative to a_0p_0 induced on A, P, and $A \times P$ respectively, as follows: for all $a, b \in A$, $p, q \in P$:

(i) $a *_A b = \xi[a, \pi(b)]$;
(ii) $p *_P q = \pi\xi[\xi(a_0, p), q]$;
(iii) $ap * bq = \xi(a, p), \pi\xi(b, q) \sim \xi(a, p) *_A \xi(b, q), p_0$.

The following notations are used for the several induced structures.

$$A^+ = \{a \mid a \in A \text{ and } a \succ_A a_0\},$$

$$P^+ = \{p \mid p \in P \text{ and } p \succ_P p_0\},$$

$$\mathcal{J}_A = \langle A, \succsim_A, *_A \rangle, \mathcal{J}_{A^+} = \textit{restriction of } \mathcal{J}_A \textit{ to } A^+,$$

$$\mathcal{J}_P = \langle P, \succsim_P, *_P \rangle, \mathcal{J}_{P^+} = \textit{restriction of } \mathcal{J}_P \textit{ to } P^+,$$

$$\mathcal{J}_{A\times P} = \langle A \times P, \succsim, * \rangle, \mathcal{J}_{A^+ \times P^+} = \textit{restriction of } \mathcal{J}_{A\times P} \textit{ to } A^+ \times P^+.$$

Similar definitions hold for A^-, P^-, etc.

Definition 3. Let A be a nonempty set, $\succsim$ a binary relation on A, and $\circ$ a partial binary operation[1] on A. The structure $\mathscr{A} = \langle A, \succsim, \circ \rangle$ is a *positive concatenatenation structure* if and only if for all a, b, c, d in A:

1. *Weak ordering:* $\succsim$ is connected and transitive;
2. *Non-triviality:* there exist p, q in A such that $p \succ q$;
3. *Local definability:* if $a \circ b$ is defined, $a \succsim c$, and $b \succsim d$, then $c \circ d$ is defined;
4. *Monotonicity:* (i) if $a \circ c$ and $b \circ c$ are defined, then $a \succsim b$ iff $a \circ c \succsim b \circ c$; (ii) if $c \circ a$ and $c \circ b$ are defined, then $a \succsim b$ iff $c \circ a \succsim c \circ b$;
5. Restricted solvability: if $a \succ b$, there exists p such that $a \succ b \circ p$;
6. *Positivity:* if $a \circ b$ is defined, then $a \circ b \succ a, b$;
7. *Archimedean:* For $a \in A, n \in I^+$ let $1a = a$ and $na = (n-1)a \circ a$ if this concatenation is defined; if not, na is undefined. The set $\{n \mid na \text{ is defined and } b \succsim na\}$ is finite.

For $a_0 \in A$, $\mathscr{A} = \langle A, \succsim, \circ, a_0 \rangle$ is a *total concatenation structure* iff:

1. $\langle A, \succsim \rangle$ is a weak order.
2. The restriction of $\mathscr{A}$ to A^+ is a positive concatenation structure.
3. The restriction of $\mathscr{A}$ to A^- but with the converse ordering $\precsim$ is a positive concatenation structure.
4. For all $a \in A, a \circ a_0 \sim a_0 \circ a \sim a$.
5. Monotonicity holds.
6. Compatibility: For $a \in A^+$ and $b \in A^-$, there exist $c, d \in A$ such that $c \circ b$ and $d \circ a$ exist, $c \circ b \succ a$, and $b \succ d \circ a$.

We next formulate what constitutes a numerical homomorphism of a positive or total concatenation structure.

Definition 4. Suppose $\mathscr{A} = \langle A, \succsim, \circ, a_0 \rangle$ is a total concatenation structure, $\odot$ a partial binary operation on Re, and ϕ a function from A into Re. Then ϕ is said to be a $\odot$*-representation of* $\mathscr{A}$ iff the following four conditions hold for all $a, b \in A$:

[1] That is, $\circ$ is defined only for some pairs $B \subseteq A \times A$. When $(a, b) \in B$, we say $a \circ b$ is defined. When $\circ$ is defined for any pair, we say $\circ$ is *closed*.

(i) $\langle \phi(A), \geq, \odot, 0 \rangle$ is a total concatenation structure.
(ii) $a \succsim b$ iff $\phi(a) \geq \phi(b)$.
(iii) If $a \circ b$ is defined, then $\phi(a) \odot \phi(b)$ is defined and

$$\phi(a \circ b) = \phi(a) \odot \mathrm{f}(b).$$

(iv) $\phi(a_0) = 0$.

For a positive concatenation structure, the representation is the same except that reference to a_0 and 0 is dropped and ϕ is into Re^+.

Theorem 1. *Every total concatenation structure has a $\odot$-representation for some numerical operation $\odot$. Moreover, if ϕ and ψ are both $\odot$-representations with $\phi(A) = \psi(A)$, and for some $a \in A$, $\phi(A) = \psi(a)$, then $\phi = \psi$.*

Proof. Narens and Luce [9] proved existence of the representation for positive concatenation structures, and this version of uniqueness is due to Cohen and Narens [2]. Let ϕ_+ be such a $\odot^+$-representation of the restriction of $\mathscr{A}$ to A^+ and ϕ_- be such a $\odot^-$-representation of the restriction of $\mathscr{A}$ to A^- with the converse order. Define ϕ and $\odot$ as follows:

$$\phi(a) = \begin{cases} \phi_+(a) & \text{if } a \in A^+, \\ 0 & \text{if } a \sim a_0, \\ -\phi_-(a) & \text{if } a \in A^-, \end{cases}$$

$$x \odot y = \phi[\phi^{-1}(x) \circ \phi^{-1}(y)].$$

Observe that

$$x \odot y = \begin{cases} x \odot^+ y & \text{if } x > 0, y > 0, \\ -x \odot^- y & \text{if } x < 0, y < 0. \end{cases}$$

It is easily verified that this is a representation, and the uniqueness follows from that proved for the positive concatenation structure. □

The next result is closely related to one given by Narens and Luce [9].

Theorem 2. *Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0, p_0 \rangle$, $a_0 \in A$, $p_0 \in P$, is a conjoint structure that is A-solvable relative to $a_0 p_0$.*

1. *If $a_0 p_0$ is minimal with respect to $\succsim$, then $\mathscr{J}_{A^+}$ and $\mathscr{J}_{A^+ \times P^+}$ are closed positive concatenation structures.*

2. *If $a_0 p_0$ is not minimal then $\mathscr{J}_A$ and $\mathscr{J}_{A \times P}$ are total concatenation structures.*

Corollary. *There exist functions of $\phi: A \to \mathrm{Re}$, $\psi: P \to \mathrm{Re}$, and a binary numerical operation $\odot$ such that*

(i) $\phi(a_0) = 0$, $\psi(p_0) = 0$.
(ii) $x \odot 0 = x$, $0 \odot y = y$.
(iii) $ap \succsim bp$ *iff* $\phi(a) \odot \psi(p) \geq \phi(b) \odot \psi(q)$.

Proof. 1. Suppose $a_0 p_0$ is minimal. It is obvious that $\succsim_A$ on A^+ is a weak order and that $*_A$ is closed. By the fact that there is some $ap \succ a_0 p_0$, then $\xi(ap) \succ_A a_0$. And by density there exists q such that $\xi(ap)p_0 \succ a_0 q \sim \xi(a_0 q)p_0 \sim a_0 p_0$, so $\langle A^+, \succsim_A, *_A \rangle$ is non-trivial. We verify the remaining axioms, where a, b, c are in A^+.

Monotonicity:

$$\begin{aligned} a *_A c \succsim_A b *_A c \quad &\text{iff} \quad \xi[a\pi(c) \succsim_A \xi[b\pi(c)] \\ &\text{iff} \quad \xi[a\pi(c)]p_0 \succsim \xi[b\pi(c)]p_0 \\ &\text{iff} \quad a\pi(c) \succsim b\pi(c) \\ &\text{iff} \quad a \succsim_A b. \end{aligned}$$

The other case is similar.

Restricted solvability: Suppose $a \succ_A b \succ_A a_0$. By density there exists a p in P such that $ap_0 \succ bp \succ bp_0$, and by monotonicity $p \succ_P p_0$, so $\xi(a_0 p) \succ_A a_0$. Observe that

$$bp \sim \xi(bp)p_0 \sim \xi[b\pi\xi(a_0 p)]p_0 \sim b *_A \xi(a_0 p)p_0,$$

and so $a \succ b *_A \xi(a_0 p)$.

Positivity: Observe that

$$a *_A bp_0 \sim \xi[a\pi(b)]p_0 \sim a\pi(b) \succ a_0\pi(b) \sim bp_0,$$

and so by monotonicity $a *_A b \succ_A b$. Since $b \succ_A a_0$,

$$a_0\pi(b) \sim bp_0 \succ a_0 p_0,$$

whence $\pi(b) \succ_P p_0$. Therefore

$$a *_A bp_0 \sim a\pi(b) \succ ap_0,$$

and so $a *_A b \succ_A a$.

Archimedean: This follows directly from the given Archimedean axiom.

Now, suppose $a_0 p_0$ is not minimal or maximal. Then $\langle (A^+ \cup \{a_0\}) \times (P^+ \cup \{p_0\}), \succsim, a_0, p_0 \rangle$ and $\langle (A^- \cup \{a_0\}) \times (P^- \cup \{p_0\}), \lesssim, a_0, p_0 \rangle$ are conjoint structures solvable relative to the minimal element a_0, p_0. Thus $\mathscr{J}_{A^+}$ and $\mathscr{J}_{A^-}$ are positive concatenation structures. To show compatibility, consider $a \in A^+$ and $b \in A^-$. Observe that

$$\begin{aligned} a \succsim_A na *_A b \quad \text{iff} \quad ap_0 &\succsim na * b, p_0 \\ &\sim \xi[na, \pi(b)], p_0 \\ &\sim na, \pi(b). \end{aligned}$$

Since $b \in A^-, \pi(b) \prec_P p_0$ and so by the Archimedean property there is some n_0 such that for all $n \geq n_0$, $na *_A b \succ_A a$. By a similar argument there is some m_0 such that for all $n \geq m_0$, $b \succ_A nb *_A a$. Therefore, $\mathscr{J}_A$ is a total concatenation structure. By Theorem 1, there is a numerical mapping ϕ and a binary $\odot$-representation of $\mathscr{J}_A$. Define $\psi = \phi\pi^{-1}$, and the conjoint representation follows immediately.

The results for $*$ follow immediately from the fact that $ap * bq \sim \xi(a, p) *_A \xi(b, q)$, p_0 and so $\mathscr{J}_{A \times P}/\sim$ is isomorphic to $\mathscr{J}_A/\sim_A$. □

The next result demonstrates that the concept of a closed total concatenation structure is exactly the correct one for the structure induced on one component of a solvable conjoint structure.

Theorem 3. *Suppose $\mathscr{A} = \langle A, \succsim, \circ, a_0 \rangle$ is a closed total concatenation structure. Then for $\mathscr{A}'$ isomorphic to $\mathscr{A}$, there exists a conjoint structure $\mathscr{C} = \langle A \times A', \succsim'', a_0, a_0' \rangle$ that is A-solvable relative to $a_0 a_0'$ and for which $\mathscr{J}_A$ is isomorphic to $\mathscr{A}$.*

Proof. Let ϕ be the isomorphism between $\mathscr{A}$ and $\mathscr{A}'$. Define $\succsim''$ on $A \times A'$ by

$$aa' \succsim'' bb' \quad \text{iff} \quad a \circ \phi^{-1}(a') \succsim b \circ \phi^{-1}(b').$$

First, we verify that $\succsim''$ satisfies the axioms of a conjoint structure. It is a weak order because $\succsim$ is. It is independent because $\circ$ is monotonic. It is solvable with $\pi(a) = \phi(a)$, since

$$a_0 \circ \phi^{-1}[\pi(a)] \sim a_0 \circ a \sim a \sim a \circ a_0 \sim a \circ \phi^{-1}(a_0')$$

is equivalent to $a_0 \pi(a) \sim'' aa_0'$. And $\xi(a, a') = a \circ \phi^{-1}(a')$ solves $\xi(a, a')a_0' \sim'' aa'$ because

$$\xi(a, a') \circ \phi^{-1}(a_0') \sim \xi(a, a') \circ a_0 \sim \xi(a, a') = a \circ \phi^{-1}(a').$$

It is dense by the restricted solvability of the positive concatenation components.

Next, we verify that $\succsim_A''$ is $\succsim$.

$$\begin{aligned} a \succsim_A'' b \quad & \text{iff} \quad aa' \succsim ba' \\ & \text{iff} \quad a \circ \phi^{-1}(a') \succsim b \circ \phi^{-1}(a') \\ & \text{iff} \quad a \succsim b. \end{aligned}$$

To show that $\mathscr{C}$ is Archimedean, consider $a, b \succ_A'' a_0$ and $p \precsim_P'' a_0'$. By what we have just shown, $a, b \succ a_0$ and $p \precsim a_0'$. Observe that

$$\begin{aligned} ba_0' \succsim na, p \quad \text{iff} \quad b & \sim b \circ a_0 \\ & \sim b \circ \varphi^{-1}(a_0') \\ & \succsim na \circ \varphi^{-1}(p). \end{aligned}$$

Since $\varphi^{-1}(p) \precsim a_0'$, we know by compatibility that there exists some c such that $c \circ \varphi^{-1}(p) \succ b$. By the Archimedean property of $\mathscr{A}$, for some integer n, $na \succ c$, whence $na \circ \varphi^{-1}(p) \succ b$, proving the first part of the Archimedean property of the conjoint structure. The proof of the second part is similar, using the second part of comparability and the Archimedean property of the negative part of $\mathscr{A}$.

