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SEVERAL POSSIBLE MEASURES OF RISK

1. INTRODUCTION

We all speak of the 'risk' of gambles, but rarely do we explicitly define it. In some contexts it seems to be little more than a reminder that probabilities are involved; in others it seems to refer to the variance of the gamble; and in still others one gets the hint of an as yet undefined concept. In his portfolio theory of decisions, Coombs (1969) explicitly treats risk as a variable on a par with expected value, except for one thing – it is not explicitly defined. He cites several manipulations that he believes affect risk, and he uses them to effect in his empirical work (Coombs and Bowen, 1971a, b; Coombs and Huang, 1970a, b; Coombs and Meyer, 1969).

The most elegant and novel attempt to define risk explicitly is the axiomatic theory of Pollatsek and Tversky (1970). Let the domain of risk measurement be all random variables with density functions, then they show that if risk ordering satisfies some plausible constraints and if X is a random variable,

$$(1) \qquad R(X) = \theta V(X) - (1-\theta)E(X),$$

for some real θ, $0 < \theta < 1$, preserves the order and for independent random variables X and Y,

$$R(X+Y) = R(X) + R(Y).$$

The only problem is that empirically it is not correct. Coombs and Bowen (1971a) have shown that subjects detect differences in risk between gambles that have the same mean and variance.

So the problem remains. The purpose of this paper is to take up the question again, approaching it slightly differently. The method employed yields several possibilities, among which it is difficult to select by convincing a priori arguments. Data will be needed.

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2. AN ADDITIVE MODEL

It will mostly prove efficient to suppress the random variable notation and to associate risk with densities. At any moment, the particular putative measure of risk we are referring to will be denoted by $R(f)$, where f is the density function. If $\alpha > 0$ is a change in scale, then we define the density f_α of the gamble in which all outcomes have been altered by the factor α by

$$(2) \qquad f_\alpha(x) = \frac{1}{\alpha} f\left(\frac{x}{\alpha}\right).$$

Our first class of assumptions concerns what happens to risk under a change of scale. The conjecture is that if $R(f)$ is known, then $R(f_\alpha)$ is some function of $R(f)$ and α. In this paper I shall explore the two simplest possibilities, namely, that their effects are additive or that they are multiplicative. It is quite possible that the methods employed can be carried over to more complex hypotheses. The additive assumption can be formulated as follows:

ASSUMPTION 1A. There is a strictly increasing function S with $S(1) = 0$ such that for all density functions and all real $\alpha > 0$,

$$R(f_\alpha) = R(f) + S(\alpha).$$

This postulate simply says that if one moves from a domain all in dollars to one in units of one hundred dollars, then the risk of all transformed gambles will change by a constant amount depending upon the change in units. This implies one of the major conditions of Pollatsek and Tversky's theory (Axiom 6, Definition 11, p. 127, Krantz *et al.,* 1971), but it is not implied by their theory. The most obvious failing of additivity is that a gamble having zero risk is transformed by a change of amount into one with non-zero risk. The unreasonableness of this favors the multiplicative hypothesis.

The second class of assumptions concerns the nature of the aggregation of a density into a single number. Aggregation suggests some sort of an integral. Two major candidates come to mind. One, which we treat later, is that there is some transformation of the random variable, say T, and R is the expectation of the resulting random variable. The other is that the density itself undergoes a pointwise nonlinear transformation T and then it is integrated.

ASSUMPTION 2. There is a non-negative function T, $T(0) = 0$, such that for all density functions f,

$$R(f) = \int_{-\infty}^{\infty} T[f(x)]\,\mathrm{d}x.$$

Observe that it is possible to develop a suitable qualitative theory for these two assumptions. The first would be a version of additive conjoint measurement in which $A \times P$ is isomorphic to P and $A = Re^{+}$. The second also requires a version of additive conjoint measurement over partitions of random variables. The details are left to the reader.

