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DIMENSIONALLY INVARIANT NUMERICAL LAWS CORRESPOND TO MEANINGFUL QUALITATIVE RELATIONS*

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In formal theories of measurement meaningfulness is usually formulated in terms of numerical statements that are invariant under admissible transformations of the numerical representation. This is equivalent to qualitative relations that are invariant under automorphisms of the measurement structure. This concept of meaningfulness, appropriately generalized, is studied in spaces constructed from a number of conjoint and extensive structures some of which are suitably interrelated by distribution laws. Such spaces model the dimensional structures of classical physics. It is shown that this qualitative concept corresponds exactly with the numerical concept of dimensionally invariant laws of physics.

1. Introduction. The interplay between the primitive qualitative observations one can make of nature and the often developed numerical representations of them as laws in terms of dimensional variables poses at least two philosophical puzzles. When is such a numerical representation of the relevant variables possible? And given that it is, what can one say about the mathematical nature of laws formulated within that representation?

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In physics, at least, the structure of variables is the one built up from products of powers of a basic set of numerical scales such as mass, length, time, charge, etc. This familiar dimensional structure is more-or-less formally described in books on dimensional analysis. One nice axiomatization of it is due to Whitney ([15]); it can also be found in Krantz, Luce, Suppes, and Tversky ([5], pp. 460–461). So the answer to the first question becomes a matter of finding empirically valid conditions on the variables involved so that they can be mapped into the familiar dimensional structure.

One way to proceed is as follows. Axiomatize a qualitative structure of (physical) attributes as being composed of interconnected extensive and conjoint structures for the several attributes. Extensive structures are those having an operation of combining within the attribute and that operation is represented numerically by addition. Conjoint structures are those establishing a trade-off between components and that trade-off is represented numerically by multiplication of scales defined on the components. In such an axiomatization of interrelated attributes it is sufficient that the interrelations be described by certain elementary, qualitative laws. One such construction is given in Krantz *et al.* ([5], pp. 499–501); it rests on two kinds of trinary constraints which Luce ([6]) called laws of similitude and exchange. An alternative and much simpler construction rests on distribution laws and uses theorems of Narens and Luce ([9]); this development is outlined below.

Within the numerical framework of dimensional analysis, it is postulated that any law of nature satisfies the property of being dimensionally invariant. For an exact statement of what this means, see pp. 464–467 of Krantz *et al.;* intuitively, it is sufficient to say that it formalizes the idea that the form of the law does not rest upon the choice of units for the several measures. The key fact about dimensionally invariant laws—the fact that makes the method of dimensional analysis yield results of interest—is that the dimensions involved in the law can be grouped into several products of powers that are dimensionless and the law depends only on these dimensionless quantities.[1] This is the famous Π-theorem of Buckingham.

Thus, the major remaining philosophical puzzle of dimensional analysis is to explain why the numerical laws of physics (and perhaps of other sciences) must be dimensionally invariant. True, physical laws are usually dimensionally invariant, but it has never been made

[1]Professor Suppes has pointed out to me that the absolute invariance of these dimensionless quantities is reminiscent of the invariance under certain automorphisms of some geometric concepts. It may be useful to pursue this apparent connection between geometry and dimensional analysis, but I will not attempt to do it here.

clear to everyone's satisfaction why this is so. Put another way, no one has offered a qualitative definition of a possible physical law from which it follows, as a purely mathematical matter, that its numerical representation must satisfy dimensional invariance. This difficulty was discussed at some length by Krantz *et al.* ([5], pp. 504–506). See also Birkhoff ([2]), Bridgman ([3]), and Causey ([4]).

This paper proposes an answer. Informally, it is this: a relation, formulated in terms of the structure of qualitative attributes, is a candidate for a scientific law if and only if it can be expressed solely in terms of the several laws that define that qualitative structure. The essential problem in making this proposal mathematically precise is to give a clear meaning to the notion of being "expressed solely in terms of." The proposal is to treat it as invariance under the automorphisms of the qualitative structure. The test of the adequacy of that definition is whether we can prove that all such relations can be represented numerically by dimensionally invariant functions and, conversely, that all such functions define relations of this character.

2. Meaningfulness in theories of measurement. Although our primary concern is how to formulate putative qualitative laws within a structure of qualitative attributes, it will be useful first to consider a simpler, but related problem. For some time measurement theorists, working on the scaling of single attributes, have discussed the question as to which statements formulated solely in terms of a single measure can be judged to be meaningful (see Adams, Fagot, and Robinson, [1]; Pfanzagl, [10]; Robinson, [11]; Suppes, [13]; and Suppes and Zinnes, [14]. For example, everyone agrees that it is meaningful to say "the ratio of one mass to another is 5," whereas it is not meaningful either to say "the mass of this object is 5," or to say "the ratio of today's temperature (in °F or °C) to that of yesterday is 1.1."