Last, we show that $*_A''$ is $\circ$. Since $\pi(a) = \phi(a)$ and $(a, a') = a \circ \phi^{-1}(a')$,

$$a *_A b \sim \xi(a, \pi(b)) \sim \xi(a, \phi(b)) \sim a \circ \phi^{-1}\phi(b) \sim a \circ b. \qquad \square$$

Note that no particular relation need hold between the concatenations induced on A^+ and A^-. The conjoint structure must be further restricted if they are to be related.

Definition 5. Let $\mathscr{C} = \langle A \times P, \succsim, a_0 p_0 \rangle$ be a conjoint structure that is solvable relative to $a_0 p_0$. $\mathscr{C}$ is *invertible* iff for each $a \in A$ there exists $\gamma(a) \in P$ such that

(i) $a\gamma(a) \sim a_0 p_0$,
(ii) $\pi^{-1}\gamma = \gamma^{-1}\pi$,
(iii) $\pi^{-1}\gamma(a *_A b) \sim \pi^{-1}\gamma(a) *_A \pi^{-1}\gamma(b)$.

Theorem 4. *Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0 p_o \rangle$ is an $a_0 p_0$-solvable conjoint structure. $\mathscr{C}$ is invertible iff there exists a* 1 : 1 *mapping ϕ of $A/\sim$ onto $A/\sim$ such that for all $a, b \in A$*

(i) $a *_A \phi(a) \sim a_0$.
(ii) $\phi^{-1} = \phi$.
(iii) $\phi(a *_A b) = \phi(a) *_A \phi(b)$.

Under ϕ, $\mathscr{J}_{A^+}/\sim$ is isomorphic to $\mathscr{J}_{A^-}/\sim$.

Proof. The relation between ϕ and γ is $\phi = \pi^{-1}\gamma$. It is routine to show that each of these properties of Definition 5 correspond directly to those of the theorem.

We show the isomorphism. Observe that ϕ restricted to A^+ is into A^- since if $a \succ a_0$ and $\phi(a) \succsim a_0$, then by the monotonicity of $*_A$ and property (i), $a_0 \sim a *_A \phi(a) \succ a_0$, which is impossible. Similarly, ϕ restricted to A^- is into A^+. By property (ii), it is therefore onto. Property (iii) establishes that ϕ preserves $*_A$. The last thing to show is that it is order reversing. Suppose $a, b \in A^+$, then

$$a \succsim_A b \quad \text{iff} \quad a *_A \phi(a) \sim a_0 \sim b *_A \phi(b) \precsim a *_A \phi(b)$$

(by property (i) and monotonicity)

$$\text{iff} \quad \phi(a) \precsim \phi(b) \qquad \text{(by monotonicity)} \qquad \square$$

3. Additivity of the conjoint structure as reflected in its components

Our first question is: does additivity of the conjoint structure correspond to additivity of the induced positive concatenation structure? The answer is Yes. It is well known that the additivity of the operation $*_A$ corresponds to associativity of the operation (Theorem 3 of Ch. 3, in [5]); and it is also well known that the additivity of the conjoint structure corresponds to adding the Thomsen condition "if $ax \sim fq$ and $fq \sim bx$, then $ap \sim bq$" to Definition 1.

Theorem 5. *Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0, p_0 \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$. The structure satisfies the Thomsen condition iff the induced operation $*_A$ is associative.*

Corollary. *The induced operation $*_A$ is also commutative.*

Proof. Suppose the Thomsen condition holds. By definition of $*_A$, $a\pi(b) \sim a *_A b, p_0$ and $a *_A c, p_0 \sim a, \pi(c)$, so by the Thomsen condition, $a *_A c, \pi(b) \sim a *_A b, \pi(c)$, which is equivalent to $(a *_A c) *_A b \sim (a *_A b) *_A c$. Setting $a = a_0$, we see $*_A$ is commutative. Using these facts,

$$a *_A (b *_A c) \sim (b *_A c) *_A a \sim (b *_A a) *_A c \sim (a *_A b) *_A c,$$

so $*_A$ is associative.

Conversely, suppose $*_A$ is associative. By Theorem 2 and Theorem 3.3 of [5], this implies $*_A$ is commutative, from which bisymmetry follows:

$$\begin{aligned} (a *_A b) *_A (c *_A d) &\sim_A a *_A (b *_A c) *_A d && \text{(by associativity)} \\ &\sim_A a *_A (c *_A b) *_A d && \text{(by commutativity)} \\ &\sim_A (a *_A c) *_A (b *_A d) && \text{(by associativity).} \end{aligned}$$

Now, suppose $ax \sim fq$ and $fp \sim bx$, which are equivalent to $a *_A \pi^{-1}(x) \sim_A f *_A \pi^{-1}(q)$ and $f *_A \pi^{-1}(p) \sim_A b *_A \pi^{-1}(x)$. By the monotonicity of $*_A$, bisymmetry, and commutativity,

$$\begin{aligned} [a *_A \pi^{-1}(p)] *_A [\pi^{-1}(x) *_A f] &\sim_A [a *_A \pi^{-1}(x)] *_A [f *_A \pi^{-1}(p)] \\ &\sim_A [f *_A \pi^{-1}(q)] *_A [b *_A \pi^{-1}(x)] \\ &\sim_A [b *_A \pi^{-1}(x)] *_A [\pi^{-1}(q) *_A f] \\ &\sim_A [b *_A \pi^{-1}(q)] *_A [\pi^{-1}(x) *_A f], \end{aligned}$$

whence by the monotonicity of $*_A$ and its definition, $ap \sim bq$.

The Corollary, that $*_A$ is commutative, has been shown in the process of showing associativity. □

It is of some interest in this and other theorems that relate operations on one component to the conjoint structure to note just where the Archimedean axiom of $\mathscr{C}$ is used. In Theorem 5 it plays no role in deriving associativity from the Thomsen condition, but it is essential the other way because the argument rests on proving $*_A$ is commutative, which uses the Archimedean property of $\mathscr{J}_A$ and that follows from the Archimedean property of $\mathscr{C}$.

An alternative characterization arises when the conjoint structure is solvable relative to many points. It is captured as follows.

Definition 6. A conjoint structure has *invariant induced operations* iff for every $a_0p_0, a_0'p_0'$ for which it is solvable, inducing operations $*_A$ and $*_A'$, respectively, and for every $a, b, c, d \in A$,

$$a *_A b \succsim_A c *_A d \quad \text{iff} \quad a *_A' b \succsim_A c *_A' d.$$

Theorem 6. *Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure that satisfies Axioms* 1, 2, *and* 5 *of Definition* 1 *and, in addition, is unrestrictedly solvable. Then $\mathscr{C}$ satisfies the Thomsen condition iff it has invariant induced operations.*

Proof. If the Thomsen condition holds, then by Theorem 6.2 of [5], triple cancellation also holds. Now, suppose invariance of the induced operation fails, i.e., there are operators $*_A$ and $*_A'$ such that $a *_A b \succsim c *_A d$ and $c *_A' d \succ a *_A' b$. Thus we have the four equations

$$a\pi(b) \succsim c\pi(d), \qquad b\pi(d) \sim d\pi(b),$$

$$c\pi'(d) \succ a\pi'(b), \qquad d\pi'(b) \sim b\pi'(d).$$

Applying triple cancellation to the last three yields $c\pi(d) \succ a\pi(b)$, contrary to the first. So invariance of the induced operations holds.

Conversely, suppose invariance of the induced operations holds and $ax \sim fq$, $fp \sim bx$. Let $*$ be the operation defined relative to f, x, so $p = \pi(b)$, $q = \pi(a)$. Thus, the Thomsen condition is equivalent to proving commutivity, $a * b \sim b * a$. To show this, define the operation $*_A'$ relative to ap_0, where p_0 is any element of P. Observe that

$$a *_A' b, p_0 \sim a\pi'(b) \sim bp_0, \qquad b *_A' a, p_0 \sim b\pi'(a) \sim bp_0,$$

whence by transitivity and monotonicity, $a *_A' b \sim_A b *_A' a$. By invariance, $a *_A b \sim_A b *_A a$. □

As proved, the implication of invariant induced operations depends on using triple cancellation and that follows from the Thomsen condition only in the presence of the Archimedean axiom. We do not know of a proof that avoids the Archimedean axiom.

4. Factorizable automorphisms

One way to study how the induced operations relate to a solvable conjoint structure is to ask about the relations of the two classes of automorphisms, the one for induced structures $\mathscr{J}_A = \langle A, \succsim_A, *_A, a_0 \rangle$ which preserves $\succsim_A, *_A$, and a_0, and the other for $\mathscr{C} = \langle A \times P, \succsim \rangle$ preserving $\succsim$ and, possibly, $a_0 p_0$.

A major distinction to be made about automorphisms of $\mathscr{C}$ is whether or not they can be expressed in terms of separate transformations on the two factors. We know nothing of the structures for which there are no factorizable automorphisms.

Definition 7. Suppose $\mathscr{C}$ is a conjoint structure. An order automorphism α of $\mathscr{C}$ is *factorizable* iff there exist 1 : 1 functions $\theta : A \xrightarrow{\text{onto}} A$ and $\eta : P \xrightarrow{\text{onto}} P$ such that for all $a \in A$, $p \in P$, $\alpha(a, p) = \theta(a)\eta(p)$.

We denote by $\mathscr{F}$ the set of all factorizable automorphisms, by $\mathscr{F}_A$ those transformations that arise on A, i.e.,

$$\mathscr{F}_A = \{\theta \mid \theta : A \xrightarrow{\text{onto}} A \text{ and there exists } \eta : P \xrightarrow{\text{onto}} P \text{ such that } \langle\theta,\eta\rangle \in \mathscr{F}\},$$

and by $\mathscr{F}_P$ those that arise on P. Note $\mathscr{F} \subseteq \mathscr{F}_A \times \mathscr{F}_P$ but in general, $\mathscr{F} \neq \mathscr{F}_A \times \mathscr{F}_P$. In fact, if $\mathscr{F}$ is non-trivial, equality is almost never the case. For example, in the case of the additive reals, it is well known that the automorphisms are factorizable into pairs of affine transformations $(rx+s, ry+s')$ having a common unit r. Arbitrary pairs of affine transformations do not form autormorphisms.

One can argue that the factorization of a conjoint structure captures the idea of independent factors only when many automorphisms are also factorizable. As we shall see in Theorems 17, 18, and 19 several combinations of assumptions about the smoothness of a numerical representation and the existence of ratio and interval scale factorizable automorphisms whose factors are somewhat independent are sufficient to lead to the usual multiplicative representation of the physical theory of dimensions. A generalization arises when the structure has an intrinsic zero (Theorem 13).

Lemma 1. *Under function composition, $\mathscr{F}$, $\mathscr{F}_A$, and $\mathscr{F}_P$ are groups.*

Proofs. Observe, first, that members of $\mathscr{F}_A$ and $\mathscr{F}_P$ are order automorphisms of $\langle A, \succsim_A\rangle$ and $\langle P, \succsim_P\rangle$. Let ι denote the identity map of $A \times P$, ι_A that of A, and ι_P that of P. Clearly, $\iota = \langle\iota_A, \iota_P\rangle$ is factorizable and these are identities for the three sets. If $\langle\theta,\eta\rangle \in \mathscr{F}$, then $\langle\theta^{-1},\eta^{-1}\rangle \in \mathscr{F}$ since if $ap \succsim bq$ and $\theta^{-1}(a)\eta^{-1}(p) \prec \theta^{-1}(b)\eta^{-1}(q)$ we get the contradiction $ap \prec bq$ by applying $\langle\theta,\eta\rangle$ to the latter inequality. Since associativity holds for function composition and $\langle\theta^{-1},\eta^{-1}\rangle = \langle\theta,\eta\rangle^{-1}$, these are groups. $\square$

To illustrate that there exist non-additive structures with factorizable automorphisms, we consider first an important class of numerical, non-associative positive concatenation structures called unit representations [2]: if $x, y \in \mathrm{Re}^+$, the operation is of the form

$$x \oplus y = yf(x/y),$$

where f must satisfy certain properties including $f(1) > 1$ and f is strictly increasing. These structures will arise later (Theorem 13) in the representation of a certain class of conjoint structures. Observe that the n-copy operator,

$$nx = \begin{cases} x & \text{if } n = 1, \\ (n-1)x \oplus x & \text{if } n > 1, \end{cases}$$

is given by

$$nx = xf^{(n)}(1).$$

Moreover

$$\begin{aligned} n(x \oplus y &= (x \oplus y)f^{(n)}(1) = yf(x/y)f^{(n)}(1) \\ &= xf^{(n)}(1) \oplus yf^{(n)}(1) = nx \oplus ny. \end{aligned}$$

Since $f^{(n)} > 0$,

$$x \geq y \quad \text{iff} \quad x f^{(n)}(1) \geq y f^{(n)}(1) \quad \text{iff} \quad nx \geq ny,$$

so nx is order preserving. Finally, nx is onto. Thus, it is an automorphism of the unit representation.

