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AXIOMS FOR THE AVERAGING AND ADDING REPRESENTATIONS OF FUNCTIONAL MEASUREMENT *

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1. Introduction

N.H. Anderson has applied simple averaging and adding representations to numerous bodies of category data (see Anderson (1970, 1971, and 1974) for summaries and further references). The averaging representation, which seems the more useful, is as follows: there are several sets of attributes or factors, $F_1, F_2, \ldots, F_k$, and to each set is associated a numerical weight w_i and a real-valued function ϕ_i such that the numerical scale assigned to the stimulus complex $(f_1, \ldots, f_k)$, $f_i \in F_i$, is given by $\Sigma_{i=1}^k w_i \phi_i(f_i)/\Sigma_{i=1}^k w_i$. The adding representation does not involve the weights, so the scale assigned to $(f_1, f_2, \ldots, f_k)$ is of the form $\Sigma_{i=1}^k \phi_i$.

To my knowledge, no one has presented axioms for an ordering of such complexes of attributes that are sufficient to give rise to the averaging representation, and the axioms of additive conjoint measurement apply only to addition over a fixed number of factors. My purpose here is to indicate that, in fact, one such set of conditions for averaging really is known under the guise of conditional expected utility (Krantz et al. (1971) Ch. 8); Luce and Krantz (1971)), and that a simple modification of it yields the additive representation.

It is important at the outset to recognize that the desired representations are both distinct from that of ordinary additive conjoint measurement (Krantz et al. (1971) Ch. 6), the reason being that they apply to different sized subcomplexes of attributes whereas conjoint measurement does not. For example, if f_i and g_i are elements of F_i, then one can

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compare (f_2, f_4) with (g_1, g_3, g_4) which have, respectively, the averaging scale values

$$\frac{w_2\phi_2(f_2) + w_4\phi_4(f_4)}{w_2 + w_4} \quad \text{and} \quad \frac{w_1\phi_1(g_1) + w_3\phi_3(g_3) + w_4\phi_4(g_4)}{w_1 + w_3 + w_4}$$

and the additive ones

$$\phi_2(f_2) + \phi_4(f_4) \quad \text{and} \quad \phi_1(g_1) + \phi_3(g_3) + \phi_4(g_4).$$

Such comparisons are inadmissible in the theory of additive conjoint measurement.

2. Averaging axiomatization

To formulate the problem generally, suppose $N = \{1, 2, ..., n\}, n \geqslant 3$, $\mathcal{E} = 2^N$ = power set of N, and F_i, $i \in N$ are sets. Let

$$\mathcal{D} = \{\{f_i\}_{i \in A} \mid A \in \mathcal{E} - \{\emptyset\}, f_i \in F_i\},$$

which is the set of all possible complexes formed from choices among one or more sets of attributes. It is notationally convenient to think of each set $\{f_i\}_{i \in A}$ as a function f_A on A, where for $i \in A$, $f_A(i) = f_i$. Note that for $A, B \in \mathcal{E} - \{\emptyset\}$ with $A \cap B = \emptyset$, if $f_A, g_B \in \mathcal{D}$, then $f_A \cup g_B \in \mathcal{D}$.

Given an ordering $\succsim$ of $\mathcal{D}$, the first problem is to state conditions on $\succsim$ sufficient to prove existence of weights w_i and functions ϕ_i from F_i into Re, $i \in N$, such that the function u on $\mathcal{D}$, where

$$u(f_A) = \sum_{i \in A} w_i \phi_i[f_A(i)] / \sum_{i \in A} w_i, \tag{1}$$

is order preserving.

Toward this end, we introduce the notion of a standard sequence. Let K be a sequence of consecutive integers and for each $k \in K$, let $f_A^{(k)}$ be elements of $\mathcal{D}$. We say $\{f_A^{(k)}\}_{k \in K}$ is a *standard sequence* iff there is some $B \in \mathcal{E} - \{\emptyset\}$ with $A \cap B = \emptyset$ and elements $g_B^{(0)}, g_B^{(1)} \in \mathcal{D}$ such that not $g_B^{(0)} \sim g_B^{(1)}$ and for all $k, k+1 \in K$, $f_A^{(k)} \cup g_B^{(1)} \sim f_A^{(k+1)} \cup g_B^{(0)}$.

