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LEXICOGRAPHIC TRADEOFF STRUCTURES[1]

INTRODUCTION

In selecting a house from a limited domain of houses, price dominates one's considerations when the difference in price between two houses is sufficiently large. But when the two prices are sufficiently close, other factors begin to enter into consideration. We do not ignore price – it is hardly possible to pay no attention to the difference between $49,500 and $55,000 – but we begin to acknowledge that such a difference can be compensated for by factors such as location, style, lot size, spaciousness, etc.

Examples such as this exhibit three main features, two of which do not seem to be jointly captured in any of our existing decision models. First, the choice situation can be described as a conjoint structure with at least two factors affecting the decision attribute. Second, one factor, call it the first, is most important in the sense that for sufficiently large differences on that factor it is totally controlling. Third, when, however, the difference on the first factor is not too large, the second factor, and others if they exist, come into play and tradeoffs exist between the two factors, as in the usual theories of conjoint measurement.

Tversky (1969) discussed this problem. He pointed out that, to a degree, the behavior is captured by a lexicographic semiorder: a structure that is lexicographic in the sense that control shifts to the second component when indifference exists on the first; however indifference is assumed to be nontransitive and, in fact, to arise from a semiorder (see below). As Tversky notes on p. 40, " ... despite its intuitive appeal, it is based on a noncompensatory principle that is likely to be too restrictive in many contexts."

He then goes on to suggest an additive difference representation where there are functions ϕ_i assigned to the n components X_i and increasing weighting functions f_i such that x is prefererred or indifferent to y if and only if

$$\sum_{i=1}^{n} f_i\,[\phi_i(x_i) - \phi_i(y_i)] \geqslant 0,$$

where $f_i(-\delta) = -f_i(\delta)$. This, however, does not capture very well the lexicographic character of some choices.

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So there appears to be room for a model that attempts to retain the structure of a lexicographic semiorder and softens its noncompensatory feature. For simplicity, I will look at the additive conjoint tradeoff in the two factor case.

PRELIMINARIES

Let A and X be sets – the qualitative factors of the problem – and let $\succsim$ be a binary relation on A × X – the preference relation. For reasons that will become apparent, I do not assume $\succsim$ is transitive. I do, however, assume that it is *independent* in the following sense: for all a, b in A and x, y in X,

$$(a,x) \succsim (b,x) \text{ iff } (a,y) \succsim (b,y)$$

and

$$(a,x) \succsim (a,y) \text{ iff } (b,x) \succsim (b,y).$$

Note that the first property means that $\succsim$ induces an unambiguous order on A, which is denoted $\succsim_A$; similarly, the second one induces $\succsim_X$ on X.

The usual strict preference and indifference are defined by: for all a, b in A and x, y in X,

$$(a,x) \succ (b,y) \text{ iff } (a,x) \succsim (b,y) \text{ and not } (b,y) \succsim (a,x),$$
$$(a,x) \sim (b,y) \text{ iff } (a,x) \succsim (b,y) \text{ and } (b,y) \succsim (a,x).$$

Since $\succsim$ is not transitive, one must be careful about not assuming the transitivity of either $\succ$ or $\sim$. In like manner, $\succ_A$, $\sim_A$, $\succ_X$, and $\sim_X$ are defined. They too, are not necessarily transitive.

The next concept, a binary relation P_A on A, is designed to single out that part of $\succsim$ in which the A-component completely dominates: for all a, b in A,

$$aP_A b \text{ iff for every } x, y \text{ in } X, (a,x) \succ (b,y).$$

If P is any binary relation on A, we define two other relations in terms of it: for all a, b in A,

$aI(P)b$ iff not aPb and not bPa,
$aW(P)b$ iff either (i) aPb, or
(ii) $aI(P)b$ and there exists c in A such that $aI(P)c$ and Pb, or
(iii) $aI(P)b$ and there exists d in A such that aPd and $dI(P)b$.

A binary relation P on A is a *semiorder* iff for all a, b, c, d in A,

(i) not aPa,
(ii) aPb and cPd imply either aPd or cPb,
(iii) aPb and bPc imply either aPd or dPc.

If P is a semiorder, it is not difficult to show that P is transitive, but $I(P)$ is not, and that $W(P)$ is negatively transitive and asymmetric.

If P is a semiorder on A and if $aW(P)b$ and $aI(P)b$, the set

$$A(a, b) = \{c \mid aW(P)c \text{ and } cW(P)b\}$$

is called an *indifference interval.*

AXIOMS

Consider a binary relation $\succsim$ on $A \times X$ with the derived concepts $\succsim_A$, P_A, $W(P_A)$, and $A(a, b)$ defined above.