This means that, given Theorem 3, the assumption of the following result is not vacuous. The result does not rest on the Archimedean property of $\mathscr{C}$.

Theorem 7. *Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is an A-solvable conjoint structure relative to $a_0 p_0 \in A \times P$. Then the mapping of α_n defined for each integer n by: $\alpha_n(a, p) = (na, np)$ is a factorizable automorphisms of $\mathscr{C}$ iff the n-copy operator of $*_A$ is an automorphism of $\mathscr{J}_A = \langle A, \succsim_A, *_A \rangle$.*

Proof. Observe that by induction on $p *_P q = \pi[\pi^{-1}(p) *_A \pi^{-1}(q)]$, we see $np \sim_P \pi n \pi^{-1}(p)$. Thus, $\pi^{-1}(np) \sim_A n\pi^{-1}(p)$ and $n\pi(b) \sim_P \pi n \pi^{-1} \pi(b) \sim_P \pi(nb)$.

First, suppose α_n is a factorizable automorphism. Then,

$$\begin{aligned}
a *_A b \succsim_A c *_A d \quad & \text{iff} \quad a\pi(b) \succsim c\pi(d) && \text{(definition of } *_A) \\
& \text{iff} \quad na, n\pi(b) \succsim nc, n\pi(d) && (\alpha_n \text{ is order preserving}) \\
& \text{iff} \quad na, \pi(nb) \succsim nc, \pi(nd) && \text{(remark above)} \\
& \text{iff} \quad na *_A nb \succsim_A nb *_A nd && \text{(definition of } *_A \text{ and } \succsim_A)
\end{aligned}$$

By monotonicity of $*_A$,

$$\begin{aligned}
a \succsim_A b \quad & \text{iff} \quad a *_A c \succsim_A b *_A c \\
& \text{iff} \quad na *_A nc \succsim_A nb *_A nc \\
& \text{iff} \quad na \succsim_A nb,
\end{aligned}$$

so the n-copy operator is order preserving. Observe, from the definition of π that $\pi(a_0) \sim p_0$ and so

$$2a_0, p_0 \sim a_0 *_A a_0, p_0 \sim a_0, \pi(a_0) \sim a_0, p_0.$$

By induction $na_0 \sim_A a_0$ and similarly, $np_0 \sim p_0$. Thus,

$$\begin{aligned}
n(a *_A b), p_0 \sim n(a *_A b), np_0 \quad & && \text{(monotonicity of } p_0 \sim_P np_0) \\
& \sim \alpha_n(a *_A b, p_0) && (\alpha_n \text{ is a factorizable automorphism}) \\
& \sim \alpha_n(a, \pi(b)) && \text{(definition of } *_A) \\
& \sim na, n\pi(b) && \text{(definition of } \alpha_n) \\
& \sim na, \pi(nb) && \text{(first remark)} \\
& \sim na *_A nb, p_0, && \text{(definition of } *_A)
\end{aligned}$$

proving that the n-copy operator is an automorphism of $\mathcal{J}_A$.

Conversely, suppose the n-copy operator is an automorphism of $\mathcal{J}_A$, then

$$
\begin{aligned}
ap \succsim bq \quad &\text{iff} \quad a *_A \pi^{-1}(p) \succsim_A b *_A \pi^{-1}(q) \\
&\qquad\qquad\qquad\qquad \text{(definition } *_A \text{ and monotonicity)} \\
&\text{iff} \quad n(a *_A \pi^{-1}(p)) \succsim_A n(b *_A \pi^{-1}(q)) \quad (na \text{ is order preserving}) \\
&\text{iff} \quad na *_A n\pi^{-1}(p) \succsim_A nb *_A n\pi^{-1}(q) \quad (na \text{ is automorphism}) \\
&\text{iff} \quad na *_A \pi^{-1}(np) \succsim_A nb *_A \pi^{-1}(nq) \quad \text{(remark above)} \\
&\text{iff} \quad na, np \succsim nb, nq, \quad \text{(definition of } *_A \text{ and monotonicity)}
\end{aligned}
$$

proving α_n is a factorizable automorphism of $\mathscr{C}$. □

Our next result is important. It shows that if the conjoint structure is sufficiently endowed with factorizable automorphisms, then all of the induced operations are basically the same. This theorem does not use the Archimedean property of $\mathscr{C}$.

Theorem 8. *Suppose a conjoint structure $\mathscr{C} = \langle A \times P, \succsim \rangle$ is A-solvable relative to both a_0p_0 and $a_0'p_0'$, that θ is a function from A onto A with $\theta(a_0) \sim_A a_0'$, and η is a function from P onto P with $\eta(p_0) \sim_P p_0'$. Then, $\langle \theta, \eta \rangle$ is a factorizable automorphism of $\mathscr{C}$ iff $\eta = \pi'\theta\pi^{-1}$ and θ is an isomorphism from $\mathcal{J}_A = \langle A, \succsim_A, *_A, a_0 \rangle$ onto $\mathcal{J}_A' = \langle A, \succsim_A, *_A', a_0' \rangle$.*

Proof. Suppose $\langle \theta, \eta \rangle$ is a factorizable automorphism. Since, by definition of π, for all p in P, $\pi^{-1}(p)p_0 \sim a_0 p$, applying the automorphism yields

$$a_0', \pi'\theta\pi^{-1}(p) \sim \theta\pi^{-1}(p), p_0' \sim \theta\pi^{-1}(p), \eta(p_0) \sim \theta(a_0), \eta(p) \sim a_0', \eta(p),$$

whence $\eta = \pi'\theta\pi^{-1}$. Next, apply the automorphism to the definition of $*_A$, $a *_A b, p_0 \sim a, \pi(b)$, which yields

$$
\begin{aligned}
\theta(a *_A b), p_0' \sim \theta(a *_A b), \pi'\theta\pi^{-1}(p_0) &\sim \theta(a), \pi'\theta\pi^{-1}\pi(b) \\
&\sim \theta(a), \pi'\theta(b) \sim \theta(a) *_A' \theta(b), p_0',
\end{aligned}
$$

whence $\theta(a *_A b) \sim_A \theta(a) *_A' \theta(b)$. Since θ is $1:1$ and order preserving by the fact $\langle \theta, \eta \rangle$ is a factorizable automorphism, this proves θ is an isomorphism from $\mathcal{J}_A$ onto $\mathcal{J}_A'$.

Conversely, suppose θ is an isomorphism.

$$
\begin{aligned}
ap \succsim bq \quad &\text{iff} \quad a *_A \pi^{-1}(p) \succsim_A b *_A \pi^{-1}(q) \quad \text{(definition of } *_A) \\
&\text{iff} \quad \theta(a *_A \pi^{-1}(p)) \succsim_A \theta(b *_A \pi^{-1}(q)) \quad (\theta \text{ is isomorphism}) \\
&\text{iff} \quad \theta(a) *_A' \theta\pi^{-1}(p) \succsim_A \theta(b) *_A' \theta\pi^{-1}(q) \quad (\theta \text{ is isomorphism}) \\
&\text{iff} \quad \theta(a), \pi'\theta\pi^{-1}(p) \succsim \theta(b), \pi'\theta\pi^{-1}(q) \quad \text{(definition of } *_A')
\end{aligned}
$$

and so $\langle \theta, \pi'\theta\pi^{-1} \rangle$ is an automorphism of $\mathscr{C}$. □

Corollary. *Let ι_A and ι_P denote the identity maps of A and P, respectively. If $\langle\theta,\iota_P\rangle$ (respectively, $\langle\iota_A,\eta\rangle$) is a factorizable automorphism, then for some $c\in A$ (respectively, $r\in P$) $\theta\sim_A c *_A \iota_A$ (respectively, $\eta\sim_P \iota_P *_P r$).*

Proof. Fix a_0p_0 and by definition $a_0\pi(a)\sim ap_0$ for all $a\in A$. Apply the factorizable automorphism $\langle\theta,\iota_P\rangle$ and we obtain

$$\theta(a)p_0\sim\theta(a_0)\pi(a)\sim\theta(a_A) *_A a, p_0,$$

so

$$\theta(a)\sim_A\theta(a_0) *_A a. \qquad \square$$

If $a_0'=a_0$ and $p_0'=p_0$, the conclusion from Theorem 8 is that $\langle\theta,\eta\rangle$ is an automorphism of $\mathscr{C}$ iff $\eta=\pi\theta\pi^{-1}$ and θ is an automorphism of $\mathscr{J}_A$. Restricting our attention to the positive part of $\mathscr{J}_A$, $\mathscr{J}_A^+$, we know that all automorphisms of $\mathscr{C}$ that have a common fixed point form an Archimedean ordered group (Theorem 2.4 of [2]) and so commute.

Two transformations of the induced structure $\mathscr{J}_A$ which seem natural to investigate further are those generated by 'translating' each element of A by a constant, i.e., for some fixed c in A, the transformations $\iota_A *_A c$ and $c *_A \iota_A$, where ι_A is the identity map of A. If these are, indeed, isomorphisms of the induced structure and $\mathscr{C}$ is sufficiently solvable, then the next result establishes that $\mathscr{C}$ must satisfy the Thomsen condition and so, by the corollary to Theorem 5, $*_A$ is commutative and the two transformations are identical.

Theorem 9. *Suppose that $\mathscr{C}$ is a conjoint structure that is A-solvable relative to a_0p_0 and that for every a in A, p in P there is a b in A such that $bp\sim ap_0$.*

1. *$\mathscr{C}$ satisfies the Thomsen condition iff, for every c in A, r in P, $\langle c *_A \iota_A, \iota_P *_P r\rangle$ is an automrophism of $\mathscr{C}$.*
2. *Suppose θ is $1:1$ from A onto A and η is $1:1$ from P onto P. Under the conditions of part 1, $\langle\theta,\eta\rangle$ is an automorphism of $\mathscr{C}$ iff for some automorphism θ^* of $\mathscr{J}_A$,*

$$\theta=\theta(a_0) *_A \theta^* \quad\textit{and}\quad \eta=\pi\theta^*\pi^{-1} *_P \eta(p_0).$$

Proof. 1. Suppose $\langle a *_A \iota_A, \iota_p *_p p\rangle$ is an automorphism for every a in A, p in P. For b,c in A, apply it with $p=\pi(c)$ to $bp_0\sim a_0\pi(b)$ to yield

$$a *_A b, \pi(c)\sim a *_A a_0, \pi(b) *_P \pi(c)\sim a, \pi(b) *_P \pi(c).$$

Observe that

$$a_0, \pi(b *_A c)\sim b *_A c, p_0\sim b\pi(c)\sim\pi^{-1}\pi(b), \pi(c)\sim a_0, \pi(b) *_P \pi(c).$$

So, using independence,

$$(a *_A b) *_A c, p_0\sim a *_A b, \pi(c)\sim a, \pi(b) *_P \pi(c)\sim a, \pi(b *_A c)\sim a *_A (b *_A c), p_0.$$

Thus, $*_A$ is associative and so, by Theorem 5, the Thomsen condition holds.

Conversely, suppose the Thomsen condition holds, then we know $*_A$ is associative. Let $\theta = c *_A \iota_A$ and $\eta = \iota_P *_P r$. Consider

$$\begin{aligned}
ap \succsim bq \quad &\text{iff} \quad a *_A \pi^{-1}(p) \succsim_A b *_A \pi^{-1}(q) \quad (\text{definition of } *_A)\\
&\text{iff} \quad a *_A [\pi^{-1}(p) *_A \pi^{-1}(r)] \succsim_A b *_A [\pi^{-1}(q) *_A \pi^{-1}(r)]\\
&\qquad\qquad (\text{monotonicity and associativity})\\
&\text{iff} \quad a *_A \pi^{-1}(p *_P r) \succsim_A b *_A \pi^{-1}(q *_P r) \quad (\text{definition of } *_P)\\
&\text{iff} \quad (c *_A a) *_A \pi^{-1}(p *_P r) \succsim_A (c *_A b) *_A \pi^{-1}(q *_P r)\\
&\qquad\qquad (\text{monotonicity and associativity})\\
&\text{iff} \quad c *_A a, p *_P r \succsim c *_A b, q *_P r \quad (\text{definition of } *_A)\\
&\text{iff} \quad \theta(a), \eta(p) \succsim \theta(b), \eta(q), \quad (\text{definition of } \theta \text{ and } \eta)
\end{aligned}$$

So $\langle\theta, \eta\rangle$ is order preserving. We show it is onto by showing that θ (and so η) is onto. For each b in A there exists an a such that

$$bp_0 \sim a, \pi\theta(a_0) \sim a *_A \theta(a_0), p_0 \sim \theta(a_0) *_A a, p_0 \sim \theta(a), p_0.$$

2. Suppose $\langle\theta, \eta\rangle$ is an automorphism of $\mathscr{C}$. Define θ^* as the solution to

$$\theta^*(a) *_A \theta(a_0), p_0 \sim \theta^*(a), \pi\theta(a_0) \sim \theta(a)p_0.$$

Because θ is onto and 1 : 1 and unrestricted solvability holds, θ^* is onto and 1 : 1. It is order preserving because

$$\begin{aligned}
a \succsim_A b \quad &\text{iff} \quad ap_0 \succsim bp_0\\
&\text{iff} \quad \theta(a)\eta(p_0) \succsim \theta(b)\eta(p_0)\\
&\text{iff} \quad \theta(a) \succsim_A \theta(b)\\
&\text{iff} \quad \theta^*(a) *_A \theta(a_0) \succsim_A \theta^*(b) *_A \theta(a_0)\\
&\text{iff} \quad \theta^*(a) \succsim_A \theta^*(b).
\end{aligned}$$