It is pretty well agreed that the problem of meaningfulness within a single measured attribute is closely tied to knowing how two different, but equally acceptable, numerical representations of the qualitative structure are related. Meaningful numerical statements are those that remain invariant under permissible changes in the representation. That, however, is very similar to saying, in the more complex case of physical laws, that the representation must be dimensionally invariant. So, on the face of it, there appears to be a close relationship between meaningful statements in a single attribute and dimensionally invariant laws stated in terms of several attributes. Indeed, the position I shall take is that they are both special cases of a general concept of meaningfulness in relational structures, which I formulate in the next two sections.

3. Automorphisms in relational and measurement structures.

Definition 1. If A is a set, $I = \{1, \ldots, n\}$ is an index set, and S_i, i in I, are k_i-ary relations on A (i.e., subsets of A^{k_i}), then $\mathscr{A} = \langle A, (S_i)_{i \in I}\rangle$ is a *relational structure.* A relational structure with $A \subseteq Re$ is called *numerical* and the notation is altered to $\mathscr{R} = \langle R, (T_i)_{i \in I}\rangle$. If $\mathscr{A}$ is a relational structure and $\mathscr{R}$ is a numerical one, then $\langle \mathscr{A}, \mathscr{R}\rangle$ is a *measurement structure* iff

(i) both have the same index set I and for all i in I, S_i and T_i are both k_i-ary relations;

(ii) T_1 is $\geq$; and

(iii) there is a homomorphism ϕ from $\mathscr{A}$ onto $\mathscr{R}$, i.e., ϕ is a function from A onto R such that for all i in I and for all $a_1, \ldots, a_{k_i}$ in A, $S_i(a_1, \ldots, a_{k_i})$ iff $T_i(\phi(a_1), \ldots, \phi(a_{k_i}))$.

The set of all such homomorphisms is denoted $\Phi(\mathscr{A}, \mathscr{R})$.

With no loss of generality in a measurement structure we assume that S_1 is a weak order and that the resulting equivalence relation has been factored out. Thus, any ϕ in $\Phi(\mathscr{A}, \mathscr{R})$ is an isomorphism and so it is meaningful to refer to ϕ^{-1}.

In the following it will be convenient to abbreviate $S_i(a_1, \ldots, a_{k_i})$ by $S_i(a_j)$. The ambiguity of this notation should not create problems.

Definition 2. Suppose $\mathscr{A} = \langle A, (S_i)_{i \in I}\rangle$ is a relational structure. A mapping θ from A onto A is an *automorphism of* $\mathscr{A}$ if and only if it is *1:1* and for all i in I, and all a_j in A, $j = 1, \ldots, k_i$,

$$S_i(a_j) \text{ iff } S_i(\theta(a_j)).$$

The following theorem discusses the close relations between the qualitative and numerical automorphisms of a measurement structure. In particular, part (iv) shows that the automorphisms of the representation $\mathscr{R}$ of $\mathscr{A}$ correspond to what is usually referred to as the class of admissible transformations on the isomorphisms. They constitute what is sometimes called the *scale type* of the measurement structure.

Theorem 1. Suppose that $\langle \mathscr{A}, \mathscr{R}\rangle$ is a measurement structure and ϕ is in $\Phi(\mathscr{A}, \mathscr{R})$.

(i) If θ is an automorphism of $\mathscr{A}$, then $\phi\theta\phi^{-1}$ is an automorphism of $\mathscr{R}$.

(ii) If ζ is an automorphism of $\mathscr{R}$, then $\phi^{-1}\zeta\phi$ is an automorphism of $\mathscr{A}$.

(iii) ϕ' is in $\Phi(\mathscr{A},\mathscr{R})$ iff there exists an automorphism θ of $\mathscr{A}$ such that $\phi' = \phi\theta$.

(iv) ϕ' is in $\Phi(\mathscr{A},\mathscr{R})$ iff there exists an automorphism ζ of $\mathscr{R}$ such that $\phi' = \zeta\phi$.

Proof.

(i) Suppose x_j are in R, then

$$\begin{aligned} T_i(\phi\theta\phi^{-1}(x_j)) \quad &\text{iff } S_i(\theta\phi^{-1}(x_j)) && (\phi \text{ is an isomorphism}) \\ &\text{iff } S_i(\phi^{-1}(x_j)) && (\theta \text{ is an automorphism}) \\ &\text{iff } T_i(x_j), && (\phi \text{ is an isomorphism}) \end{aligned}$$

and so $\phi\theta\phi^{-1}$ is an automorphism of $\mathscr{R}$.

(ii) Similar to (i).

(iii) Suppose θ is an automorphism of $\mathscr{A}$, then for a_j in A,

$$\begin{aligned} T_i(\phi\theta(a_j)) \quad &\text{iff } S_i(\theta(a_j)) && (\phi \text{ is an isomorphism}) \\ &\text{iff } S_i(a_j), && (\theta \text{ is an automorphism}) \end{aligned}$$

whence $\phi\theta$ is an isomorphism.

Conversely, suppose ϕ' is an isomorphism and let $\theta = \phi^{-1}\phi'$.

$$\begin{aligned} S_i(\theta(a_j)) \quad &\text{iff } T_i(\phi\theta(a_j)) && (\phi \text{ is an isomorphism}) \\ &\text{iff } T_i(\phi'(a_j)) && (\text{definition of } \theta) \\ &\text{iff } S_i(a_j), && (\phi' \text{ is an isomorphism}) \end{aligned}$$

so θ is an automorphism.