Theorem 1. *Suppose N, $\mathcal{E}$, $\mathcal{D}$, and $\succsim$ are as above. The following conditions on $\succsim$ are sufficient for the representation given in eq. 1: For all $A, B \in \mathcal{E} - \{\emptyset\}$ and all $f_A, f'_A, f_A^{(k)}, g_B, h_B^{(k)} \in \mathcal{D}$* ($k$ an integer)

(1) *$\succsim$ is a weak order for which $\mathcal{D}/\sim$ has more than one element.*

(2) *If $A \cap B = \emptyset$ and $f_A \sim g_B$, then $f_A \cup g_B \sim f_A$.*

(3) (*Independence*) *If $A \cap B = \emptyset$, then $f_A \succsim f'_A$ iff $f_A \cup g_B \succsim f'_A \cup g_B$.*

(4) *If $\{f_A^{(k)}\}_{k \in K}$ and $\{h_A^{(k)}\}_{k \in K}$ are standard sequences and for some $j, j+1 \in K$, $f_A^{(j)} \sim h_A^{(j)}$ and $f_A^{(j+1)} \sim h_A^{(j+1)}$, then $f_A^{(k)} \sim h_A^{(k)}$ for all $k \in K$.*

(5) (*Archimedean*) *Every strictly bounded standard sequence is finite.*

(6) (*Restricted solvability*).

(i) *For each $A \in \mathcal{E} - \{\emptyset\}$ and $g_B \in \mathcal{D}$, there exists f_A for which $f_A \sim g_B$.*

(ii) *If* $A \cap B = \emptyset$ *and* $h_A^{(1)} \cup g_B \succsim f_{A \cup B} \succsim h_A^{(0)} \cup g_B$, *then there exists* $h_A \in \mathcal{D}$ *such that* $h_A \cup g_B \sim f_{A \cup B}$.

The weights are unique up to multiplication by a positive constant and the ϕ_i *are unique up to the common linear transformation* $\alpha\phi_i + \beta, \alpha > 0$

Note that Axioms 1–5 are necessary if eq. 1 holds and u is not a constant. Axiom 6 is structural and, in essence, requires the sets F_i be represented densely in the reals.

If Axiom 6 is replaced by a general solvability assumption, namely, that if $A \cap B = \emptyset$, $f_{A \cup B}, g_B \in \mathcal{D}$ then there exists $h_A \in \mathcal{D}$ such that $f_{A \cup B} \sim h_A \cup g_B$, then Axiom 4 can be proved from the other axioms.

It is trivial to show that these assumptions are a special case of the definition on p. 380 of Krantz et al. (1971) with $\mathcal{C} = \bigcup_{i=1}^n F_i$ and $\mathcal{N} = \{\emptyset\}$. By Theorem 1 on p. 381 we know there exists a finitely additive probability function P on $\mathcal{E}$ and a real-valued function u on $\mathcal{D}$ such that u is order preserving and for $A, B \in \mathcal{D} - \{\emptyset\}$ with $A \cap B = \emptyset$,

$$u(f_A \cup g_B) = u(f_A)\, P(A/A \cup B) + u(g_B)\, P(B/A \cup B)\,.$$

If we set $w_i = P(\{i\})$ and $\phi_i = u[f\{i\}]$, then eq. 1 follows by induction.

As an axiomatization of utility theory, the conditional expected utility model is controversial (see, e.g., Balch (1974); Balch and Fishburn (1974); Krantz and Luce (1974); Spohn (1977)). Much of the controversy concerns the interpretation of union in expressions such as $f_A \cup g_B$ when f_A, g_B, and $f_A \cup g_B$ are thought of as uncertain decisions. In the present interpretation, as complexes of attributes, no comparable difficulty of interpreting $\cup$ exists. If f_A and g_B are two complexes of disjoint attributes, then $f_A \cup g_B$ is simply their set theoretic union, which clearly is a complex of attributes.

3. Adding representation

Let us turn now to the question of conditions under which an additive representation exists. Specifically, we seek conditions under which there exist real-valued functions ϕ_i on F_i such that

$$v(f_A) = \sum_{i \in A} \phi_i[f_A(i)] \tag{2}$$

is order preserving. Observe that a complex f_A can be thought of as a complex on all of N in which the outcomes on the components in $N - A$ are null elements, say e_i. If we set $\phi_i(e_i) = 0$, then eq. 2 becomes something like the ordering representation of additive conjoint measurement except for the fact that the scales now have a fixed zero and so only admit similarity transformations.