THEOREM 1. *If Assumptions* 1A *and* 2 *hold, then there are constants* $A > 0$ *and* $B \geqslant 0$ *such that for all densities* f,

$$\text{(3)} \qquad R(f) = -A\int_{-\infty}^{\infty} f(x)\log f(x)\,\mathrm{d}x + B.$$

Proof. For all $\alpha, \beta > 0$, observe that

$$(f_\alpha)_\beta(x) = \frac{1}{\beta} f_\alpha\left(\frac{x}{\beta}\right) = \frac{1}{\alpha\beta} f\left(\frac{x}{\alpha\beta}\right) = f_{\alpha\beta}(x).$$

Applying Assumption 1A to this yields

$$\begin{aligned} R(f) + S(\alpha\beta) &= R(f_{\alpha\beta}) \\ &= R(f_\alpha) + S(\beta) \\ &= R(f) + S(\alpha) + S(\beta). \end{aligned}$$

Since S is strictly increasing, it is well known (Aczél, 1966) there is $A > 0$ such that

$$\text{(4)} \qquad S(\alpha) = A\log\alpha.$$

Turning to Assumption 2 and using (4)

$$\begin{aligned} \text{(5)} \qquad R(f_\alpha) &= \int_{-\infty}^{\infty} T\left[\frac{1}{\alpha} f\left(\frac{x}{\alpha}\right)\right]\mathrm{d}x \\ &= \alpha\int_{-\infty}^{\infty} T\left[\frac{1}{\alpha} f(y)\right]\mathrm{d}y \end{aligned}$$

$$= R(f) + A \log \alpha$$

$$= \int_{-\infty}^{\infty} T[f(x)]\, \mathrm{d}x + A \log \alpha.$$

Consider the following special density:

$$f(x) = \begin{cases} \alpha & \text{for } x \in (0, 1/\alpha) \\ 0 & \text{elsewhere.} \end{cases}$$

Substituting in (5),

$$\alpha \int_0^{1/\alpha} T(1)\, \mathrm{d}x = \int_0^{1/\alpha} T(\alpha)\, \mathrm{d}x + A \log \alpha,$$

so

$$T(\alpha) = \alpha[B - A \log \alpha], \qquad B = T(1) \geqslant 0.$$

Substituting this in Assumption 2 yields (3). ◇

3. DISCUSSION

Since this possible measure of risk, which is the continuous analogue of entropy or uncertainty, is quite different from that of Pollatsek and Tversky, it is interesting to see just where the differences lie. Conceptually, the major differences are two. First, Pollatsek and Tversky begin their study using sums of independent random variables (convolutions of densities) which leads to an extensive structure. They assume that the measure of risk is the additive representation of that structure. That is to say, the risk of the sum of two independent random variables is the sum of their separate risks. This is by no means an obviously true property of risk – one might well imagine interactions that would reduce or aggravate the risk. In fact, many psychologists believe the risk of a gamble that is repeated n times is less than n times the risk of the gamble played once. To show that this need not be the case for (3), consider the example where $A = 1, B = 0$, and

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0\quad, & x < 0. \end{cases}$$

Then substituting in (3) we find

$$R(f) = 1 - \log \lambda$$

and

$$R(f \circ f) - 2R(f) = C + \log \lambda - 1,$$

where of course $f \circ f$ is the gamma distribution of order 2 and C is Euler's constant. Since $\log \lambda$ runs from $-\infty$ to ∞ as λ goes from 0 to ∞, there is no assurance that $R(f \circ f) < 2R(f)$.

A second difference from Pollatsek and Tversky is that while their theory uses changes of unit, what they assume about the changes is a good deal weaker than Assumption 1A and, as is readily shown, is implied by Assumption 1A. In fact, Assumption 1A is false in their system since from (1) and (2) it follows that

$$R(f_\alpha) = \theta\alpha^2 V(f) - (1-\theta)\alpha E(f),$$

which is neither additive nor multiplicative. Indeed, the impact of the variances is accelerated relative to the mean.