(iv) Similar to (iii).

4. Meaningful relations. The intuition of meaningfulness in relational structure $\langle A, (S_i)_{i\in I}\rangle$ is simple enough: to be meaningful a relation should, in some sense, be expressible in terms of the several relations S_i, $i \in I$, that define the structure. It is, however, not easy to capture this concept directly, and an indirect definition seems to be most satisfactory.

Much is revealed about a structure by knowing its automorphisms—those 1:1 transformations of itself that leave the defining relations unchanged.[2] It seems clear that whenever a relation is expressible

[2]As F. Roberts has pointed out to me, an ambiguity exists concerning exactly which transformations I mean on the underlying structure since, just prior to Definition 2, I factored out the equivalence relation. The intention is endomorphisms here, not automorphisms.

in terms of the defining relations of the structure it must be invariant under the automorphisms of the structure. What is less transparent, and apparently not fully understood, is the sense in which a relation that is invariant under the automorphisms of a structure is expressible in some manner in terms of the defining relations.

The problem can be illustrated by the notion of independence in probability theory. Let $\langle X, \mathscr{E}, \succcurlyeq\rangle$ be a qualitative probability structure (see Chapter 5 of Krantz *et al.*) with an additive numerical representation P. One usually defines independence $\perp_p$ by, for A, B in $\mathscr{E}$,

$$A \perp_p B \text{ iff } P(A.B) = P(A)P(B).$$

However, there are structures with two (or more) representations P and P' for which $\perp_p$ is not the same as $\perp_{p'}$. Thus, although $\perp_p$ is invariant under automorphisms, by Padoa's principle (Suppes, [12], p. 169), it cannot be expressed as an explicit function of the defining relations of $\langle X, \mathscr{E} \succcurlyeq\rangle$. (For this reason, I believe it to be a mistake not to include independence as a primitive of qualitative probability that is to be mapped into multiplication under the homomorphism.)

Despite the uncertainty as to the implications for definability of a relation that is invariant under automorphisms, I shall accept the latter as a suitable definition of meaningfulness. Formally, we state this as follows:

Definition 3. Suppose $\mathscr{A} = \langle A, (S_i)_{i\in I}\rangle$ is a relational structure and S is a relation on A. S is *$\mathscr{A}$-meaningful* if and only if for every automorphism θ of $\mathscr{A}$, and all a_j in A, $j = 1, \ldots, k$,

$$S(a_j) \text{ iff } S(\theta(a_j)).$$

The intuitive motivation for this concept of meaningfulness—the idea that a relation can be expressed in terms of those that define a relational structure—I have already discussed. To a considerable extent, the decision as to whether the definition is successful rests upon the results that can be proved. In particular, we will show, in the context of measurement structures, that the addition of a meaningful relation to the defining relations of the structure does not in any way alter the set of homomorphisms (Theorem 3). And in the context of dimensional analysis we show that meaningfulness is equivalent to dimensional invariance (Theorem 5).

5. Meaningful relations in measurement structures.

Theorem 2. Suppose $\mathscr{A} = \langle A, (S_i)_{i\in I}\rangle$ and $\mathscr{R} = \langle R, (T_i)_{i\in I}\rangle$ are relational structures such that $\langle \mathscr{A}, \mathscr{R}\rangle$ is a measurement structure. Suppose S and T are relations on A and R, respectively, such that for some ϕ in $\Phi(\mathscr{A}, \mathscr{R})$ and all a_j in A, $j = 1, \ldots,$ k,

$S(a_j)$ iff $T(\phi(a_j))$.

Then, S is $\mathscr{A}$-meaningful if and only if T is $\mathscr{R}$-meaningful.

Proof. Suppose S is $\mathscr{A}$-meaningful. Let ζ be any $\mathscr{R}$-automorphism. By Theorem 1.ii $\phi^{-1}\zeta\phi$ is an automorphism in $\mathscr{A}$ and so

$$\begin{array}{lll} T(\zeta(x_j)) & \text{iff } S(\phi^{-1}\zeta(x_j)) & (\phi \text{ is an isomorphism}) \\ & \text{iff } S(\phi^{-1}\zeta\phi\phi^{-1}(x_j)) & \\ & \text{iff } S(\phi^{-1}(x_j)) & (S \text{ is } \mathscr{A}\text{-meaningful}) \\ & \text{iff } T(x_j), & (\phi \text{ is an isomorphism}) \end{array}$$

so T is $\mathscr{R}$-meaningful. The converse is similar.

The import of this theorem is that the concept of meaningfulness is preserved perfectly under the isomorphisms of a measurement structure. The next result establishes that, as defined, a meaningful relation does not impose any further constraint on a measurement structure.