AXIOM 1. *$\succsim$ is reflexive, connected, and independent.*

AXIOM 2. *P_A is a semiorder.*

AXIOM 3. *$W(P_A)$ is identical to $\succ_A$.*

AXIOM 4. *For all a, b in A, if $A(a, b)$ is an indifference interval and $\succsim_{ab}$ denotes the restriction of $\succsim$ to $A(a, b) \times X$, then $\langle A(a, b) \times X, \succsim_{ab}\rangle$ is an additive conjoint structure satisfying Definition* 6.7 *of Krantz et al.* (1971).

AXIOM 5. *There exists a finite or countable subset of A,*

$$B = \{ \ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots\}$$

such that for all b_{i-1}, b_i, b_{i+1} in B,

(i) $b_i \succ_A b_{i-1}$,
(ii) *$A(b_{i-1}, b_{i+1})$ is an indifference interval,*
(iii) *for a in A, there exist b_{i-1}, b_i in B with $b_i \succsim_A a \succsim_A b_{i-1}$.*

AXIOM 6. *For every a in A, there exists some b in A such that $bI(P_A)a$ and for any c in A with $c \succ_A b$, then $cP_A a$.*

Before stating and proving the representation theorem, it may be useful to consider the meaning of the axioms. Axiom 1 is unexceptional in this context; the property of independence is discussed at some length in Krantz et al. (1971). Axiom 2 states that the strictly dominating part of $\succsim$ behaves as one would expect – it is transitive and one does not find a strict preference spanned by one of its indifference intervals. Moreover, indifference $I(P_A)$ is not transitive. Axiom 3 says that the order induced on A via the independence of $\succsim$ is identical to the weak order induced by the lexicographic aspect of $\succsim$, which aspect is, according to Axiom 2, a semiorder. It is not easy to see exactly what this means except to say that the same numerical scale will serve to represent both $\succsim_A$ and P_A. As will be shown in the theorem, these three axioms are necessary consequences of the representation.

The last three axioms are, in part, structural. Axiom 4 states that within any indifference interval a tradeoff exists between the two components which is additive in character. As will be seen, the usual necessary parts of an additive conjoint axiomatization are also necessary in this context. Axiom 5 requires that the indifference intervals on A span all of A and thoroughly overlap one another; this permits us to use the uniqueness of additive conjoint representations to piece together scales over A and X. And finally, Axiom 6 requires that the set of elements indifferent to a given one be closed from above. This is not really essential, but it makes the representation slightly simpler.

There are two kinds of defects in these axioms. First, the structural ones are probably somewhat stronger than is really needed to prove the representation. Second, and far more important, the necessary axioms are simple only in terms of a number of defined concepts – $\succsim_A$, P_A, and $W(P_A)$ – and they would be really quite messy in terms of $\succsim$. It would be desirable to develop a system, leading to much the same representation, which is comparatively simple in terms of the primitive $\succsim$. Professor Peter Fishburn has recently made progress along the latter lines.

REPRESENTATION THEOREM

THEOREM. *Suppose* $\langle A \times X, \succsim \rangle$ *satisfies axioms* 1–6. *Then there exist real functions* ϕ_A *and* δ_A *on* A *and* ϕ_X *on* X *such that for all* a, b *in* A *and* x, y *in* X,

1. $$\delta_A(a) = \sup_{\substack{b \\ bI(P_A)a}} [\phi_A(b) - \phi_A(a)] > 0,$$

2. $aP_A b$ *iff* $\phi_A(a) > \phi_A(b) + \delta_A(b)$,
3. $aW(P_A)b$ *iff* $\phi_A(a) > \phi_A(b)$,
4. $(a,x) \succsim (b,y)$ *iff either* $\phi_A(a) > \phi_A(b) + \delta_A(b)$ *or*
   $-\delta_A(a) \leqslant \phi_A(a) - \phi_A(b) \leqslant \delta_A(b)$ *and*
   $\phi_A(a) + \phi_X(x) \geqslant \phi_A(b) + \phi_X(y)$.

If ϕ_A', δ_A', *and* ϕ_X' *form another representation with the same properties, then there are constants* $\alpha > 0$, β_A, *and* β_X *such that*

$$\phi_A' = \alpha\phi_A + \beta_A, \ \delta_A' = \alpha\delta_A, \textit{and } \phi_X' = \alpha\phi_X + \beta_X.$$

If such a representation exists, then Axioms 1, 2, 3, *and the non-structural part of* 4 *must hold.*[2]

Proof. By Axiom 4 and Theorem 6.2 of Krantz et al. (1971), there exists an additive representation of each indifference interval $A(a, b) \times X$ which is an interval scale on each component but with a common unit. By Axiom 5.ii there are successive indifference intervals with nontrivial regions of overlap. Forcing the local scales to agree on the regions of overlap yields ϕ_A and ϕ_X with the asserted uniqueness properties.