The remaining axioms (Definition 11, p. 127, Krantz *et al.*, 1971) are satisfied by (3). To show Axiom 5, observe that

$$f \circ \alpha^*(t) = f(t-\alpha),$$

which when substituted in (2) yields $R(f \circ \alpha^*) = R(f)$. Axiom 6, as we noted, follows immediately from Assumption 1A. And Axiom 7, continuity, is well known to hold for (3).

It is sometimes of interest to partition a gamble (random variable) into two parts and to ask how the risk of the whole compares to the parts. So let f and g be disjoint random variables, i.e., for all x,

$$f(x)g(x) = 0.$$

Let

$$h(x) = \frac{1}{2}\begin{cases} f(x) & \text{if } f(x) \neq 0 \\ g(x) & \text{if } g(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then

$$R(h) = \int_{-\infty}^{\infty} T[h(x)]\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} T[\tfrac{1}{2}f(x)]\,\mathrm{d}x + \int_{-\infty}^{\infty} T[\tfrac{1}{2}g(x)]\,\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} T\left[\tfrac{1}{2}f\left(\frac{y}{2}\right)\right] dy + \frac{1}{2}\int_{-\infty}^{\infty} T\left[\tfrac{1}{2}g\left(\frac{y}{2}\right)\right] dy$$

$$= \tfrac{1}{2}R(f_{1/2}) + \tfrac{1}{2}R(g_{1/2})$$

$$= \frac{R(f)+R(g)}{2} - A\log 2$$

$$< \frac{R(f)+R(g)}{2}.$$

For a normal density η with mean μ at variance σ^2, substituting in (3) yields

$$R(\eta) = A\left[\tfrac{1}{2} + \log(\sqrt{2\pi}\sigma)\right] + B.$$

This result adds further doubt to (3) as a measure of risk since it says that risk grows only with variance and is independent of the mean. Some have argued that risk is evaluated in terms of relative, not absolute, variability. The question is whether that is relative to their absolute level of wealth or to the expected value of the gamble being discussed.

Note that this problem is inherent in Assumption 2, for if f_μ is a shift family of distributions in the sense that $f_\mu(x-\mu) = f_0(x)$, then

$$\int_{-\infty}^{\infty} T[f_\mu(x)]\,dx = \int_{-\infty}^{\infty} T[f_\mu(y-\mu)]\,dy = \int_{-\infty}^{\infty} T[f_0(y)]\,dy.$$

4. MULTIPLICATIVE MODEL

The probable failures of the additive model suggest modifying at least one of the assumptions. We first consider Assumption 1A. Many feel that the impact of changing units is more likely multiplicative rather than additive. In that case, we should consider the following assumption:

ASSUMPTION 1M. For all density functions f and all real $\alpha > 0$, there is an increasing function S with $S(1) = 1$ such that

$$R(f_\alpha) = S(\alpha)R(f).$$

THEOREM 2. *If Assumptions* 1M *and* 2 *hold, then there are constants* $\theta > 0$ *and* $A > 0$ *such that*

$$(6) \qquad R(f) = A\int_{-\infty}^{\infty} f(x)^{1-\theta}\,\mathrm{d}x.$$

Proof. Following the proof of Theorem 1, one first shows $S(\alpha) = \alpha^\theta$ for some $\theta > 0$. Then substituting $R(f_\alpha) = \alpha^\theta R(f)$ in Assumption 2 and using the same special choices for f yields (6). ◇

As is easily verified, the axioms of Definition 11 (Krantz *et al.*, 1971) are satisfied. It is less clear what happens for convolutions. If f is $n(\mu, \sigma)$, then

$$R(n) = B\frac{\sigma\theta}{(1-\theta)^{1/2}}(2\pi)^{\theta/2}.$$

And so

$$R(n \circ n) - 2R(n) = B\frac{\sigma\theta}{(1-\theta)^{1/2}}(2\pi)^{\theta/2}(2^\theta - 2).$$

In this case, we see that $\theta < 1$ insures a decreasing growth of risk. The following example, due to Michael Cohen, shows that the desired inequality is not true of (6) with $\theta \leqslant 1$. Define

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,1] \cup [2,3] \\ 0 & \text{elsewhere.} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{elsewhere.} \end{cases}$$

