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THREE AXIOM SYSTEMS FOR ADDITIVE SEMIORDERED STRUCTURES*

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Abstract. Axioms are provided for extensive, probability, and (two-component, additive) conjoint structures which are semiordered, rather than weakly ordered. These are sufficient to construct the usual representation for the natural, induced weak order and a tight, threshold representation for the semiorder itself. The fundamental difficulty is in finding axioms that are simple in the primitive semiorder, rather than simple in its induced weak order; from this point of view, the results are only partially successful.

1. Introduction. A theory of measurement is any axiomatization of an ordered algebraic structure that, on the one hand, has at least one model with a plausible empirical interpretation and, on the other hand, each model is homomorphic to some numerical structure. The most familiar examples are from physics and are of three types. First are the extensive structures, with interpretations as length, mass, and the like, in which a set of objects is ordered by $\gtrsim$ ($a \gtrsim b$ means a has at least as much length, or mass, etc. as b) and has a binary operation $\circ$ ($a \circ b$ means some relevant combination of a and b). In the numerical representation, $\gtrsim$ maps to $\geqq$ and $\circ$ to $+$. Second are simple conjoint structures, with interpretations as momentum, kinetic energy, and the like, in which a Cartesian product is ordered by $\gtrsim$. In the numerical representation, $\gtrsim$ is mapped into $\geqq$ and the product structure into products of numbers. Third are probabilistic structures in which an algebra of sets (events) is ordered by $\gtrsim$. Again $\gtrsim$ maps into $\geqq$ and the union of disjoint sets into $+$. Careful expositions of such structures are given in Krantz, Luce, Suppes and Tversky (1971) and in Pfanzagl (1968).

When an ordering $\gtrsim$ is represented by $\geqq$, it follows immediately that $\gtrsim$ must be a weak order—i.e., connected and transitive. This is generally conceded to be an idealization, and some proposals aimed at achieving greater realism exist. One is probability models in which $a \gtrsim b$ is replaced by the probability that a has at least as much of the attribute as b; another remains algebraic, but invokes less stringent concepts of order and a more complex numerical representation. The most familiar, and probably the earliest, example of such structures comes from sensory psychophysics. The concept of a sensory threshold, or just noticeable difference (jnd), is simply a representation in which two stimuli are distinguishable if and only if they are at least one jnd apart. As in physics, this numerical representation was widely used long before its algebraic counterpart was axiomatized (Luce (1956)).

A complete representation theory exists for a weakened concept of order called a *semiorder* (see below for a summary). What is still lacking is an adequate theory for semiorders over more structured sets, corresponding to those arising in

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extensive, probability, and conjoint measurement. I report three such axiom systems and their corresponding representations here. As will become clear, all three results are unsatisfactory from an empirical point of view. I publish them mainly to stimulate others to consider this difficult problem in the hope that they will arrive at rather more satisfactory axiomatizations.

2. Semiorders. One way to think about generalizations of a weak order $\langle \mathscr{A}, \succsim \rangle$, where $\mathscr{A}$ is a set and $\succsim$ a relation on $\mathscr{A}$, is to decompose it into its symmetric, $\langle \mathscr{A}, \sim \rangle$, and asymmetric, $\langle \mathscr{A}, \succ \rangle$, parts; the former, which is the symmetric complement of the latter, is an equivalence relation, whereas the latter is a strict partial order. Most generalizations retain the two basic properties of the asymmetric part, namely, asymmetry and transitivity, but weaken the transitivity of its symmetric complement. In practice, it is easier to axiomatize the asymmetric part and simply define its symmetric complement as:

$$a \sim b \quad \text{iff} \quad \text{neither } a \succ b \quad \text{nor} \quad b \succ a.$$

A structure $\langle \mathscr{A}, \succ \rangle$ is an *interval order* (Fishburn (1970a, b)) if and only if

(i) it is irreflexive;

and

(ii) for all $a, b, c, d \in \mathscr{A}$, $a \succ b$ and $c \succ d$ imply either $a \succ d$ or $c \succ b$.

It is a *semiorder* (Luce (1956), Scott and Suppes (1958)) if, in addition,

(iii) for all $a, b, c, d \in \mathscr{A}$, $a \succ b$ and $b \succ c$ imply either $a \succ d$ or $d \succ c$.

If one is willing to use both $\succ$ and $\sim$, the axioms of a semiorder are more briefly stated: $\sim$ is the symmetric complement of $\succ$; $\sim$ is reflexive; $\succ \sim \succ \subset \succ$; and $\succ^2 \cap \sim^2 = \varnothing$.