Theorem 3. Suppose $\langle \mathscr{A}, \mathscr{R} \rangle$ is a measurement structure and S a relation on A. S is $\mathscr{A}$-meaningful if and only if there is relation T on R of the same order as S such that, for

$$\mathscr{A}' = \langle A, (S_i)_{i\in I}, S\rangle, \mathscr{R}' = \langle R, (T_i)_{i\in I}, \mathrm{T}\rangle,$$

$\langle \mathscr{A}', \mathscr{R}' \rangle$ is a measurement structure and $\Phi(\mathscr{A}', \mathscr{R}') = \Phi(\mathscr{A}, \mathscr{R})$.

Proof. Suppose S is $\mathscr{A}$-meaningful and let ϕ_0 be in $\Phi(\mathscr{A}, \mathscr{R})$. Define T by:

$T(x_j)$ iff $S(\phi_0{}^{-1}(x_j))$.

$\langle \mathscr{A}', \mathscr{R}' \rangle$ is a measurement structure since ϕ_0 is an isomorphism. Clearly $\Phi(\mathscr{A}', \mathscr{R}') \subseteq \Phi(\mathscr{A}, \mathscr{R})$. Suppose ϕ is in $\Phi(\mathscr{A}, \mathscr{R})$. By Theorem 1.iii, $\phi_0{}^{-1}\phi = \theta$ is an automorphism, and so

$$\begin{array}{ll} S(a_j) & \text{iff } S(\phi_0{}^{-1}\phi(a_j)) \\ & \text{iff } T(\phi(a_j)), \end{array}$$

whence ϕ is in $\Phi(\mathscr{A}', \mathscr{R}')$.

Conversely, suppose T exists such that $\Phi(\mathscr{A}', \mathscr{R}') = \Phi(\mathscr{A}, \mathscr{R})$. Let θ be any automorphism of $\mathscr{A}$. By Theorem 1.iii, if ϕ is an isomorphism of these structures, so is $\phi\theta$, whence

$$\begin{array}{ll} S(\theta(a_j)) & \text{iff } T(\phi\theta(a_j)) \\ & \text{iff } S(a_j), \end{array}$$

and so S is $\mathscr{A}$-meaningful.

6. Space of distributive attributes. We are ready now to turn our attention to the problem of the qualitative equivalent of dimensionally invariant laws. To this end, we proceed as follows. First, in this section we formulate the concept of a space of distributive (qualitative) attributes which are interrelated by elementary distribution laws. It follows rather directly that such a space can be isomorphically imbedded in a structure of physical quantities as defined by Whitney (15) (see Definition 10.1 of [5], pp. 460–461). The concept of a dimensionally invariant law is defined in such a structure (see [5] pp. 465–466). In the following section we first show that such a space of distributive attributes is a relational structure (*Lemma 1*), and so the concept of a meaningful relation (*Definition 3*) exists in such a space. Finally, we prove that, under the isomorphism between the space of distributive attributes and the structure of physical quantities, meaningful relations correspond exactly to dimensionally invariant laws.

Suppose A and P are sets and $\geqslant$ is a binary relation on $A \times P$. We say $\langle A \times P, \geqslant\rangle$ is a *solvable conjoint structure* if the following three properties hold for all a, b in A and p, q in P.

1. *Weak ordering:* $\geqslant$ is transitive and connected.
2. *Independence:* (i) $(a,p) \geqslant (a,q)$ iff $(b,p) \geqslant (b,q)$, and (ii) $(a,p) \geqslant (b,p)$ iff $(a,q) \geqslant (b,q)$.
3. *Solvable:* Given any three of a,b in A and p,q in P, the fourth exists such that $(a,p) \geqslant (b,q)$ and $(b,q) \geqslant (a,p)$, which we abbreviate by $(a,p) \sim (b,q)$ throughout the rest of this paper.

Note that property 2 induces a natural weak ordering $\geqslant_A$ on A and $\geqslant_P$ on P.

As is discussed in [9], somewhat weaker versions of solvability suffice.

Denote by o_A, o_P, and o binary operations (not necessarily closed) on, respectively, A, P, and $A \times P$. We say that such an operation is *distributive* if for all a,b,c,d in A and p,q,r,s in P and whenever the indicated operations are defined the appropriate following property holds:

o_A: if $(a,p) \sim (c,q)$ and $(b,p) \sim (d,q)$, then $(a o_A b,p) \sim (c o_A d,q)$,

o_P: if $(a,p) \sim (b,r)$ and $(a,q) \sim (b,s)$, then $(a,p o_P q) \sim (b,r o_P s)$.

o: case 1. $(a,p)\mathrm{o}(a,q) \sim (a,r)$ iff $(b,p)\mathrm{o}(b,q) \sim (b,r)$, or
case 2. $(a,p)\mathrm{o}(b,p) \sim (c,p)$ iff $(a,q)\mathrm{o}(b,q) \sim (c,q)$, or both.