Thus,

$$f \circ g(x) = \begin{cases} \frac{1}{2}(1 - |x-k|), & x \in [k-1, k+1], \quad k = 1, 3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$f \circ f(x) = \begin{cases} \frac{1}{4}(1 - |x-k|), & x \in [k-1, k+1], \quad k = 1, 5 \\ \frac{1}{2}(1 - |x-3|), & x \in [2,4] \\ 0 & \text{elsewhere.} \end{cases}$$

For fixed θ, (6) yields

$$R(f) = 2^\theta, \qquad R(g) = 1,$$

$$R(f \circ g) = 2^{1+\theta}/(2-\theta), \qquad R(f \circ f) = \frac{2^\theta + 4^\theta}{2-\theta}.$$

Thus,

$$R(f)+R(g)-R(f\circ g) = 1-\frac{\theta}{2-\theta}2^{\theta}$$

and

$$2R(f)-R(f\circ f) = \frac{2^{\theta}(3-2\theta-2^{\theta})}{2-\theta}$$

both of which are negative for θ near 1.

5. AN ALTERNATIVE AGGREGATION RULE

Another drawback to (6) is that if a density function is split at any point and the two portions shifted left and right, respectively, by any amounts, the risk is unchanged. That is highly counter intuitive. It arises in large part from the aggregation rule, Assumption 2. Moreover, that rule excludes, for example, any of the moments as possible solutions to the problems. So it seems appropriate to consider the following assumption in lieu of Assumption 2.

ASSUMPTION 3. There is a function T such that for all densities f

$$R(f) = \int_{-\infty}^{\infty} T(x)f(x)\,\mathrm{d}x = E[T(X)].$$

THEOREM 3. *If Assumptions* 1A *and* 3 *hold, then for some constants* $A>0$ *and* $B>0$,

$$R(f) = A\int_{-\infty}^{\infty} (\log x)f(x)\,\mathrm{d}x + B = AE(\log X)+B. \tag{7}$$

Proof. As in Theorem 1, $S(\alpha)=A\log\alpha$. Substituting Assumption 3 into 1A and rewriting yields

$$\int_{-\infty}^{\infty} T(\alpha x)f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} T(x)f(x)\,\mathrm{d}x + A\log\alpha.$$

Using the same special case for f,

$$C = \int_{-\infty}^{\infty} T(x)\,\mathrm{d}x = \int_{0}^{1/\alpha} T(\alpha x)\alpha\,\mathrm{d}x \tag{8}$$

$$= \int_0^{1/\alpha} T(x)\alpha \, dx + A \log \alpha.$$

Differentiating,

$$\int_1^{1/\alpha} T(x) \, dx = \frac{1}{\alpha} T\left(\frac{1}{\alpha}\right) - \frac{A}{\alpha}.$$

Substituting this for the integral in (8) and rewriting yields

$$T(\alpha) = B + A \log \alpha,$$

and (7) follows from Assumption 3. ◇

We observe that (7) exhibits $R(f \circ g) \leqslant R(f) + R(g)$. This follows from the fact that $\log (X + Y) \leqslant \log X + \log Y$, and so,

$$\begin{aligned} R(f) + R(g) &= AE(\log X) + B + AE(\log Y) + B \\ &\geqslant AE[\log (X + Y)] + B \\ &= R(f \circ g). \end{aligned}$$

THEOREM 4. *If Assumptions* 1M *and* 3 *hold, then for some* $\theta > 0$ *and* $A > 0$

$$(9) \qquad R(f) = A \int_{-\infty}^{\infty} x^{\theta} f(x) \, dx = AE(X^{\theta}).$$

Proof. As in Theorem 2, $S(\alpha) = \alpha^{\theta}, \theta > 0$. Proceeding as in Theorem 3,

$$C = \int_0^1 T(x) \, dx = \alpha^{\theta+1} \int_0^{1/\alpha} T(x) \, dx.$$

Differentiating and simplifying,

$$T(x) = Ax^{\theta},$$

which yields the result. ◇

Since $R(f)$ is proportional to the θ raw moment, it is clear that except for $\theta = 1$, the mean, $R(f \circ g) - R(f) - R(g)$ can be either positive or negative.