Next we show $\theta^*(a *_A b) \sim_A \theta^*(a) *_A \theta^*(b)$. From the definition of π,

$$\begin{aligned}
bp_0 \sim a_0\pi(b) \quad &\text{iff} \quad \theta(b), \eta(p_0) \sim \theta(a_0), \eta\pi(b)\\
&\qquad\qquad (\langle\theta, \eta\rangle \text{ is factorizable automorphism})\\
&\text{iff} \quad \theta(b) *_A \pi^{-1}\eta(p_0) \sim_A \theta(a_0) *_A \pi^{-1}\eta\pi(b)\\
&\qquad\qquad (\text{definition of } *_A)\\
&\text{iff} \quad \theta(a) *_A \theta(b) *_A \pi^{-1}\eta(p_0) \sim_A \theta(a) *_A \theta(a_0) *_A \pi^{-1}\eta\pi(b)\\
&\qquad\qquad (\text{monotonicity and associativity of } *_A).
\end{aligned}$$

From the definition of $*_A$,

$$
\begin{aligned}
a *_A b, p_0 \sim a, \pi(b) \quad &\text{iff} \quad \theta(a *_A b), \eta(p_0) \sim \theta(a), \eta\pi(b) \quad (\text{apply } \langle\theta, \eta\rangle) \\
&\text{iff} \quad \theta(a *_A b) *_A \pi^{-1}\eta(p_0) \sim_A \theta(a) *_A \pi^{-1}\eta\pi(b) \\
&\qquad\qquad (\text{definition of } *_A) \\
&\text{iff} \quad \theta(a *_A b) *_A \theta(a_0) *_A \pi^{-1}\eta(p_0) \\
&\qquad \sim_A \theta(a) *_A \theta(a_0) *_A \pi^{-1}\eta\pi(b) \\
&\qquad (\text{monotonicity, associativity and commutativity of } *_A)
\end{aligned}
$$

By transitivity and monotonicity

$$\theta(a) *_A \theta(b) \sim \theta(a *_A b) *_A \theta(a_0).$$

Substitute the definition of θ^*,using monotonicity, associativity and commutativity of $*_A$ and the conclusion follows.

In like manner, define η^* and show that it is an automorphism of $\mathscr{J}_P$.

We show $\langle\theta^*, \eta^*\rangle$ is an automorphism of $\mathscr{C}$ and so, by Theorem 8, $\eta^* = \pi\theta^*\pi^{-1}$.

$$
\begin{aligned}
ap \succsim bq \quad &\text{iff} \quad \theta(a), \eta(p) \succsim \theta(b), \eta(q) \quad (\langle\theta, \eta\rangle \text{ is an automorphism}) \\
&\text{iff} \quad \theta^*(a) *_A \theta(a_0) *_A \pi^{-1}\eta(p) \succsim_A \theta^*(b) *_A \theta(a_0) *_A \pi^{-1}\eta(q) \\
&\qquad\qquad (\text{def. of } \theta^* \text{ and monotonicity}) \\
&\text{iff} \quad \theta^*(a), \eta(p) \succsim \theta^*(b), \eta(q) \quad (\text{monotonicity and def. of } *_A) \\
&\text{iff} \quad \pi\theta^*(a) *_P \eta(p) \succsim_P \pi\theta^*(b) *_P \eta(q) \quad (\text{def. of } *_P) \\
&\text{iff} \quad \pi\theta^*(a) *_P \eta^*(p) *_P \eta(p_0) \succsim_P \pi\theta^*(b) *_P \eta^*(q) *_P \eta(p_0) \\
&\qquad\qquad (\text{def. of } \eta^* \text{ and monotonicity}) \\
&\text{iff} \quad \theta^*(a), \eta^*(p) \succsim \theta^*(b), \eta^*(q) \\
&\qquad\qquad (\text{monotonicity and def. of } *_P).
\end{aligned}
$$

The above proof also establishes the converse that $\langle\theta, \eta\rangle$ is a factorizable automorphism when $\langle\theta^*, \eta^*\rangle$ is. □

In part 1, the proof of the necessity of the Thomsen condition involves showing $*_A$ is associative and then using Theorem 5, and so the Archimedean property of $\mathscr{C}$, to draw the conclusion. The sufficiency of the Thomsen condition and part 2 do not draw upon the Archimedean property of $\mathscr{C}$.

5. Distributive operation on a component

Much physical measurement involves an interplay between conjoint structures and an empirical operation that forms a positive concatenation structure (usually, an extensive one) on one of the components. This operation is totally distinct from those induced by the conjoint structures, as can be shown using Theorem 4.3 of Narens [7]. Yet there is a close tie between the representations, including the fact that automorphisms of the operation appear as factors of factorizable automorphisms of the conjoint structure.

The following property, which relates the operation $\circ_A$ on A to the conjoint structure $\mathscr{C}$, together with $\langle A, \succsim_A, \circ_A\rangle$ having a ratio scale representation has been shown [7, 9] to yield the usual representation of the conjoint structure as products of powers of the representations of its components.

Definition 8. Suppose $\mathscr{C} = \langle A \times P, \succsim\rangle$ is a conjoint structure and that $\circ_A$ is a partial binary operation on A. Then, $\circ_A$ is *distributive* iff for all $a, b, c, d \in A$ such that $a \circ_A b$ and $c \circ_A d$ are defined and all $p, q \in P$, if $ap \sim cq$ and $bp \sim dq$, then $a \circ_A b, p \sim c \circ_A d, q$.

An automorphism θ of a structure $\langle A, \succsim, \ldots\rangle$ is called *positive* iff for all $a \in A$, $\theta(a) \succ a$ and *negative* iff θ^{-1} is positive. Cohen and Narens ([2], Theorem 2.1) showed that for a positive concatenation structure, every non-trivial automorphism is either positive or negative.

Theorem 10. *Suppose $\mathscr{C} = \langle A \times P, \succsim\rangle$ is a conjoint structure that is A-solvable relative to a_0p_0 and that for every $a \in A$ and $p \in P$, there is a $b \in A$ such that $bp \sim ap_0$. Suppose $\circ_A$ is a closed binary operation on A such that $\mathscr{A} = \langle A, \succsim_A, \circ_A\rangle$ is a positive concatenation structure. The following three statements are equivalent:*

1. *$\circ_A$ is distributive.*
2. *If θ is an automorphism of $\mathscr{A}$, then for some $c \in A$, $\theta = \iota_A *_A c$; and for every $c \in A$, $\iota_A *_A c$ is an automorphism of $\mathscr{A}$.*
3. (a) *For θ an automorphism of $\mathscr{A}$, both $\langle\theta, \iota_P\rangle$ and $\langle\iota_A, \pi\theta\pi^{-1}\rangle$ are factorizable automorphism of $\mathscr{C}$,*[2] *and*

   (b) *$\mathscr{A}$ is homogeneous*[3] *in the sense that for each $a, b \in A$, there exists an automorphism θ of $\mathscr{A}$ such that $\theta(a) \sim_A b$.*

Corollary. *Under the hypotheses and conclusions of the theorem, the following are true:*

[2] The former property Narens [7] called component $\mathscr{A}$-invariance and the latter component $\mathscr{P}$-invariance.

[3] This concept, which was studied by Cohen and Narens [2], Definition 3.1 and Narens [7], was generalized by Narens [8]. The generalized concept is introduced in the next section, and it plays an essential role throughout the remainder of the paper.

(i) *The Thomsen condition holds.*

(ii) *Unrestricted solvability holds.*

(iii) *Automorphisms satisfy one-point uniqueness in the sense that if θ and θ' are automorphisms and for some $a \in A$, $\theta(a) = \theta'(a)$, then $\theta = \theta'$.*

(iv) *$\mathscr{C}$ has an additive representation.*

Proof. 1 implies 2. Suppose $\circ_A$ is distributive. If $\theta_c = \iota_A *_A c$, $c \in A$, we show θ_c is an automorphism. It is onto by the solvability assumption. It is order preserving because $*_A$ is monotonic. By distributivity, $a\pi(c) \sim_A a *_A c, p_0$ and $b\pi(c) \sim_A b *_A c, p_0$ imply

$$(a \circ_A b) *_A c, p_0 \sim a \circ_A b, \pi(c) \sim (a *_A c) \circ_A (b *_A c), p_0,$$

whence by the definition of θ_c and independence of $\mathscr{C}$.

$$\theta_c(a \circ_A b) \sim_A \theta_c(a) \circ_A \theta_c(b).$$

Conversely, suppose θ is an automorphism of $\mathscr{A}$. By Theorem 2.1 of [2], θ is either the identity, positive, or negative. If it is positive, select any $a \in A^+$ and then since $\mathscr{J}_{A^+}$ is a positive concatenation structure (Theorem 2) there exists $c \in A^+$ such that $\theta(a) \sim_A a *_A c \sim_A \theta_c(a)$. Since θ_c is also an automorphism, Theorem 2.1 of [2] proves $\theta = \theta_c$. If θ is negative, the proof is similar with $a, c \in A^-$.

2 implies 1. Suppose $ap \sim cq$ and $bq \sim dq$. Observe,

$$\begin{aligned} ap \sim cq \quad &\text{iff} \quad a *_A \pi^{-1}(p), p_0 \sim c *_A \pi^{-1}(q), p_0 \\ &\text{iff} \quad \theta_{\pi^{-1}(p)}(a) \sim_A \theta_{\pi^{-1}(q)}(c). \end{aligned}$$

And

$$bp \sim dq \quad \text{iff} \quad \theta_{\pi^{-1}(p)}(b) \sim_A \theta_{\pi^{-1}(q)}(d).$$

Since $\theta_{\pi^{-1}(p)}$ and $\theta_{\pi^{-1}(q)}$ are automorphisms and $\circ_A$ is monotonic,

$$\begin{aligned} \theta_{\pi^{-1}(p)}(a \circ_A b) &\sim_A \theta_{\pi^{-1}(p)}(a) \circ_A \theta_{\pi^{-1}(p)}(b) \\ &\sim_A \theta_{\pi^{-1}(q)}(c) \circ_A \theta_{\pi^{-1}(q)}(d) \sim_A \theta_{\pi^{-1}(q)}(c \circ_A d) \end{aligned}$$

whence, $a \circ_A b, p \sim c \circ_A d, q$, proving distributivity.

2 implies 3. We first establish the Thomsen condition. Consider any $a, b, c \in A$. Using the above notation, if θ_b and θ_c are automorphisms of $\mathscr{A}$, so then is $\theta_c\theta_b$ and, by hypothesis, for some $d \in A$, $\theta_c\theta_b = \theta_d$. Observe,

$$d \sim_A a_0 *_A d \sim_A \theta_d(a_0) \sim_A \theta_c\theta_b(a_0) \sim_A (a_0 *_A b) *_A c \sim_A b *_A c.$$

So

$$a *_A (b *_A c) \sim_A a *_A d \sim_A \theta_d(a) \sim_A \theta_c\theta_b(a) \sim_A (a *_A b) *_A c,$$

and so $*_A$ is associative. Thus, by Theorem 5, $\mathscr{C}$ satisfies the Thomsen condition and $*_A$ is commutative.

(a) Let θ be an automorphism of $\mathscr{A}$ which, by hypothesis is of the form $\theta = \iota_A *_A c \sim_A c *_A \iota_A$. By Part 1 of Theorem 9,

$$\langle c *_A \iota_A, \iota_P *_P p_0 \rangle = \langle \iota_A *_A c, \iota_P \rangle = \langle \theta, \iota_P \rangle$$

is a factorizable automorphism of $\mathscr{C}$. Observe,

$$\begin{aligned}\pi\theta\pi^{-1}(p) &= \pi(\iota_A *_A c)\pi^{-1}(p) && \text{(hypothesis)}\\ &\sim_P \pi[\pi^{-1}(p) *_A c]\\ &\sim_P \pi\xi[\pi^{-1}(p), \pi(c)] && (\text{definition of } *_A)\\ &\sim_P \pi\xi[\xi(a_0, p), \pi(c)] && (\text{definition of } \xi \text{ and } \pi^{-1})\\ &\sim_P p *_P \pi(c) && (\text{definition of } *_P)\\ &\sim_P (\iota_P *_P \pi(c))(p),\end{aligned}$$

and so by Part 1 of Theorem 9,

$$\langle \iota_A, \pi\theta\pi^{-1}\rangle = \langle a_0 *_A \iota_A, \iota_P *_P \pi(c)\rangle$$

is a factorizable automorphism of $\mathscr{C}$.

(b) For $a, b \in A$, $\theta_a(a_0) \sim_A a_0 *_A a \sim_A a$ and $\theta_b(a_0) \sim_A b$, so $\theta_b\theta_a^{-1}(a) \sim_A b$, establishing that $\mathscr{A}$ is homogeneous.