Examples of distributive operations are ubiquitous in physics. Whenever numerical variables x,y, and z are related by an equation of the form $z = x^{\alpha} y^{\beta}$, then any operation represented by addition is distributive. For example, suppose the x-variable has a + operation, then the given equations are:

$$x_1{}^{\alpha}y^{\beta} = u_1{}^{\alpha}v^{\beta} \text{ and } x_2{}^{\alpha}y^{\beta} = u_2{}^{\alpha}v^{\beta}.$$

Taking the $1/\alpha$ root,

$$\begin{aligned}(x_1 + x_2)y^{\beta/\alpha} &= x_1y^{\beta/\alpha} + x_2y^{\beta/\alpha}\\ &= u_1v^{\beta/\alpha} + u_2v^{\beta/\alpha}\\ &= (u_1 + u_2)v^{\beta/\alpha}.\end{aligned}$$

Raising this to the α power yields the numerical analogue of the qualitative property defined. What Narens and Luce showed was that the equations must be of this form of products of powers whenever distribution laws obtain. We make their result precise.

A solvable conjoint structure $\langle A \times P, \succcurlyeq\rangle$ is called a *distributive triple* iff

(i) at least two of the operations o_A, o_P, and o exist and each is distributive;

(ii) whichever of the structures $\langle A, \succcurlyeq_A, o_A\rangle$, $\langle P, \succcurlyeq_P, o_P\rangle$, $\langle A \times P, \succcurlyeq, o\rangle$ exist are extensive structures with additive representations ϕ_A, ϕ_P, ϕ, respectively (see [5], p. 84, for the exact definition).

Any distributive triple has an order preserving representation of the form

$$\phi(a,p) = \alpha\phi_A(a)^{\beta_A}\phi_P(p)^{\beta_P},$$

where the ϕ's are additive representations whenever the corresponding operation exists, and this is unique up to multiplication by a positive constant and to being raised to a positive power ([9], pp. 213–220). Put another way, the existence of distribution laws forces the conjoint structure to be additive in the sense that the logarithm of the multiplicative representation exists and is additive.

Definition 4. Let $\mathscr{Q}$ consist of finitely many relational structures $\mathscr{A}_i = \langle A_i, (S_{ij})_{j\in I_i}\rangle$, $i = 1, \ldots, D$. $\mathscr{Q}$ is said to be a *space of distributive attributes* iff

(i) for each $\mathscr{A}_i$ *in* $\mathscr{Q}$ there is a numerical relational structure $\mathscr{R}_i$ such that $\langle \mathscr{A}_i, \mathscr{R}_i\rangle$ forms a measurement structure (Definition 1);

(ii) for each $\mathscr{A}_i$ in $\mathscr{Q}$ there are two other structures in $\mathscr{Q}$ such that the three constitute a distributive triple.

Theorem 4. If $\mathscr{Q}$ is a space of distributive attributes, then there exists a subset $\mathscr{B}$ of $\mathscr{Q}$ with the following properties:

(i) All structures in $\mathscr{B}$ are extensive.

(ii) No set of three structures from $\mathscr{B}$ forms a distributive triple.

(iii) Let $\mathscr{B}$ have B members and let ϕ_i, $i = 1, \ldots, B$, be additive representations of the members of $\mathscr{B}$. For any structure in $\mathscr{Q}$ but not in $\mathscr{B}$ with a representation ψ, which is additive if the structure is extensive, there exist unique constants $\alpha > 0$ and ρ_i, $i = 1, \ldots, B$ such that

$$\psi = \alpha \prod_{i=1}^{B} \phi_i^{\rho_i}. \tag{1}$$

Proof. Let $\mathscr{E}$ denote the subset of extensive structures in $\mathscr{Q}$. For any attribute in $\mathscr{Q}$ but not in $\mathscr{E}$ there exist, by Definition 4, two extensive structures that complete a distributive triple. The Narens-Luce theorem states that its representation will be of the form given in Equation 1. And for any triple of attributes in $\mathscr{E}$ that form a distributive triple, one can be eliminated in favor of the other two, again satisfying Equation 1. Continue this inductively until a set $\mathscr{B}$ is found for which no distributive triple holds wholly within $\mathscr{B}$. Thus $\mathscr{B}$ satisfies parts (i) and (ii).

The imbedding of this representation into a structure of physical quantities as defined in [5] is exactly as in their Theorem 10.11, p. 501.

7. Dimensional invariance. In the theory of dimensional analysis, a *similarity* is defined to be, in essence, any change of units of the structures in $\mathscr{B}$ along with the corresponding changes of units induced by Equation 1. If $x_1, \ldots, x_n$ are n variables in the structure of physical quantities and f is a function from them into the reals, f is said to be *dimensionally invariant* provided that for every similarity ζ

$$f(x_1, \ldots, x_n) = 0 \text{ iff } f[\zeta(x_1), \ldots, \zeta(x_n)] = 0.$$

It is assumed that all physical laws are dimensionally invariant. For precise and general definitions of similarity and dimensional invariance see pp. 465–466 of [5], and for a discussion of the problem of explaining why physical laws are dimensionally invariant, see pp. 504–506.

As an example of these notions, let a thin square plate of area A be placed at angle α in a fluid of density d, viscosity μ, and moving at velocity v. Let L be the resulting lift. By assuming that the relation relating these variables (of known dimensions in terms of length, mass, and time) is dimensionally invariant, it is not difficult to show that it must be of the form

$$L = \nu^2 dA (\sin \alpha)^2 \cos \alpha\, G(R \sin \alpha),$$

where G is an unknown function and $R = \nu ld/\mu$ is the so-called Reynolds number. It is easily verified that both R and $L/\nu^2 dA$ are dimensionless quantities and so they are absolutely invariant under any similarity, hence the law is indeed dimensionally invariant.