6. GENERALIZATION

Since there are two major types of assumptions, there are two likely places for generalization. I have not had any clever insights about replacing addition or multiplication in Assumption 1; however, it is quite natural to consider modifying Assumption 2 to read

$$R(f) = U\left\{\int_{-\infty}^{\infty} T[f(x)]\,\mathrm{d}x\right\},$$

where $T(0) = 0$ and U is strictly increasing. If, for example, one checks through the proof of Theorem 1 introducing this change, one finds

$$T(\alpha) = \alpha U^{-1}[C - \log \alpha].$$

Without some further assumptions, specifying U in some fashion, it is not possible to proceed. For example, if one knows that

$$U(x) = \frac{1}{1-r}\log x, \quad r \neq 1,$$

then it is easy to see that for appropriate choices of constants,

$$R(f) = \frac{1}{1-r}\log\int_{-\infty}^{\infty} f(x)^r\,\mathrm{d}x,$$

which is the continuous analogue of what Aczél and Daróczy (1975) call Rényi entropies.

Similar arguments can be carried out for the other three cases, and for $U = \log$, one obtains

$$\begin{aligned} R(f) &= \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{cf(x)^{-\theta}}\,\mathrm{d}x && \text{(1M \& 2)}\\ &= CE(X^A) && \text{(1A \& 3)}\\ &= E[(\theta X^\theta + 1)\,\mathrm{e}^{X^\theta}]. && \text{(1M \& 3)} \end{aligned}$$

7. CONCLUDING REMARKS

The problem is how to decide if any of the four possibilities arrived at,

$$\int f(x) \log f(x)\, \mathrm{d}x, \qquad (1A \,\&\, 2)$$

$$\int f(x)^{1-\theta}\, \mathrm{d}x, \qquad (1M \,\&\, 2)$$

$$E(\log X), \qquad (1A \,\&\, 3)$$

$$E(X^{\theta}), \qquad (1M \,\&\, 3)$$

is correct as a measure of risk. We have noted that some believe that $R(f \circ f) < 2R(f)$. This is known to hold [actually more generally as $R(f \circ g) \leqslant R(f) + R(g)$] for the third and to fail for the first, second, and the fourth.

If this restriction is not accepted, then one must approach the problem empirically. Probably the most satisfying procedure would be first to perform the qualitative, additive conjoint measurement tests of Assumptions 1A and 1M and of Assumptions 2 and 3. If either of these fails, then we know we must turn to a more complex theory.

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REFERENCES

Aczél, J., *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

Aczél, J. and Daróczy, Z., *On Measures of Information and Their Characterizations*, Academic Press, New York, 1975.

Coombs, C. H., 'Portfolio theory: a theory of risky decision making', *La Decision*, Centre National de la Recherch Scientifique, Paris, 1969.

Coombs, C. H. and Bowen, J. N., 'A test of VE-theories of risk and the effect of the central limit theorem', *Acta Psychologica* 35 (1971a), 15–28.

Coombs, C. H. and Bowen, J. N., 'Additivity of risk in portfolios', *Perception and*

Psychophysics 10 (1971b), 43–46.
Coombs, C. H. and Huang, L. C., 'Polynomial psychophysics of risk', *Journal of Mathematical Psychology* 7 (1970a), 317–338.
Coombs, C. H. and Huang, L. C., 'Tests of a portfolio theory of risk preference', *Journal of Experimental Psychology* 85 (1970b), 23–29.
Coombs, C. H. and Meyer, D. E., 'Risk-preference in coin-toss games', *Journal of Mathematical Psychology* 6 (1969), 514–527.
Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A., *Foundations of Measurement, I*, Academic Press, New York, 1971.
Pollatsek, A. and Tversky, A., 'A theory of risk', *Journal of Mathematical Psychology* 7 (1970), 540–553.