JOURNAL OF MATHEMATICAL PSYCHOLOGY 16, 91–93 (1977)

# A Note on Sums of Power Functions[1]

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In studying the applicability of additive conjoint measurement to binaural loudness summation, Levelt *et al.* (1972) found that the two component functions each grow as a power function of auditory intensity. Moreover, the exponents, which differed slightly for the two ears, were in the range found in magnitude estimation. For a statistically more sophisticated approach to the problem, which casts some doubt on the strict additivity of loudness, see Falmagne (1976).

The purpose of this note is to establish that power functions must follow if three conditions are met: (i) Additivity holds: (ii) the functions are sufficiently smooth; and (iii) for identical ears a loudness match is not disrupted by increasing all of the intensities by a constant factor (adding the same number of decibels throughout). If the two ears differ, then in order to preserve a match, the factor on one ear is a power function of that applied to the other ear. The precise result is the following:

THEOREM. *Suppose F and G are strictly increasing, twice differentiable functions with positive first derivatives defined on the positive reals.*

1. *Suppose there exist unique positive integers m, n such that for all positive integers i and all positive numbers x, y, u, v,*

$$F(x) + G(y) = F(u) + G(v), \tag{1}$$

*implies*

$$F(i^m x) + G(i^n y) = F(i^m u) + G(i^n v). \tag{2}$$

*Then there are constants* $\alpha_F, \alpha_G, \beta > 0$ *and* $\gamma_F, \gamma_G$ *such that*

$$F(x) = \alpha_F x^{\beta/m} + \gamma_F$$
$$G(x) = \alpha_G x^{\beta/n} + \gamma_G .$$

2. *If for all positive integers m, n, i, Eq.* (1) *implies Eq.* (2), *then there are constants* $\alpha_F, \alpha_G > 0$ *and* $\gamma_F, \gamma_G$ *such that*

$$F(x) = \alpha_F \log x + \gamma_F$$
$$G(x) = \alpha_G \log x + \gamma_G .$$

[1] This work was supported in part by a grant from the National Science Foundation to Harvard University. I thank Professor J.-C. Falmagne for his helpful comments.





ISSN 0022-2496

*Proof.* Let $x, y$ be arbitrary positive numbers, and let $\epsilon$ be sufficiently small that by the fact that $F$ and $G$ are strictly increasing and continuous there is $\phi(\epsilon)$ such that

$$F(x + \epsilon) + G(y - \phi(\epsilon)) = F(x) + G(y).$$

By Eq. (2),

$$F(i^m x + i^m \epsilon) + G(i^n y - i^n \phi(\epsilon)) = F(i^m x) + G(i^n y).$$

By the existence of the second derivative,

$$F(x + \epsilon) = F(x) + \epsilon F'(x) + O(\epsilon^2).$$

So

$$\epsilon F'(x) + O(\epsilon^2) = \phi(\epsilon)\, G'(y) + O(\phi(\epsilon)^2)$$

$$i^m \epsilon F'(i^m x) + i^m O(\epsilon^2) = i^n \phi(\epsilon)\, G'(i^n y) + i^n O(\phi(\epsilon)^2).$$

Divide the last equation by the previous one,

$$\frac{i^m F'(i^m x) + i^m O(\epsilon^2)/\epsilon}{F'(x) + O(\epsilon^2)/\epsilon} = \frac{i^n G'(i^n y) + i^n O(\phi(\epsilon)^2)/\phi(\epsilon)}{G'(y) + O(\phi(\epsilon)^2)/\phi(\epsilon)}.$$

Observe that $\phi(\epsilon) \to 0$ and $O(\epsilon^2)/\epsilon \to 0$ as $\epsilon \to 0$; so there must be a constant $k$ such that

$$\frac{i^m F'(i^m x)}{F'(x)} = k = \frac{i^n G'(i^n y)}{G'(y)}.$$

Setting $x = 1$, we see $i^m = kF'(1)/F'(i^m)$, whence

$$\frac{F'(i^m x)}{F'(1)} = \frac{F'(i^m)}{F'(1)} \frac{F'(x)}{F'(1)}.$$

By Lemma 10.13 of Krantz *et al.* (1971, p. 493),

$$F'(x) = F'(1) x^\rho.$$

So integrating

$$\begin{aligned} F(x) &= \alpha_F x^{\beta_F} + \gamma_F, \qquad && \rho \neq -1, \quad \beta_F = \rho + 1, \\ &= \alpha_F \log x + \gamma_F, && \rho = -1. \end{aligned}$$

Similarly,

$$G(x) = \alpha_G x^{\beta_G} + \gamma_G,$$

or

$$G(x) = \alpha_G \log x + \gamma_F.$$

Going back to Eq. (2), it is easily seen that it holds for all $m, n$ with the logarithm and that it holds for a unique $m$ and $n$ if and only if the power expression is true with $\beta_F = \beta/m$ and $\beta_G = \beta/n$.

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RECEIVED: February 18, 1977

*Printed in Belgium*