Why should one assume physical laws to be dimensionally invariant? This was interpreted by Krantz *et al.* ([5], p. 505) as the need for a qualitative definition of a law and the proof that every such law has a corresponding dimensionally invariant numerical representation. The present proposal is to define a relational structure that corresponds to a space of distributive attributes and then treat as a putative physical law any meaningful relation in that structure, where of course meaningful means invariance under automorphisms. A specific example of this approach in a particular physical context is McKinsey and Suppes [8].

Definition 5. Suppose $\mathscr{Q}$ is a space of distributive attributes (Definition 4) composed of relational structures $\mathscr{A}_i = \langle A_i, (S_{ij})_{j \in I}\rangle$, $i = 1, 2, \ldots, D$, interlocked by distribution laws. Define the relational structure $\mathscr{Q}^*$ on $A = \cup_{i=1}^{D} A_i$ as having the following relations:

$S^*_{ij}(a_k)$ iff $a_k \in A_i$ and $S_{ij}(a_k)$,

and for a distributive triple $\mathscr{A}_i$, $\mathscr{A}_j$, $\mathscr{A}_k$ where $A_i = A_j \times A_k$, the orderings are $\succcurlyeq_i$, $\succcurlyeq_j$, and $\succcurlyeq_k$ respectively, and distributiveness is on $\mathscr{A}_j$, then for a, b, c, d, p, q in A,

$S^*_{ijk}(a, b, c, d, p, q)$ iff $a, b, c, d \in A_j$, p, $q \in P$
$(a,p) \sim_i (c,q)$, $(b,p) \sim_i (d,q)$, and $(ao_j b,p) \sim_i (co_j d,q)$.

Lemma 1. Suppose $\mathscr{Q}$ is a space of distributive attributes. Then θ is an automorphism of $\mathscr{Q}^*$ (Definition 5) iff

(i) the restriction of θ to A_i, θ_i, is an automorphism of $\mathscr{A}_i$, and

(ii) if $\mathscr{A}_i$, $\mathscr{A}_j$, $\mathscr{A}_k$ form a distributive triple, where $\mathscr{A}_i$ is the conjoint structure, then (θ_j, θ_k) is an automorphism of $\mathscr{A}_i$.

Proof. Suppose θ is an automorphism and θ_i its restriction to $\mathscr{A}_i$. Since all relations of $\mathscr{Q}^*$ are invariant under θ, those of $\mathscr{A}_i$ in particular are; moreover, since the relations of $\mathscr{A}_i$ are defined on A_i, the invariance can hold only if θ_i maps into A_i. So θ_i is an automorphism.

Now consider a distributive triple $\mathscr{A}_i$, $\mathscr{A}_j$, $\mathscr{A}_k$, and consider the mapping

$$\zeta_i(a,p) = (\theta_j(a), \theta_k(p)).$$

To show that ζ_i is an automorphism of $\mathscr{A}_i$, select any representation ϕ_i of $\mathscr{A}_i$. By Narens and Luce [9], we know there are representations ϕ_j of $\mathscr{A}_j$ and ϕ_k of $\mathscr{A}_k$, unique up to multiplication by a constant such that

$$\phi_i = \phi_j^{\beta_j}\, \phi_k^{\beta_k}.$$

Using the uniqueness results and Theorem 1,

$$\begin{aligned}\phi_i \zeta_i(a,p) &= [\phi_j \theta_j(a)]^{\beta_j}\, [\phi_k \theta_k(p)]^{\beta_k}\\ &= \alpha_j^{\beta_j}\, \phi_j(a)^{\beta_j}\, \alpha_k^{\beta_k}\, \phi_k(p)^{\beta_k}\\ &= \alpha\, \phi_i(a,p),\end{aligned}$$

where $\alpha = \alpha_j^{\beta_j}\, \alpha_k^{\beta_k}$. But, by Theorem 1, $\zeta_i = \phi_i^{-1} \alpha\, \phi_i$ is an automorphism of $\mathscr{A}_i$.

Conversely, suppose a function θ satisfies properties (i) and (ii). The only thing that needs to be shown is that the relations corresponding to distributive laws are invariant under θ. Let