3 implies 2. We first show that the Thomsen condition holds.[4] Suppose $ax \sim fq$ and $fp \sim bx$. By homogeneity, choose θ and θ' such that $\theta(f) = a$ and $\theta'(a) = \theta(b)$. Using 3(a) as the hypothesis,

$$ap \sim \theta(f)p \sim \theta(b)x \sim \theta'(a)x \sim \theta'(f)q.$$

By Theorem 2.4 of [2], the autormorphisms of $\mathscr{A}$ are commutative, so

$$\theta'(f) = \theta'\theta^{-1}(a) = \theta'\theta^{-1}\theta'^{-1}\theta(b) = b,$$

whence $ap \sim bq$. Let θ be any automorphism of $\mathscr{A}$. Since by 3(a), $\langle\theta, \iota_p\rangle$ is a factorizable automorphism of $\mathscr{C}$ and $\mathscr{C}$ satisfies the Thomsen condition, we know from Part 2 of Theorem 9 that $\theta = \theta(a_0) *_A \theta^*$, where θ^* is an automorphism of $\mathscr{J}_A$, and

$$\begin{aligned}\iota_P &= \pi\theta^*\pi^{-1} *_P \iota_P(p_0)\\ &\sim_P \pi\theta^*\pi^{-1} *_P p_0 \sim_P \pi\theta^*\pi^{-1},\end{aligned}$$

so $\theta^* = \iota_A$. Thus, since $*_A$ is commutative (the Thomsen condition and the Corollary to Theorem 5),

$$\theta = \theta(a_0) *_A \iota_A \sim_A \iota_A *_A \theta(a_0).$$

Conversely, suppose $\theta_c = \iota_A *_A c$, $c \in A$. By 3(b) there exists an automorphism θ of $\mathscr{A}$ such that $\theta(a_0) \sim_A c$. As we have just seen, $\theta \sim_A \iota_A *_A \theta(a_0) \sim_A \iota_A *_A c = \theta_c$, and so θ_c is an automorphism. □

Proof of Corollary. (i) The Thomsen condition was established in the proof of 3 implies 2.

[4] The following proof is taken from Falmange and Narens [3].

(ii) Suppose $a \in A$, $q \in P$ are given. By the hypothesis of Theorem 10, there exists f such that

$$fq \sim \xi(a, p)p_0 \sim a, p.$$

Next, suppose $a, b \in A$, $p \in P$ are given. By the hypothesis of Theorem 10, there exists f such that $ap \sim f\pi(b)$. From $fp_0 \sim a_0\pi(f)$ and $a_0\pi(b) \sim bp_0$, the Thomsen condition yields $f\pi(b) \sim b\pi(f)$, and so $x \sim \pi(f)$ solves $ap \sim bx$.

(iii) Suppose θ, θ' are automorphisms and $\theta(a) = \theta'(a)$. By part 2 of the theorem, there exist $c, c' \in A$ such that $\theta = \iota_A *_A c$ and $\theta' = \iota_A *_A c'$. Thus, $a *_A c \sim_A a *_A c'$. By the monotonicity of $*_A$ (Theorem 2), $c \sim_A c'$ and so $\theta = \theta'$.

(iv) The representation follows from Theorem 4.1 of [7]. □

The Archimedean property of $\mathscr{C}$ is used, via Theorem 5 to prove that the Thomsen condition follows from 2, in establishing that 3 is a consequence of 2. The proof that 3 implies the Thomsen condition does not use the Archimedean property of $\mathscr{C}$, although that of $\mathscr{A}$ is essential.

By Definition 3.3 (and Theorem 3.2) of [2], $\mathscr{A}$ is a fundamental unit structure, and so has a ratio scale representation, iff $\mathscr{A}$ is homogeneous and Dedekind complete. Thus, if the conditions of Theorem 10 hold, and if $\mathscr{A}$ is Dedekind complete and $\circ_A$ is distributive, then $\mathscr{A}$ is a fundamental unit structure.

Theorem 10 improves considerably previous results. Narens and Luce [9] showed that if $\mathscr{A}$ is an associative positive concatenation structure (extensive structure) and $\circ_A$ is distributive, then $\mathscr{C}$ satisfies the Thomsen condition. Narens ([7], Theorem 4.1) replaced the hypothesis on $\mathscr{A}$ by the assumption $\mathscr{A}$ has a ratio scale representation, and showed $\mathscr{C}$ satisfies the Thomsen condition. We have shown that these hypotheses are entirely redundant. The fact that the only if part of statement 2 of Theorem 10 is a consequence of statement 1 of Theorem 10 is Lemma 4.1 of [7].

An interesting case of a conjoint structure $\mathscr{C}$ that has a binary operation $\circ_A$ with $\mathscr{A} = \langle A, \succsim_A, \circ_A \rangle$ a positive concatenation structure, but is not covered by Theorem 10, is relativistic velocity. If we denote by D the usual measures of distance, by V those of relativistic velocity, and by T those of time, then $D = V \times T$ is ordered by $d(v, t) = vt$. And velocity concatenation is given by,

$$u \circ_A v = \frac{u + v}{1 + uv/c^2},$$

where c is the velocity of light. As in Theorem 10 and its corollary, $\mathscr{C}$ satisfies the Thomsen condition and $\mathscr{A}$ is homogeneous. But unlike most physical examples, it is easy to verify that the automorphisms of velocity $\mathscr{A}$, do not enter at all as factors of the factorizable automorphisms of distance, $\mathscr{C}$, and equivalently that $\circ_A$ is not distributive.

It would be interesting to arrive at a generalization of Theorem 10 that covers relativistic velocity, but we do not have one.

6. Homogeneous numerical relational structures

The best known and most successful forms of measurement exhibit two significant features. First, the qualitative structure is mapped *onto* a real interval, often Re^+ or Re. Second, the representation is unique either up to similarity transformations – ratio scales – or up to affine transformations – interval scales. Assuming the former – a real representation – one problem is to find conditions for such ratio and interval representations to exist. Narens [7, 8] raised this question and formulated the following answer in terms of conditions on the automorphisms of the structure.

Definition 9. Suppose $\mathscr{A} = \langle A, \succsim, S_1, \ldots, S_k \rangle$ is a relational structure (S_i are relations on nonempty A) and G is a subgroup of its automorphisms. G acting on $\mathscr{A}$ satisfies *N-point homogeneity* iff for every $a_1, \ldots, a_N$ and $b_1, \ldots, b_N \in A$ with $a_{i+1} \succ a_i$, $b_{i+1} \succ b_i$, $i = 1, \ldots, N-1$, there exists $\theta \in G$ such that $\theta(a_i) = b_i$, $i = 1, \ldots, N$. G acting on $\mathscr{A}$ satisfies *N-point uniqueness* iff for all $\theta, \theta' \in G$ and $a_1, \ldots, a_N \in A$, $a_{i+1} \succ a_i$, $i = 1, \ldots, N-1$, if $\theta(a_i) = \theta'(a_i)$, then $\theta \equiv \theta'$.

Lemma 2. (i) *If a structure satisfies N-point homogeneity, then it satisfies M-point homogeneity for every integer $M \leq N$.*

(ii) *If a structure satisfies N-point uniqueness, then it satisfies M-point uniqueness for every integer $M \geq N$.*

(iii) *If a structure satisfies N-point homogeneity and M-point uniqueness, then $M \geq N$.*

Proof. (i) and (ii) are immediate. (iii) Suppose $M < N$. Select two sequences $a_1 \prec \cdots \prec a_M \prec a_{M+1} \prec \cdots \prec a_N$ and $a_1 \prec \cdots \prec a_M \prec b_{M+1} \prec \cdots \prec b_N$, where $b_i \neq a_i$, $i = M+1, \ldots, N$. By N-point homogeneity, there is an automorphism α that takes the first into the second. But since α agrees with the identity on $a_1, \ldots, a_M$, it follows from M-point uniqueness that α is the identity, in which case $b_i = \alpha(a_i) = a_i$, contrary to choice. So $M \geq N$. $\square$

Theorem 11 (Narens, [7, 8]). *Suppose a numerical structure $\mathscr{R} = \langle R, \geq, S_1, \ldots, S_k \rangle$ on a real open interval R satisfies both N-point homogeneity and N-point uniqueness.*

1. *If $N = 1$, then there is a homomorphism of $\mathscr{R}$ onto* Re^+ *such that its automorphisms are similarity transformations (rx, where $r > 0$).*
2. *If $N = 2$, there is a homomorphism of $\mathscr{R}$ onto* Re *such that its automorphisms are affine transformations ($rx + s$ where $r > 0$).*
3. *No real structure with $N \geq 3$ exists.*

7. Component homogeneity and uniqueness in conjoint structures with factorizable automorphisms

Although much of what is ultimately asserted about the possible numerical representations of Dedekind complete conjoint structures with factorizable automorphisms is true under homogeneity and uniqueness as just defined for orderings, some of the following proofs are valid only under the following conditions defined for cartesian products.

Definition 10. Suppose $\mathscr{C}=\langle A\times P, \succsim\rangle$ is a conjoint structure. Its factorizable automorphisms satisfy *component N-point homogeneity* iff for every $a_1,\dots,a_N$, $b_1,\dots,b_N\in A$, $p_1,\dots,p_N$, $q_1,\dots,q_N\in P$ such that for all $i=1,\dots,N-1$, $a_{i+1}\succ_A a_i$, $b_{i+1}\succ_A b_i$, $p_{i+1}\succ_P p_i$, and $q_{i+1}\succ_P q_i$, then there are factorizable automorphisms $\langle\theta,\eta\rangle$ and $\langle\theta',\eta'\rangle$ such that

$$\theta(a_i)=b_i,\quad i=1,\dots,N,\qquad \eta(p_N)=q_N$$

$$\theta'(a_N)=b_N,\qquad \eta'(p_i)=q_i,\quad i=1,\dots,N.$$

They satisfy *component N-point uniqueness* iff for $a_1,\dots,a_N,p_1,\dots,p_N$ with $a_{i+1},p_{i+1}\succ a_ip_i$, $i=1,\dots,N-1$, and factorizable automorphisms $\langle\theta,\eta\rangle,\langle\theta',\eta'\rangle$ with $\theta(a_i)=\theta'(a_i)$, $\eta(p_i)=\eta'(p_i)$, $i=1,\dots,N$, if $a_{i+1}\succ_A a_i$, $i=1,\dots,N-1$, then $\theta=\theta'$, or if $p_{i+1}\succ_P p_i$, $i=1,\dots,N-1$, then $\eta=\eta'$.

Note that if the factorizable automorphisms satisfy component N-point homogeneity, they satisfy N-point homogeneity; whereas if they satisfy N-point uniqueness they satisfy component N-point uniqueness.

Definition 11. Suppose $\mathscr{C}=\langle A\times P, \succsim\rangle$ is a conjoint structure with a non-trivial set $\mathscr{F}$ of factorizable automophisms, then point $a_0p_0\in A\times P$ is an *intrinsic zero* of $\mathscr{C}$ iff for every $\langle\theta,\eta\rangle\in\mathscr{F}$, $\theta(a_0)=a_0$ and $\eta(p_0)=p_0$.

The following theorem is stated asymmetrically with results about the components formulated only for the A-component; parallel results hold for the P-component.

Theorem 12. *Suppose $\mathscr{C}$ is an unrestrictedly solvable conjoint structure with $\mathscr{F}$ its group of factorizable automorphisms and $\mathscr{F}_A$ and $\mathscr{F}_P$ its component groups.*

1. *$\mathscr{F}$ on $\mathscr{C}$ satisfies either component* 1- *or* 2-*point uniqueness.*
2. *Suppose $\mathscr{C}$ has an intrinsic zero, then*

   (i) *$\mathscr{F}$ on $\mathscr{C}$ satisfies component* 1-*point uniqueness.*

   (ii) *If $\mathscr{F}$ on $\langle A^+\times P^+, \succsim\rangle$ satisfies component* 1-*point homogeneity, then $\mathscr{F}_A$ is the automorphism group of $\mathscr{J}_{A^+}$ and it satisfies* 1-*point homogeneity.*

   (iii) *If $\mathscr{F}_A$ on $\mathscr{J}_{A^+}$ satisfies* 1-*point homogeneity, then $\mathscr{F}_A$ on $\mathscr{J}_{A^-}$ also satisfies* 1-*point homogeneity and both satisfy* 1-*point uniqueness.*

3. *Suppose $\mathscr{F}$ on $\mathscr{C}$ satisfies component* 1*-point homogeneity and that for some* $a_0 \in A$, $p_0 \in P$,

$$\mathscr{F}_A^0 = \{\theta \,|\, \langle\theta, \eta\rangle \in \mathscr{F} \text{ are such that } \theta(a_0) = a_0,\ \eta(p_0) = p_0\}$$

satisfies 1*-point homogeneity on* $\langle A^+, \succsim_A\rangle$. *Then:*

(i) $\mathscr{F}_A$ *on* $\langle A, \succsim_A\rangle$ *satisfies* 2*-point homogeneity.*

(ii) *If, in addition,* $\mathscr{F}$ *has the property that for* $a, b \in A$ *and* $\theta, \theta' \in \mathscr{F}_A$ *with* $\theta(a) = \theta'(a)$ *and* $\theta(b) = \theta'(b)$, *there exist* $\eta, \eta' \in \mathscr{F}_P$ *and* $p \in P$ *such that* $\eta(p) = \eta'(p)$ *and* $\langle\theta, \eta\rangle, \langle\theta', \eta'\rangle \in \mathscr{F}$, *then* $\mathscr{F}_A$ *on* $\langle A, \succsim_A\rangle$ *satisfies* 2*-point uniqueness.*

4. *Suppose $\mathscr{F}$ on $\mathscr{C}$ satisfies component* 2*-point homogeneity. Then conclusions* (i) *and* (ii) *hold.*

Proof. 1. Suppose $\langle\theta, \eta\rangle, \langle\theta', \eta'\rangle \in \mathscr{F}$ agree at ap and bq, $a \prec_A b$. Let $\alpha = \theta^{-1}\theta'$, $\lambda = \eta^{-1}\eta'$. Observe, $\alpha(a) = a$, $\lambda(p) = p$. By Theorem 8, α is an automorphism of $\mathscr{J}_{A^+} = \langle A^+, \succsim_A, *_{ap}\rangle$. Since $\theta(b) = \theta'(b)$, $\alpha(b) = b$. By Theorem 2 and Theorem 2.1 of [2] α is the identity map, so $\theta = \theta'$ over A^+. For $c \in A^-$, by solvability there is $c^+ \in A^+$ and $c^+ *_{ap} c \sim a$. So

$$\theta(c^+) *_{ap} \theta(c) \sim \theta(c^+ *_{ap} c) \sim \theta(a) \sim \theta'(a) \sim \theta'(c^+ *_{ap} c)$$
$$\sim \theta'(c^+) *_{ap} \theta'(a) \sim \theta(c^+) *_{ap} \theta'(c),$$

so $\theta'(c) = \theta(c)$. By Theorem 8, $\eta = \eta'$, and so component 2-point uniqueness holds. This does not exclude the possibility of component 1-point uniqueness.