Define δ_A by Statement 1. Axioms 5 and 6 insure the existence of the sup. By Axiom 5.ii and iii, $\delta_A > 0$. By the definition of δ_A and Axiom 6, Statement 2 follows immediately. By Axiom 1 and its construction, ϕ_A preserves the order $\succsim_A$ and hence, by Axiom 3, it preserves the order $W(P_A)$. Statement 4 follows by the construction.

The uniqueness of δ_A follows from that of ϕ_A and Statement 1.

We next establish the necessity of Axioms 1, 2, 3, and the non-structural parts of 4.

1. The reflexiveness and connectedness of $\succsim$ follow immediately from Statement 4. To show independence, suppose first $(a, x) \succsim (b, x)$. If $aP_A b$, then $(a, y) \succsim (b, y)$. Otherwise, the usual argument for an additive representation works. Next, suppose $(a, x) \succsim (a, y)$, then the only possibility is the additive part of Statement 4 and so $(b,x) \succsim (b,y)$.

2. P_A is a semiorder: By Statement 2, not $aP_A a$. Next, suppose $aP_A b$, $cP_A d$, and not $aP_A d$, then by Statement 2,

$$\phi_A(b) + \delta_A(b) < \phi_A(a) \leqslant \phi_A(d) + \delta_A(d) < \phi_A(c),$$

whence cPb. Finally, suppose $aP_A b$ and $bP_A c$. Consider any d. If $\phi_A(d) \geqslant \phi_A(b)$, then since $\phi_A(b) > \phi_A(c) + \delta_A(c)$, dPc. Otherwise, $\phi_A(d) < \phi_A(b)$. Suppose $\phi_A(d) + \delta_A(d) > \phi_A(b) + \delta_A(b)$, then by definition of sup and Statements 1 and 2, there exists f with $fI(P_A)d$ and $fP_A b$. Thus, $dW(P_A)b$, and so by

Statement 3, $\phi_A(d) > \phi_A(b)$, contrary to choice. So $\phi_A(d) + \delta_A(d) \leqslant \phi_A(b) + \delta_A(b) < \phi_A(a)$, whence $aP_A d$.

3. By Statement 4, ϕ_A preserves the order $\succsim_A$ and by Statement 3 ϕ_A preserves the order of $W(P_A)$. Thus, these two orders are identical.

4. By Statement 4, the nonstructural parts of Axiom 4 hold, as shown in Krantz et al.

It is worth noting that $\succsim$ on $A \times X$ need not be transitive when this representation holds. For example, choose a, b, c in A such that $aI(P_A)b$, $bI(P_A)c$, and $aP_A c$, and choose x, y, z in X such that

$$\phi_A(c) + \phi_X(z) > \phi_A(b) + \phi_X(y) > \phi_A(a) + \phi_X(x),$$

then by Statements 2 and 4,

$$(c, z) \succ (b, y), (b, y) \succ (a, x), \text{ and } (a, x) \succ (c, z).$$

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NOTES

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[2] By considering several representations of this sort, Peter Fishburn and I have concluded that an interesting general form for the representation is the existence of functions ϕ_A on A, ϕ_P on P, and g on $A \times A$ such that

$$(a,x) \succsim (b,g) \text{ iff } \phi_A(a) - \phi_A(b) \geqslant g(a, b)\, [\phi_P(y) - \phi_P(x)].$$

In the case at hand,

$$g(a, b) = \begin{cases} 1 \text{ if } aI(P_A)b \\ 0 \text{ otherwise} \end{cases}$$

Tversky's lexicographic semiorder is

$$g(a, b) = \begin{cases} \infty \text{ if } aI(P_A)b \\ 0 \text{ otherwise,} \end{cases}$$

where $g(a, b) = \infty$ is interpreted as dividing the equation through by $g(a, b)$ and then letting $g(a, b)$ go to ∞. Fishburn is especially interested in the case where g is into $[0,1]$ and can be written

$$g(a, b) = g[\,|\phi_A(a) - \phi_A(b)|\,].$$

Tversky's 2-component additive difference model is somewhat different in that

$$(a,x) \succsim (b,y) \text{ iff } \phi_A(a) - \phi_A(b) \geqslant f_A^{-1} f_P\, [\phi_P(y) - \phi_P(x)],$$

where $f_A^{-1} f_P$ is a function independent of a and b and operating on $\phi_P(y) - \phi_P(x)$.

REFERENCES

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