$$\begin{aligned}
S(a,b,c,d,p,q) \quad &\text{iff } (a,p) \sim_i (c,q),\\
&\phantom{\text{iff }} (b,p) \sim_i (d,q),\\
&\phantom{\text{iff }} (ao_j b,p) \sim_i (co_j d,q) \qquad \text{(Definition of } S)\\
&\text{iff } (\theta_j(a),\theta_k(p)) \sim_i (\theta_j(c),\theta_k(q)),\\
&\phantom{\text{iff }} (\theta_j(b),\theta_k(p)) \sim_i (\theta_j(d),\theta_k(q)),\\
&\phantom{\text{iff }} (\theta_j(ao_j b),\theta_k(p)) \sim_i (\theta_j(co_j d),\theta_k(q))\\
&\phantom{\text{iff }} \text{(Parts } i \text{ and } ii)\\
&\text{iff } (\theta_j(a),\theta_k(p)) \sim_i (\theta_j(c),\theta_k(q))\\
&\phantom{\text{iff }} (\theta_j(b),\theta_k(p)) \sim_i (\theta_j(d),\theta_k(q))\\
&\phantom{\text{iff }} (\theta_j(a) o_j \theta_j(b),\theta_k(p)) \sim_i (\theta_j(c) o_j \theta_j(d),\theta_k(q))\\
&\phantom{\text{iff }} \text{(Part } i \text{ for an extensive structure)}\\
&\text{iff } S(\theta_j(a),\theta_j(b),\theta_j(c),\theta_j(d),\theta_k(p),\theta_k(q))\\
&\phantom{\text{iff }} \text{(Definition of } S)\\
&\text{iff } S(\theta(a),\theta(b),\theta(c),\theta(d),\theta(p),\theta(q))\\
&\phantom{\text{iff }} \text{(Definition of } \theta_j, \theta_k).
\end{aligned}$$

Theorem 5. Suppose $\mathscr{Q}$ is a space of distributive attributes $\mathscr{A}_i$, $i = 1, 2, \ldots, D$, and $\langle A, A^+, *\rangle$ is the structure of physical quantities arising from the isomorphisms ϕ_i, $i = 1, \ldots, B$, as provided in Theorem 10.11 of [5]. Let $S = (S_1, \ldots, S_D)$, where S_i is a relation of order k_i on A_i. Let P_i be the positive part of the numerical representation ψ_i of $\mathscr{A}_i$. Define f from $\displaystyle\bigtimes_{i=1}^{D} P_i^{k_i}$ into Re as follows: for x_{ij} in P_i, $j = 1, \ldots, k_i$,

$$f((x_{ij})_{i=1,\ldots,D}) = 0 \text{ iff } \psi_i^{-1}(x_{ij})_{j=1,\ldots,k_i} \in S_i,\ i = 1, \ldots, D.$$

Then, f is dimensionally invariant if and only if S *is* $\mathscr{Q}^*$-meaningful.

Proof. Suppose f is dimensionally invariant, i.e., for any similarity ζ

$$f(x_{ij}) = 0 \text{ iff } f(\zeta(x_{ij})) = 0.$$

(For exact definitions, see pp. 465–466 of [5].) Let $\phi = (\phi_1, \ldots, \phi_B)$ be the collection of isomorphisms used to construct the space of physical attributes. Let θ be any automorphism of $\mathscr{Q}^*$. We show that $\zeta = \phi\theta\phi^{-1}$ is a similarity. For the given measurement structures, $\zeta_i = \phi_i\theta_i\phi_i^{-1}$ is an automorphism of $\mathscr{A}_i$ by Lemma 1 and Theorem 1.i. For the base dimensions, which are extensive and so ratio scales, it is a constant, which we continue to write as ζ_i. Let ρ be any dimension characterized by the numbers $\rho_1, \ldots, \rho_B$, and let ψ_ρ be a representation of that dimension, then by Theorem 10.11 of [5],

$$\begin{aligned}\zeta\psi_\rho &= (\zeta_1\phi_1)^{\rho_1}(\zeta_2\phi_2)^{\rho_2}\cdots(\zeta_b\phi_b)^{\rho_B}\\ &= \zeta_1^{\rho_1}\zeta_2^{\rho_2}\cdots\zeta_b^{\rho_b} * \phi_1^{\rho_1}\phi_2^{\rho_2}\cdots\phi_B^{\rho_B}\\ &= \zeta_1^{\rho_1}\zeta_2^{\rho_2}\cdots\zeta_B^{\rho_B} * \psi_\rho.\end{aligned}$$

Therefore, by Theorems 10.3 and 10.11, ζ is a similarity. So for $a_{ij} \in A_i$, $i = 1, \ldots, D$

$$\begin{aligned} a_{ij} \in S_i \quad &\text{iff } f(\psi_i(a_{ij})) = 0 && \text{(definition of } f)\\ &\text{iff } f(\psi_i\theta_i\psi_i^{-1}\psi_i(a_{ij})) = 0 && \text{(dimensional invariance)}\\ &\text{iff } \psi_i^{-1}\psi_i\theta_i(a_{ij}) \in S_i && \text{(definition of } f)\\ &\text{iff } \theta_i(a_{ij}) \in S_i, \end{aligned}$$

and so S is $\mathscr{Q}^*$-meaningful.