2. (i) Obvious from 1.

(ii) By Theorem 8, it is immediate that $\mathscr{F}_A$ restricted to A^+ is contained in the automorphism group of $\mathscr{J}_{A^+}$.

Suppose $a, b \in A^+$. Since $ap_0, bp_0 \succ a_0 p_0$, by hypothesis there exists $\langle\theta, \eta\rangle \in \mathscr{F}$ such that

$$bp_0 = \langle\theta, \eta\rangle(ap_0) = \theta(a)\eta(p_0) = \theta(a)p_0,$$

and so $\theta(a) = b$, establishing 1-point homogeneity over $\mathscr{J}_{A^+}$. By Theorem 2.1 of [2], the automorphism group of $\mathscr{J}_{A^+}$ satisfies one-point uniqueness and so the inclusion is not proper.

(ii) For each $a \in A^-$, define a^+ as the solution to $a^+\pi(a) \sim a_0 p_0$. By monotonicity, $a^+ \in A^+$. By Theorem 8, we know that the automorphisms of $\mathscr{J}_{A^+}$ and $\mathscr{J}_{A^-}$ are the restrictions of $\mathscr{F}_A$ to A^+ and A^-, so by 1-point homogeneity of $\mathscr{J}_{A^+}$ there exists for each $b \in A^-$ a $\theta \in \mathscr{F}_A$ such that $\theta(a^+) = b^+$. Since $\theta(a_0) = a_0$, Theorem 8 asserts $\langle\theta, \pi\theta\pi^{-1}\rangle$ is a factorizable automorphism, from which we see

$$b^+\pi(b) \sim a_0 p_0 \sim \theta(a^+), \pi\theta\pi^{-1}\pi(a) \sim b^+\pi\theta(a).$$

By monotonicity and the 1 : 1 property of π, $b = \theta(a)$. The 1-point uniqueness of both structures follows from Theorem 2.1 of [2].

3. (i) Suppose $a, b, c, d \in A$, $a \succ_A b$, $c \succ_A d$. By component 1-point homogeneity, there exist $\langle\theta, \eta\rangle, \langle\theta', \eta'\rangle \in \mathscr{F}$ such that $\theta(b) = a_0$ and $\theta'(d) = a_0$. By monotonicity

$\theta(a) \succ_A a_0$ and $\theta'(c) \succ_A a_0$. By the 1-point homogeneity of $\mathscr{F}_A^0$, there exists $\beta \in \mathscr{F}_A^0$ such that $\beta\theta(a) = \theta'(c)$. Set $\alpha = \theta'^{-1}\beta\theta$, then

$$\alpha(a) = \theta'^{-1}\beta\theta(a) = \theta'^{-1}\theta'(c) = c,$$

$$\alpha(b) = \theta'^{-1}\beta\theta(b) = \theta'^{-1}\beta(a_0) = \theta'^{-1}(a_0) = d.$$

(ii) Suppose $\theta, \theta' \in \mathscr{A}$ are such that $\theta(a) = \theta'(a)$, $\theta(b) = \theta'(b)$. By hypothesis there exist η, η' and p in P such that $\langle\theta, \eta\rangle, \langle\theta', \eta'\rangle \in \mathscr{F}$ and $\eta(p) = \eta'(p)$. So, $\langle\theta, \eta\rangle$ and $\langle\theta', \eta'\rangle$ agree at ap and bp, whence by component 2-point uniqueness $\theta = \theta'$. □

8. Real conjoint representations

In this and the following sections we shall suppose that the conjoint structure is Dedekind complete and so has a representation onto the real numbers; moreover, we shall suppose the induced structures are also isomorphisms to the real numbers. So we are working with a function $F: \mathrm{Re} \times \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}$ that is strictly increasing in each of its arguments. In the following sections, we shall make various smoothness assumptions about F; it is hoped that future work will show how to weaken some of them.

We would like to work out all possible real conjoint structures with rich sets of automorphisms, but our efforts are incomplete, partly because Theorem 11 does not provide a complete classification of possibilities. In what follows we work out a number of important cases, but there is more to be done in order to understand fully all conjoint structures with rich sets of factorizable automorphisms.

Definition 12. A function $F: \mathrm{Re} \times \mathrm{Re} \to \mathrm{Re}$ is a C^n *conjoint representation* of the conjoint structure $\langle \mathrm{Re} \times \mathrm{Re}, \succsim \rangle$, where for $x, y, u, v \in \mathrm{Re}$,

$$xy \succsim uv \quad \text{iff} \quad F(x, y) \geq F(u, v),$$

provided:

(i) $F(x, \cdot)$ and $F(\cdot, y)$ map Re onto Re.

(ii) F is strictly increasing and continuous in each variable.

(iii) F is C^n, i.e., it is continuously differentiable of order n.

Such a representation is *additive* iff there exist strictly increasing f_1, f_2 and $f: \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}$ such that

$$F(x, y) = f[f_1(x) + f_2(y)].$$

We say 0 is an identity iff for all $x \in \mathrm{Re}$,

$$F(x, 0) = F(0, x) = x.$$

In terms of such a numerical representation, a factorizable automorphism is a strictly increasing function $\alpha_{\theta,\eta}: \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}$ together with $1:1$ $\theta, \eta: \mathrm{Re} \to \mathrm{Re}$ such that for all $x, y \in \mathrm{Re}$,

$$F[\theta(x), \eta(y)] = \alpha_{\theta,\eta}[F(x, y)] \tag{1}$$

Lemma 3. *Suppose F is a C^0 conjoint representation and f_1, f_2, and f are strictly monotonic increasing functions from* Re *onto* Re. *Then $F^* = f[F(f_1, f_2)]$ is a C^0 conjoint representation and the two groups of factorizable automorphisms are isomorphic.*

Proof. The first assertion is obvious.

The isomorphic mapping from the group of factorizable automorphisms of F to those of F^* is $\langle\theta, \eta\rangle \to \langle f_1^{-1}\theta f_1, f_2^{-1}\eta f_2\rangle$. Observe,

$$\begin{aligned} F^*[f_1^{-1}\theta f_1(x), f_2^{-1}\eta f_2(y)] &= fF[f_1 f_1^{-1}\theta f_1(x), f_2 f_2^{-1}\eta f_2(y)] \\ &= fF[\theta f_1(x), \eta f_2(y)] \\ &= f\alpha_{\theta\eta} f^{-1} fF[f_1(x), f_2(y)] \\ &= f\alpha_{\theta\eta} f^{-1} F^*(x, y), \end{aligned}$$

and so $f\alpha_{\theta\eta}f^{-1}$ is a factorizable automorphism of F^*. It is clear this mapping is 1 : 1 and onto. □

It is often convenient in what follows to assume 0 is an identity of F in the sense that

$$F(x, 0) = F(0, x) = x.$$

This rescaling is always possible by letting f_1 and f_2 be such that

$$F[f_1^{-1}(x), 0] = x, \qquad F[0, f_2^{-1}(y)] = y$$

and considering $F(f_1, f_2)$. According to the Lemma, this rescaling does not affect the factorizable automorphisms.

9. Real conjoint representations with an intrinsic zero and component one-point homogeneity and one-point uniqueness on its positive part

Theorem 13. *Suppose $\mathscr{C}$ has a real conjoint representation and $\mathscr{F}$ is its group of factorizable automorphisms. If $\mathscr{C}$ has an intrinsic zero a_0p_0, is unrestrictedly solvable, and $\mathscr{F}$ on $\langle A^+ \times P^+, \succsim\rangle$ satisfies component* 1*-point homogeneity and* 1*-point uniqueness, then there exist functions $\phi_A : A \xrightarrow{\text{onto}} \mathrm{Re}$, $\phi_P : P \xrightarrow{\text{onto}} \mathrm{Re}$, $\phi_A(a_0) = 0$, $\phi_P(p_0) = 0$, and a function $F : \mathrm{Re} \times \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}$ that is strictly increasing in both variables such that $F(\phi_A, \phi_P)$ is a representation of $\mathscr{C}$ and F is of the following form: There exist strictly increasing functions $f_+ : \mathrm{Re} \xrightarrow{\text{onto}} [1, \infty)$ and $f_- : \mathrm{Re} \xrightarrow{\text{onto}} (-\infty, -1]$, where f_+ and $-f_-$ satisfy the properties of a unit representa-*

tion (Theorem 3.3 *of* [2]) *and*

$$F(x,y)=\begin{cases}|y|\,f_{\text{sign}\,y}(x/|y|), & y\neq 0,\\ x & y=0.\end{cases}$$

Proof. By Theorem 12, both $\mathscr{C}$ and its components satisfy 1-point homogeneity and uniqueness under $\mathscr{F}$, $\mathscr{F}_A$, $\mathscr{F}_P$ on, respectively, $\mathscr{C}/\sim$, $\langle A^+, \succsim_A\rangle$, $\langle A^-, \succsim_A\rangle$, $\langle P^+, \succsim_P\rangle$, and $\langle P^-, \succsim_P\rangle$. So, by Theorem 11, each has a real representation with a_0 and p_0 mapping onto 0 in which the automorphisms of $\mathscr{F}$, $\mathscr{F}_A$ and $\mathscr{F}_P$ are represented by multiplication by positive constants. Thus, there exists $G:\mathrm{Re}\times\mathrm{Re}\xrightarrow{\text{onto}}\mathrm{Re}$, $\varrho:\mathrm{Re}^+\times\mathrm{Re}^+\to\mathrm{Re}^+$ such that for all $x, y\in\mathrm{Re}$, and all pairs $r, r'\in\mathrm{Re}^+$ such that they are factors of a factorizable automorphism

$$G(rx, r'y)=\varrho(r,r')G(x,y). \tag{2}$$

Let π denote the solutions relative to the intrinsic zero a_0p_0, then according to Theorem 8, the factorizable automorphisms must satisfy

$$r'y=\pi[r\pi^{-1}(y)].$$

Set $y=1$, solve for r, and substitute back to obtain

$$\frac{\pi^{-1}(r'y)}{\pi^{-1}(1)}=\frac{\pi^{-1}(r')}{\pi^{-1}(1)}\,\frac{\pi^{-1}(y)}{\pi^{-1}(1)}\,.$$

Since π^{-1} is monotonic, it is well known that for some β,

$$\pi^{-1}(y)=\pi^{-1}(1)y^{1/\beta},$$

whence

$$r'=r^{\beta}.$$

Setting

$$\varrho(r)=\varrho(r,r^{\beta}),$$

Eq. (2) becomes

$$G(rx, r^{\beta}y)=\varrho(r)G(x,y). \tag{3}$$

Observe

$$G(0, r^{\beta}y)=\varrho(r)G(0,y).$$

Setting $y=1$, solving for $\varrho(r)$, and substituting back and solving yields

$$\varrho(r)=r^{\beta\gamma}$$

for some γ. So the functional equation to be solved is

$$G(rx, r^{\beta}y)=r^{\beta\gamma}G(x,y). \tag{4}$$

For $y\neq 0$, Eq. (4) yields

$$\begin{aligned}G(x,y)&=G[|y|^{1/\beta}x/|y|^{1/\beta}, (\text{sign } y)(|y|^{1/\beta})^{\beta}]\\ &=(|y|^{1/\beta})^{\beta\gamma}G[x/|y|^{1/\beta}, (\text{sign } y)1]\\ &=|y|^{\gamma}G[x/|y|^{1/\beta}, (\text{sign } y)1].\end{aligned}$$

For $y=0$,

$$G(x,0)=G[(\operatorname{sign} x)|x|,0]=|x|^{\beta\gamma}G[(\operatorname{sign} x)1,0].$$

Note that by the fact G is strictly increasing in each variable

$$G(-1,0)<G(0,0)=0<G(1,0)$$

and for $z<0<y$,

$$|z|^{\gamma}G(0,-1)=G(0,z)<G(0,0)=0<G(0,y)=y^{\gamma}G(0,1),$$

and so

$$G(0,-1)<0<G(0,1).$$

Observe that change of variable involving multiplication by positive constants and taking powers (suitably correcting for minus signs) does not affect the representation. In particular,

$$F(x,y)=\operatorname{sign} G(x,y)\left|G\left\{\frac{x}{|G[(\operatorname{sign} x)1,0]|^{1/\beta\gamma}},(\operatorname{sign} y)|y|^{\beta}\right\}\right|^{1/\beta\gamma}$$

$$=\begin{cases} yf_{\operatorname{sign} y}(x/|y|), & y\neq 0,\\ x, & y=0,\end{cases}$$

where

$$f_{\operatorname{sign} y}(x(|y|)=(\operatorname{sign} y)|G\{x/|G[(\operatorname{sign} x)1,0]|^{1/\beta\gamma}|y|,(\operatorname{sign} y)1\}|^{1/\beta\gamma}$$

To show that f is a unit representation, observe that $F(x,0)=F(0,\pi(x))$ holds iff $\pi(x)=x$ and $F(x *_A y,0)=F(x,\pi(y))$ iff

$$x *_A y=|y|f_{\operatorname{sign} y}(x/|y|).$$

Since the induced structures $\mathscr{J}_{A^+}$ and $\mathscr{J}_{A^-}$ are Dedekind complete, 1-point homogeneous, positive concatenation structures, it follows from Theorem 3.1 of [2] that both f_+ and $-f_-$ are unit representations. □

If the conjoint structure is invertible (Definition 5) then f_+ and f_- may be chosen so that $f_+(x)=-f_{-1}(-x)$. The proof of this simple fact is left to the reader.