Theorem 2. *Suppose* N, $\mathcal{E}$, $\mathcal{D}$, *and* $\sim$ *are as above. The following conditions on* $\succsim$ *are sufficient for the representation given by* eq. 2: *Axioms* 1, 3, 5 *and* 6(i) *of Theorem* 1 *together with*

(7) *There exist* $e_i \in F_i$ *such that for every* $A \in \mathcal{E} - \{\emptyset\}$ *with* $A \cap \{i\} = \emptyset$ *and every* $f_A \in \mathcal{D}$, $e_i \cup f_A \sim f_A$.
The scales ϕ_i *of* eq. 2 *are ratio scales with* $\phi_i(e_i) = 0$ *and a common unit.*

The proof has been pretty much suggested above. Let $e_A \in \mathcal{D}$ be such that $e_A(i) = e_i$ for $i \in A$. By induction on Axiom 7, $e_{\bar{A}} \cup f_A \sim f_A$. Thus $\langle \mathcal{D}, \succsim \rangle$ is mapped onto $\langle \mathsf{X}_{i=1}^n F_i, \succsim \rangle$. The remaining axioms, restricted to $\langle \mathsf{X}_{i=1}^n F_i, \succsim \rangle$, satisfy Definition 6.13, Krantz et al. (1971), and so by Theorem 6.13 there is an additive representation. If we choose $\phi_i(e_i) = 0$, the scales are restricted to ratio ones (with a common unit) and

$$v(e_{\bar{A}} \cup f_A) = \sum_{i \in \bar{A}} \phi_i(e_i) + \sum_{i \in A} \phi_i[f_A(i)]$$

$$= v(f_A)$$

is order preserving.

It is perhaps worth noting that the union of the axioms of the two theorems is inconsistent. This can be shown as follows. Suppose $f_A \in \mathcal{D}$. By Axiom 6(i), there is no loss of generality in supposing $A \neq N$. Again, by Axiom 6(i), there exists $g_{N-A} \sim f_A$ so by Axiom 2, $f_A \cup g_{N-A} \sim g_{N-A}$. By eq. 2, $f_A \sim e_A \sim e_X$, which is contrary to Axiom 1.

4. Discussion

Krantz et al. (1971, pp. 26–31, 329–347) and Krantz and Tversky (1971) discuss the empirical testing of measurement axiomatizations. As they point out, it is impossible to test an Archimedean axiom (5), and one usually also decides on the basis of the empirical context whether or not the structural axioms (6 and 7) are appropriate. In the present models, Axiom 4, which concerns the compatibility of standard sequences, is not likely to be directly tested, especially since it is a logical consequence of the rest of the structure if Axiom 6 is strengthened. This leaves us with Axioms 1 and 3, which are common to adding and averaging, and Axiom 2 which is unique to averaging. Since Axioms 1 and 3 have arisen in the study of additive conjoint measurement, the problem of testing them has been discussed carefully (e.g., Krantz and Tversky (1971)). The distinguishing feature of averaging is Axiom 2 which would be tested simply by finding equivalent but disjoint complexes and then seeing whether or not their union is equivalent to each of them.

Eternally an optimist, I trust that these axiom systems may clarify to a degree the relations between axiomatic and functional measurement.

For one, they should make transparent just how different the adding and averaging representations are, and Theorem 1 shows unambiguously that, with an adequate amount of data, the weights of the averaging model are perfectly identifiable. Schönemann, Cafferty and Rotton (1973) discussed the latter question, arguing that (at least in the case $n = 2$) there is no difference between adding and averaging. Anderson (1973) diagreed. As far as I can tell, Schönemann et al. assumed a fixed number of components, in which case there is no identifiable difference, whereas Anderson has always had in mind models that apply to stimuli with from 1 to n components, in which case adding and averaging are, as we have seen, completely distinct.

For another, it provides one axiomatic basis for the averaging representation. What is not yet clear is how to axiomatize averaging in the actual finite factorial designs usually used. One wants something along the lines of finite additive conjoint measurement (see Krantz et al. (1971) Ch. 9).

As to the rather contentious relations between axiomatic and functional measurement, these results add little that is new. As always, the axioms formulate conditions on ordinal data that are sufficient to insure that the data can be represented by particular numerical structures and it provides constructive techniques to estimate that structure. As always, functional measurement asks whether some numerical usually category, data are well accounted for by a particular numerical structure and it provides a way to estimate the parameters of that structure. Both address the nature of empirical data, both are concerned with the adequacy of numerical representations of these data, and both evidence strong theoretical concerns about how the several factors of the stimuli combine. Anderson seems to feel otherwise. As I see it, they differ only in the type of data assumed – ordinal versus numerical –, on the focus of empirical testing – checking axioms versus fitting a representation –, and on the treatment of error – medians (Falmagne (1976)) or analysis of variance. These differences seem less a matter of scientific values or moral virtue than of scientific strategy and taste.

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