Conversely, suppose S is $\mathscr{Q}^*$-meaningful and ζ is a similarity. By Theorems 10.3 and 10.11 of [5], if ψ_ρ denotes a dimension with base exponents $\rho_1, \ldots, \rho_B$, then

$$\zeta\psi_\rho = \zeta_1^{\rho_1}\cdots\zeta_B^{\rho_B} * \psi_\rho.$$

From this, Lemma 1, and the fact that for these extensive structures multiplication by a constant is an automorphism, ζ is an automorphism of the dimensional structure. For each of the given structures $\mathscr{A}_i$, Theorem 1, iii implies there is an automorphism θ_i of $\mathscr{A}_i$ such that $\zeta_i = \psi_i\theta_i\psi_i^{-1}$. Then for $x_{ij} \in P_i$, $i = 1, \ldots, D$,

$$\begin{aligned} f(\zeta_i(x_{ij})) = 0 \quad &\text{iff } \psi_i^{-1}\zeta_i(x_{ij}) \in S_i && \text{(definition of } f)\\ &\text{iff } \psi_i^{-1}\psi_i\theta_i\psi_i^{-1}(x_{ij}) \in S_i && \text{(definition of } \theta)\\ &\text{iff } \psi_i^{-1}(x_{ij}) \in S_i && (S \text{ is } \mathscr{Q}^*\text{-meaningful})\\ &\text{iff } f(x_{ij}) = 0, && \text{(definition of } f) \end{aligned}$$

and so f is dimensionally invariant.

It may be useful to illustrate the theorem in a simple, concrete case such as the force on a plate in a moving fluid, which was mentioned previously. The space of distributive attributes must include the following measurement structures together with a number of distributive laws interlocking the last five with the first three.

$\mathscr{A}_L$ —length measurement with the homomorphism ϕ_L,
$\mathscr{A}_M$ — mass measurement with the homomorphism ϕ_M,
$\mathscr{A}_T$ —time measurement with the homomorphism ϕ_T,
$\mathscr{A}_A$ —area measurement with the homomorphism $\phi_A = \phi_L^2$,
$\mathscr{A}_\nu$ —velocity measurement with the homomorphism $\phi_\nu = \phi_L \phi_\tau^{-1}$,
$\mathscr{A}_F$ —force measurement (e.g., masses acted on by gravity) with the homomorphism $\phi_F = \phi_L \phi_M \phi_T^{-2}$.
$\mathscr{A}_d$ — density measurement (fluids and gases) with the homomorphism $\phi_d = \phi_L^{-3} \phi_M$,
$\mathscr{A}_\mu$ —viscosity measurement (fluids and gases) with the homomorphism $\phi = \phi_L^{-1} \phi_M \phi_T^{-1}$.

Note that the domains of $\mathscr{A}_d$ and $\mathscr{A}_\mu$ are the same, fluids and gases, so $A_d = A_\mu = A_f$, for fluid. Now, if we place a plate at some fixed angle in a fluid moving at a constant velocity, the raw qualitative data is the 5—tuple of qualitative observations

$(a_L, a_A, a_\nu, a_F, a_f) =$ (length of plate, area of plate, velocity of plate relative to the fluid, force on plate, fluid).

The set of all such observations constitutes the qualitative law, the relation s. It is found to be represented by the dimensionally invariant numerical law

$$0 = 1 - \phi_A(a_a)\phi_\nu(a_\nu)^2\phi_F^{-1}(a_F)\phi_d(a_f)F[\phi_L(a_L)\phi_\nu(a_\nu)\phi_d(a_f)\phi_\mu^{-1}(a_f)],$$

where F depends both on the configuration of the plate and its angle in the fluid.

The example includes a most interesting type of measurement structure, whose underlying relational structure is called a physical system and whose homomorphisms are called dimensional constants, namely, the systems of fluids and the dimensional constants density and viscosity. Such classes of systems exhibit a property called physical similarity which allows one to give the class a dimensionally invariant representation by associating to it a system measure, whose value for a particular system is its dimensional constant. This special, but very important, case of dimensional invariance has been treated in detail by Causey ([4]), Krantz *et al.* ([5], pp. 506–513), and Luce ([7]).

8. Conclusion. The philosophical problem raised was: given a physical (or perhaps other) law that can be formulated as a constraint on the combinations of dimensional quantities which can occur, why should it be dimensionally invariant? The answer proposed is that if one examines the qualitative structure underlying the dimensional quantities, the set of dimensionally invariant laws corresponds exactly to the set of meaningful qualitative relations. To understand this answer, one must specify exactly the nature of the underlying qualitative structure and the concept of a meaningful relation in that structure.

The structure is taken to be a system of extensive and conjoint structures closely interconnected by a number of distribution laws. This definition is adequate for classical physics, but it is too restrictive for modern physics (e.g., relativistic velocity does not relate to distance *via* a distribution law) and probably for other sciences.

A relation within a relational structure, including this special case, is defined to be meaningful if and only if it is invariant under the automorphisms of the structure. Intuitively, this means that the qualitative relation corresponding to a (dimensionally invariant) physical law does not further constrain the relations among the dimensions in terms of which it is expressed. Put another way, whenever a law is discovered that is not dimensionally invariant within the current structure of dimensions, it is introduced as a new constraint among them. What is perhaps surprising is that these dimensional constraints never involve more than three dimensions. Since this follows rather automatically from the fact that every numerical representation of an attribute can be written as products of powers of a finite set of base representations, the real surprise is that fact. Should new concepts, whether in physics or the other sciences, ever lead outside this simple dimensional structure, then the nature of the constraints could become far more complex.

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