10. Real conjoint representations with component one-point homogeneity

The goal of this section is to show that with F a C^4 representation, component one-point homogeneity is sufficient to force additivity.

We need the following known result.

Theorem 14. (Differentiation under integrals.) *Suppose F is a C^1 conjoint representation that is uniformly continuous and F_x is continuous on a domain $(x-\varepsilon,x+\varepsilon)\times(u-\varepsilon,v+\varepsilon)$. Then for $z\in(x-\varepsilon,x+\varepsilon)$,*

$$Q(z)=\int_u^v F(x,y)\,\mathrm{d}y$$

is continuously differentiable and

$$Q'(z) = \int_u^v F_x(z, y)\, \mathrm{d}y.$$

The first major result of this section shows that a factorizable automorphism $\langle \theta, \eta \rangle$ of a sufficiently smooth representation is completely determined by knowing its values at 0 together with $\theta'(0)$.

Theorem 15. *Suppose F is a C^2 conjoint representation for which 0 is the identity and for all x, y $F_x(x, y) \neq 0$ and $F_y(x, y) \neq 0$. If $\langle \theta, \eta \rangle$ is a factorizable automorphism of F, then θ and η are C^1 and satisfy the following differential equations:*

$$\theta'(x) = \frac{\theta'(0)}{F_y(x,0)} \frac{F_y[\theta(x), \eta(0)]}{F_x[\theta(x), \eta(0)]} \frac{F_x[\theta(0), \eta(0)]}{F_y[\theta(0), \eta(0)]}, \tag{5}$$

$$\eta'(y) = \frac{\theta'(0)}{F_x(0, y)} \frac{F_x[\theta(0), \eta(y)]}{F_y[\theta(0), \eta(y)]}. \tag{6}$$

Proof. Integrate the defining relation of a factorizable automorphism, Eq. (1), over the second variable:

$$W(x) = \int_u^v F[\theta(x), \eta(y)]\, \mathrm{d}y = \int_u^v \alpha_{\theta\eta}[F(x, y)]\, \mathrm{d}y. \tag{7}$$

Letting $z = F(x, y)$ and $g(x, z)$ be the solution to $z = F[x, g(x, z)]$, which exists by the solvability of F and the implicit function theorem, we make the change of variable in the right integral:

$$W(x) = \int_{F(x,u)}^{F(x,v)} \frac{\alpha_{\theta\eta}(z)}{F_y[x, g(x, z)]}\, \mathrm{d}z.$$

Theorem 14, the chain rule of differentiation, the implicit function theorem, and the fact that F is C^2 yields that W is C^1.

Let

$$L(v) = \int_0^v F[v, \eta(x)]\, \mathrm{d}x.$$

Observe, L is a strictly monotonic C^1 function and by Theorem 14 it has a positive derivative everywhere. By the inverse function theorem, L^{-1} is C^1 and therefore $L^{-1}W(x) = \theta(x)$ (Eq. (7)) is as well. A similar argument shows that η is C^1.

Using Eq. (1) and the fact that 0 is the identity,

$$\alpha_{\theta\eta}(x) = \alpha_{\theta\eta} F(x, 0) = F[\theta(x), \eta(0)].$$

Since F and θ are C^1, $\alpha_{\theta\eta}$ is also C^1. So, from Eq. (1),

$$F_x[\theta(x), \eta(y)]\theta'(x) = \alpha'_{\theta\eta}[F(x, y)] F_x(x, y), \tag{8a}$$

$$F_y[\theta(x), \eta(y)]\eta'(y) = \alpha'_{\theta\eta}[F(x, y)] F_y(x, y). \tag{8b}$$

If $\alpha'_{\theta\eta}(z)=0$, then from the fact that for each x and z there is a y with $z=F(x, y)$ and Eq. (8a) we conclude $\theta'\equiv 0$. As this is impossible, $\alpha'_{\theta\eta}\neq 0$ and so,

$$\frac{\theta'(x)}{\eta'(y)}=\frac{F_x(x, y)}{F_y(x, y)}\frac{F_y[\theta(x), \eta(y)]}{F_x[\theta(x), \eta(y)]}. \tag{9}$$

Let $y=0$ and note $F_x(x, 0)=1$,

$$\theta'(x)=\frac{\eta'(0)}{F_y(x, 0)}\frac{F_y[\theta(x), \eta(0)]}{F_x[\theta(0), \eta(0)]}.$$

Letting $x=y=0$ yields

$$\eta'(0)=\theta'(0)\frac{F_x[\theta(0), \eta(0)]}{F_y[\theta(0), \eta(0)]}.$$

Substituting yields Eq. (5). Equation 6 follows from Eqs. (5) and (9). □

Explicit formulas may be obtained for θ and η by integrating Eqs. (5) and (6). Define

$$R_v(z)=\int_0^z \frac{F_x(x, v)}{F_y(x, v)}\,\mathrm{d}x, \qquad S_u(z)=\int_0^z \frac{F_y(u, y)}{F_x(u, y)}\,\mathrm{d}y,$$

then

$$\theta(x)=R^{-1}_{\eta(0)}\left\{\theta'(0)\frac{F_y[\theta(0), \eta(0)]}{F_x[\theta(0), \eta(0)]}R_0(x)+R_{\eta(0)}[\theta(0)]\right\},$$

$$\eta(y)=S^{-1}_{\theta(0)}\{\theta'(0)S_0(x)+S_{\theta(0)}[\eta(0)]\}.$$

The next result formulates a condition for the additivity of F that is somewhat different from the well known result of Scheffé [10] (see [5], Sec. 6.5.3). It has two advantages: it only assumes F is C^1 instead of C^2, and it is often very easy to verify.

Theorem 16. *Suppose F is a C^1 conjoint representation. Let*

$$\Psi(x, y)=\log F_x(x, y)/F_y(x, y). \tag{10}$$

Then F is additive iff there exist functions ψ_1 and ψ_2 such that e^{ψ_1} and e^{ψ_2} are integrable functions on any bounded domain and

$$\Psi(x, y)=\psi_1(x)+\psi_2(y). \tag{11}$$

Proof. Suppose F is additive, i.e., for C^1 *functions* f, f_1, and f_2,

$$F(x, y)=f[f_1(x)+f_2(y)].$$

Calculating the two first partials of F,

$$\frac{F_x(x, y)}{F_y(x, y)}=\frac{f'[f_1(x)+f_2(y)]f_1'(x)}{f'[f_1(x)+f_2(y)]f_2'(y)}=\frac{f_1'(x)}{f_2'(y)}.$$

Taking logarithms, Eq. (11) follows.

Suppose Eq. (11) holds. Consider

$$H(x, y) = \int_0^x e^{\psi_1(z)}\, dz + \int_0^y e^{-\psi_2(z)}\, dz. \tag{12}$$

Since

$$H_x(x, y) = e^{\psi_1(x)} \quad \text{and} \quad H_y(x, y) = e^{-\psi_2(y)},$$

it follows from Eqs. (10) and (11) that

$$\frac{F_x(x, y)}{F_y(x, y)} = \frac{H_x(x, y)}{H_y(x, y)}, \tag{13}$$

and we show this implies F is a function of H and so, by Eq. (12), is additive. Consider the system of differential equations

$$\dot{x}(t) = H_y \quad \text{and} \quad \dot{y}(t) = -H_x.$$

Thus, H is a hamiltonian of this system and so H is constant along its trajectories. Indeed, the fact that $H_x H_y \neq 0$ means that the trajectories of the above differential equations are uniquely parameterized by this condition. But for Eq. (11) to hold, H must also be constant along the same trajectories, proving F is a function of H. □

Corollary 1. *The following constitutes an additive representation of F. Let* $\psi_1'(x) = \Psi_x(x, y)$, $\psi_2'(y) = \Psi_y(x, y)$,

$$f_1(x) = \int_0^x \exp\left[\int_0^u \psi_1'(v)\, dv\right] du,$$

$$f_2(y) = \frac{F_y(0,0)}{F_x(0,0)} \int_0^y \exp\left[-\int_0^u \psi_2'(v)\, dv\right] du,$$

$$f(z) = \int_0^z F_x[f_1^{-1}(u), 0] \int_0^{f_1^{-1}(u)} \exp[-\psi_1(x)\, dz]\, du + F(0,0).$$

Proof. Integrating ψ_1' and ψ_2' using the definition of Ψ and assuming F has an additive representation yields

$$f_1(x) = f_1'(0) \int_0^x \exp \int_0^u \psi_1'(v)\, dv\, du + f_1(0),$$

$$f_2(y) = f_2'(0) \int_0^y \exp\left[-\int_0^u \psi_2'(v)\, dv\right] du + f_2(0).$$

By scaling and shifting we may arrange that $f_1(0) = 0$, $f_2(0) = 0$, and $f_1'(0) = 1$. We thus obtain the representation for f_1. To obtain that for f_2 observe that

$$\frac{F_y(0,0)}{F_x(0,0)} = \frac{f_2'(0)}{f_1'(0)} = f_2'(0).$$

To get the representation of f, observe

$$F_x(x, y) = f'[f_1(x) + f_2(y)]f_1'(x).$$

If we let $y=0$ and set $u=f_1(x)$, we obtain

$$f'(u) = F_x[f_1^{-1}(u), 0]/f_1'[f_1^{-1}(u)].$$

Computing f_1' from the expression for f_1, substituting, and integrating yields the result. $\square$

Corollary 2. *Suppose F is C^3, $F_x \neq 0$, and $F_y \neq 0$. Then F is additive iff $\Psi_{xy} \equiv 0$.*

Proof. Obviously $\Psi_{xy} \equiv 0$ is equivalent to Eq. (11). $\square$

Theorem 17. *Suppose F is a C^4 conjoint representation for which 0 is the identity, component 1-point homogeneity holds, $F_x \neq 0$, and $F_y \neq 0$. Then, F is additive.*

Proof. For any factorizable automorphism θ, η, Eqs. (5), (6), and (10) yield

$$\frac{\theta'(x)}{\eta'(y)} = \exp\{\Psi[\theta(0), \eta(0)] - \Psi[\theta(x), \eta(0)] - \Psi[\theta(0), \eta(y)] + \Psi(x, 0) + \Psi(0, y)\}$$

and from Eqs. (9) and (10),

$$\frac{\theta'(x)}{\eta'(y)} = \exp\{\Psi(x, y) - \Psi[\theta(x), \eta(y)]\}.$$

Equating and rearranging,

$$\begin{aligned} \Psi(x, y) - \Psi(x, 0) - \Psi(0, y) = \Psi[\theta(x), \eta(y)] &- \Psi[\theta(x), \eta(0)] \\ &- \Psi[\theta(0), \eta(y)] + \Psi[\theta(0), \eta(0)]. \end{aligned} \tag{14}$$

Since F is C^3, we may take the second partial of Eq. (14),

$$\Psi_{xy}(x, y) = \Psi_{xy}[\theta(x), \eta(y)]\theta'(x)\eta'(y). \tag{15}$$

From Eq. (15) and the assumption of component one-point homogeneity of the factorizable automorphisms, it follows that if $\Psi_{xy} = 0$ at one point, it is everywhere 0. So, we assume $\Psi_{xy} \neq 0$ everywhere and derive a contradiction.

Since F is C^4, take the two third partials

$$\Psi_{xxy}(x, y) = \Psi_{xy}[\theta(x), \eta(y)]\theta''(x)\eta'(y) + \Psi_{xxy}[\theta(x), \eta(y)]\theta'(x)^2\eta'(y),$$

$$\Psi_{xyy}(x, y) = \Psi_{xy}[\theta(x), \eta(y)]\theta'(x)\eta''(y) + \Psi_{xyy}[\theta(x), \eta(y)]\theta'(x)\eta'(y)^2.$$

Set $x=y=0$, by component 1-point homogeneity choose $\theta(0)=x, \eta(0)=y$, and set

$$\beta_1 = \Psi_{xxy}(0, 0) \quad \text{and} \quad \beta_2 = \Psi_{xyy}(0, 0).$$

Then,

$$\Psi_{xxy}(x,y)=\frac{\beta_1-\Psi_{xy}(x,y)\theta''(0)\eta'(0)}{\theta'(0)^2\eta'(0)}, \tag{16a}$$

$$\Psi_{xyy}(x,y)=\frac{\beta_2-\Psi_{xy}(x,y)\theta'(0)\eta''(0)}{\theta'(0)\eta'(0)^2}. \tag{16b}$$

In Eqs. (5) and (6), introduce Ψ notation, differentiate, and set $x=y=0$:

$$\frac{\theta''(0)}{\theta'(0)}=-\Psi_x(x,y)\theta'(0)-\beta_3, \tag{17a}$$

$$\frac{\eta''(0)}{\eta'(0)}=\Psi_y(x,y)\eta'(0)-\beta_3, \tag{17b}$$

where

$$\beta_3=-\Psi_x(0,0)=\Psi_y(0,0)=-F_{xy}(0,0).$$

Setting $x=y=0$ and $\theta(0)=x$, $\eta(0)=y$ in Eq. (15) yields

$$\Psi_{xy}(0,0)=\Psi_{xy}(x,y)\theta'(0)\eta'(0).$$

Substitute $\eta'(0)$ from Eq. (6) and solve for $\theta'(0)$ recall $\Psi_{xy}\neq 0$):

$$\theta'(0)=[\Psi_{xy}(0,0)/\Psi_{xy}(x,y)]^{1/2}e^{-\Psi(x,y)/2}. \tag{18a}$$

Using Eq. (9)

$$\eta'(0)=[\Psi_{xy}(0,0)/\Psi_{xy}(x,y)]^{1/2}e^{+\Psi(x,y)/2}. \tag{18b}$$

Substitute Eqs. (17) and (18) in (16) and set

$$\gamma_i=\frac{\beta_i+\beta_3\psi_{xy}(0,0)}{\Psi_{xy}(0,0)^{3/2}}$$

to obtain

$$\Psi_{xxy}(x,y)=\gamma_1\Psi_{xy}(x,y)^{3/2}e^{\Psi(x,y)/2}+\Psi_x(x,y)\Psi_{xy}(x,y), \tag{19a}$$

$$\Psi_{xyy}(x,y)=\gamma_2\Psi_{xy}(x,y)^{3/2}e^{-\Psi(x,y)/2}-\Psi_y(x,y)\Psi_{xy}(x,y). \tag{19b}$$

Taking the partial of Eq. (19a) with respect to y and that of (19b) with respect to x, both of which exist because all terms on the right are at least C^1, and equating them because they are the same thing yields

$$2\Psi_{xy}^2=\Psi_{xxy}[-\Psi_y+\tfrac{3}{2}\gamma_2e^{-\Psi/2}\Psi_{xy}^{1/2}]+\Psi_{xyy}[-\Psi_x-\tfrac{3}{2}\gamma_1e^{\Psi/2}\Psi_{xy}^{1/2}]$$
$$-\tfrac{1}{2}\gamma_1\Psi_{xy}^{3/2}e^{\Psi/2}\Psi_y-\tfrac{1}{2}\gamma_2\Psi_{xy}^{3/2}e^{-\Psi/2}\Psi_x.$$

Substitute for Ψ_{xxy} and Ψ_{xyy} from Eq. (17) and collect terms to show $\Psi_{xy}\equiv 0$. This contradiction means $\Psi_{xy}\equiv 0$ and so by Corollary 2 to Theorem 15, F is additive. □

11. Real conjoint representations with component one and a half-point homogeneity

The case of an additive structure is well known to entail transformations involving three free parameters, the unit of the two transformations being the same. This suggests considering component homogeneity in which one of the factors satisfies 2-point homogeneity and the other only 1-point; we call this *component* $1-\frac{1}{2}$*-point homogeneity*. As might be guessed, this permits us to relax the smoothness requirement on F.

Theorem 18. *Suppose F is a C^3 conjoint representation for which 0 is the identity and component $1-\frac{1}{2}$-point homogeneity holds, $F_x \neq 0$, and $F \neq 0$. Then, F is additive.*

Proof. Consider Eq. (14) of Theorem 17; its derivation only used C^2. Take the partial with respect to x:

$$\Psi_x(x,y) - \Psi_x(x,0) = \Psi_x[\theta(x), \eta(y)]\theta'(x) - \Psi_x[\theta(x), \eta(0)]\theta'(x).$$

By component $1-\frac{1}{2}$-point homogeneity, choose θ and η so that $\theta(x)=0$ and

$$\begin{matrix} \eta(0)=0 \\ \eta(y)=1 \end{matrix} \quad \text{if } y>0 \quad \text{and} \quad \begin{matrix} \eta(0)=1 \\ \eta(y)=0 \end{matrix} \quad \text{if } y<0.$$

Thus,

$$\Psi_x(x,y) = \Psi_x(x,0) + \theta'(x)(\text{sign } y)[\Psi_x(0,1) - \Psi_x(0,0)],$$

which depends only on x except for sign y. Thus, for any $y > \varepsilon > 0$,

$$\Psi_{xy}(x,y) = 0.$$

By Eq. (15) and component 1-point homogeneity, this means $\Psi_{xy} \equiv 0$, and so F is additive. □

12. Real conjoint representations with two-point uniqueness of the structure and its factors and component two-point homogeneity

Theorem 19. *Suppose F is a C^0 conjoint representation that is unrestrictedly solvable, that its group $\mathcal{F}$ of factorizable automorphisms satisfy both 2-point uniqueness (and hence component 2-point uniqueness) and component 2-point homogeneity (and hence 2-point homogeneity), and that $\mathcal{F}_A$ on $\langle A, \succsim_A \rangle$ and $\mathcal{F}_P$ on $\langle P, \succsim_P \rangle$ satisfy 2-point uniqueness. Then, F is additive.*

Proof. The proof follows an approach similar to that of Theorem 13, but with similarities replaced by affine transformations. Since $\mathcal{F}$ satisfies both 2-point homogeneity and uniqueness, by Theorem 11 we may rescale F in such a way that the factorizable automorphisms are affine transformations. By part 4 of Theorem 12 and the hypothesis, $\mathscr{F}_A$ on $\langle A, \succsim_A\rangle$ and $\mathscr{F}_P$ on $\langle P, \succsim_P\rangle$ each satisfy both 2-point homogeneity and uniqueness, and so a further rescaling leads to this representation as affine transformations. Thus, there is no loss of generalizability in assuming F is rescaled so that there are functions $\varrho: \mathrm{Re}^+ \times \mathrm{Re} \times \mathrm{Re}^+ \times \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}^+$ and $\sigma: \mathrm{Re}^+ \times \mathrm{Re} \times \mathrm{Re}^+ \times \mathrm{Re} \xrightarrow{\text{onto}} \mathrm{Re}$ such that for some $r, r' \in R^+$ and some $s, s' \in \mathrm{Re}$, and all $x, y \in \mathrm{Re}$

$$F(rx+s, r'x+s') = \varrho(r,s,r',s')F(x,y) + \sigma(r,x,r',s'). \tag{20}$$

For any real x_0, x_0', y_0, y_0' by component 1-point homogeneity there is a factorizable automorphism $(x,y) \to (rx+s, r'y+s')$ such that

$$rx_0 + s = x_0', \qquad r'y_0 + s' = y_0'. \tag{21}$$

Let π and π' be the solutions defined relative to $x_0 y_0$ and $x_0' y_0'$, respectively. By Theorem 8, we know this automorphism must satisfy

$$r'y + s' = \pi'[r\pi^{-1}(y) + s]. \tag{22}$$

Substituting Eq. (21) into Eq. (22) yields

$$\pi'^{-1}[r'(y-y_0) + y_0'] = r[\pi^{-1}(y) - x_0] + x_0'. \tag{23}$$

Define

$$f(x) = \pi^{-1}(x+y_0) - x_0, \qquad f(y)' = \pi'^{-1}(y+y_0') - x_0', \tag{24}$$

thus Eq. (23) becomes

$$f(r'x)' = rf(x). \tag{25}$$

Set $x=1$, solve for r and substitute:

$$f(r'x)' = f(r')'f(x)/f(1).$$

Note that by Eq. (25), with $x=1$, the value $r = f(1)'/f(1)$ corresponds to $r'=1$. Set $r'=1$, solve for $f(x)$, and substitute:

$$f(r'x)' = f(r')'f(x)'/f(1)'.$$

Thus, for some β

$$\frac{f(x)}{f(1)} = \frac{f(x)'}{f(1)'} = \begin{cases} x^{1/\beta}, & x \geq 0, \\ -|x|^{1/\beta}, & x < 0, \end{cases} \tag{26}$$

$$r' = r^\beta.$$

Observe that β appears to depend upon x_0, x_0', y_0, y_0'. However, since any π can be paired with any π', and so any r with any r', it follows that all the β's must be the same.

We next show that for any real s, $(x,y) \to (x+s, y)$ is a factorizable automorphism. Fix x_0, y_0 and let $x_0' = x_0 + s$ and $y_0' = y_0$. The question is whether the condition from

Theorem 8, namely for all y

$$y = \pi'[\pi^{-1}(y) + s],$$

is satisfied. From Eq. (25) with $r = 1$ and $x = 1$, we see that $f(1)' = f(1)$. Using this and Eqs. (24) and (26) we see

$$\pi^{-1}(y) = x_0 + f(1)\begin{cases}(y - y_0)^{1/\beta}, & y \geq y_0,\\ -(y_0 - y)^{1/\beta}, & y < y_0,\end{cases}$$

$$\pi'(z) = y_0 + f(1)^{-\beta}\begin{cases}(z - x_0 - s)^{\beta}, & z \geq x_0 + s,\\ -(x_0 + s - z)^{\beta}, & z < x_0 + s.\end{cases}$$

Thus,

$$\begin{aligned}\pi'[\pi^{-1}(y) + s] &= y_0 + f(1)^{-\beta}\begin{cases}[\pi^{-1}(y) + s - x_0 - s]^{\beta}, & y \geq \pi(x_0) = y_0,\\ -[x_0 + s - \pi^{-1}(y) - s]^{\beta}, & y < y_0\end{cases}\\ &= y_0 + f(1)^{-\beta}\begin{cases}[f(1)(y - y_0)^{1/\beta}]^{\beta}, & y \geq y_0,\\ -[f(1)(y_0 - y)^{1/\beta}]^{\beta}, & y < y_0\end{cases}\\ &= y.\end{aligned}$$

From this and Eq. (22) we have

$$F(x + s, y) = \varrho(s)F(x, y) + \sigma(s).$$

With no loss of generality (Lemma 3), assume 0 is an identity of F. If we set $y = 0$,

$$s + x = F(x + s, 0) = \varrho(s)F(x, 0) + \pi(s) = \varrho(s)x) + \sigma(s).$$

Next set $x = 0$ and use the above relation,

$$F(s, y) = \varrho(s)F(0, y) + \sigma(s) = \varrho(s)y + \sigma(s) = s + y,$$

proving F is additive. $\square$

13. Concluding remarks

Abstracting certain features of physical measurement, we have examined those conjoint structures in which there are non-trivial factorizable automorphisms. One important result is the fact that $\langle\theta, \eta\rangle$ is a factorizable automorphism iff $\eta = \pi'\theta\pi^{-1}$ where π and π' are solutions of the structure mapping the first component onto the second and θ is an isomorphism between induced structures (Theorem 8). It turned out that the conjoint structure is additive iff translations of the identity of the induced concatenation structure form factorizable automorphisms (Theorem 9). And this in turn was shown to be very closely tied in with the concept of a distributive operation on one of the components (Theorem 10). The final sections combine together two additional features of physical measurement, namely, that the structures are Dedekind complete, and so have representations onto real intervals, and that the automorphisms exhibit what we have called component homogeneity and uniqueness properties. In case the structure has an intrinsic zero and satisfies component 1-point homogeneity and 1-point uniqueness on its positive part, then it has a representation that is expressed in terms of unit structures (Theorem 13). The remaining theorems (15, 17, 18, and 19) combine in several ways assumptions about the smoothness of F with properties of the factorizable automorphisms that force F to be additive. In the course of this, we establish a condition for additivity of F which is related to but entails less smoothness on F than the classical result of Scheffé [10] (Theorem 16).

All of these results rest on invoking some degree of component 1- or 2-point homogeneity, which has the major effect of allowing one to mix various automorphisms on the components in order to get the same transformation on the structure. There are, however, interesting structures for which the factorizable automorphisms satisfy 1- or 2-point homogeneity, but not component 1- or 2-point homogeneity. For example,

$$F(x,y)=\begin{cases}\alpha x+(1-\alpha)y, & x>y,\ 0<\alpha<1,\\ \beta x+(1-\beta)y, & x\le y,\ 0<\beta<1,\end{cases}$$

has a factorizable automorphism of the form $x\to rx+s$, $y\to ry+s$, $F\to rF+s$ in which exactly the same affine transformation is applied to each component. This suggests working out the theory when only homogeneity, not component homogeneity is assumed. This has been done. As it is intricate, lengthy, and the work of just one of us (Cohen) it will be published separately.

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