One reason that semiorders are of interest for measurement is the existence of a unique induced weak order which may be handled in the usual way. (Interval orders, which I do not discuss further, induce two such orders.) We introduce this ordering in the following definitions: Let $\langle \mathscr{A}, \succ \rangle$ be an irreflexive relation, then for all $a, b \in \mathscr{A}$,

$a\, R\, b$ iff, for all $c \in \mathscr{A}$, $c \succ a$ implies $c \succ b$ and $b \succ c$ implies $a \succ c$,

$a \approx b$ iff, for all $c \in \mathscr{A}$, $a \sim c$ is equivalent to $b \sim c$,

$a\, I\, b$ iff $a\, R\, b$ and $b\, R\, a$,

$a\, P\, b$ iff $a\, R\, b$ and not $b\, R\, a$.

It is not difficult to show that $\approx$ is an equivalence relation and that for a semiorder

$a\, P\, b$ iff either $a \succ b$, or

there exists $c \in \mathscr{A}$ such that $a \succ c$ and $c \sim b$, or

there exists $c \in \mathscr{A}$ such that $a \sim c$ and $c \succ b$.

The following theorems have been proved.

THEOREM (Luce (1956)). *If $\langle \mathscr{A}, \succ \rangle$ is a semiorder, then $\langle \mathscr{A}, R \rangle$ is a weak order.*

THEOREM (Roberts (1968), (1971)). *If $\langle \mathscr{A}, \succ \rangle$ is an irreflexive relation, then it is a semiorder iff there exists a weak order $\langle \mathscr{A}, R' \rangle$ compatible with it in the following sense: for all $a, b, c \in \mathscr{A}$,*

(i) *$a\, R'\, b$, $b\, R'\, c$, and $a \sim c$ imply both $a \sim b$ and $b \sim c$,*

(ii) *$a \succ b$ implies $a\, R'\, b$.*

Moreover, $\langle \mathscr{A}, R \rangle$ is the unique weak order that is both compatible with it and for which I is $\approx$.

THEOREM (Roberts (1968), (1971)). *An irreflexive relation* $\langle \mathscr{A}, \succ \rangle$ *is a semiorder for which* $\langle \mathscr{A}, R \rangle$ *has a countable order-dense subset iff there exist real-valued functions* ϕ *and* δ, *where* $\delta \geqq 0$, *such that for all* $a, b \in \mathscr{A}$:

(i) $\phi(a) = \phi(b)$ *iff* $a \approx b$;
(ii) $a \succ b$ *implies* $\phi(a) \geqq \phi(b) + \delta(b)$, *and* $\phi(a) > \phi(b) + \delta(b)$ *implies* $a \succ b$;
(iii) $\delta(a) = \sup\limits_{\substack{b \\ b \sim a}} [\phi(b) - \phi(a)]$;
(iv) $\phi(a) \geqq \phi(b)$ *iff* $\phi(a) + \delta(a) \geqq \phi(b) + \delta(b)$.

The latter formulates the general ordinal representation of semiorders. We say that properties (i) and (ii) define a *threshold representation*; it is called *tight* when property (iii) holds and *monotonic* when property (iv) holds. Note that in a threshold representation, $a R b$ iff $\phi(a) \geqq \phi(b)$. Dropping the monotonicity property leads to interval orders. It should be noted that some of the theorems in the literature concern nontight representations. For example, when $\mathscr{A}$ is finite the above theorem can be modified as follows: The condition of order density is automatically satisfied and a representation can be constructed in which δ is constant (but not in general tight). For general surveys of this literature, see Fishburn (1970c) and Roberts (1970).

In replacing weak orders by semiorders over structured sets, the key issue is to formulate the interrelations between the ordering and the structure. But is it not obvious how to do this? Simply require that the induced weak order satisfy the usual axioms. This will not do for at least three reasons.

First, if one takes such a system of axioms and translates each axiom back into the primitive $\succ$, the resulting system is cumbersome.

Second, and far more crucial, some statements about P involve the existence of elements in the system. Existence statements have the unhappy property of being untestable in principle in infinite structures: when we have failed to find an element with the required property, we do not know whether it is because it does not exist or simply because we have not looked far enough. Although most measurement systems include one or more existential axioms which cannot really be tested and are usually accepted or rejected on more-or-less a priori grounds, they also include universal axioms. These are generally believed to contain most of the empirical meat of the system, and they are testable in the following sense. Each such axiom establishes some constraint on a certain (small) finite set of elements from the system, and that constraint can be checked empirically for any such set. These axioms are not always testable in the sense that the number of such sets, and so constraints, may be infinite. In the present case, if we simply demand that the induced weak order satisfy one of these usual systems of axioms, then a direct translation back into the underlying semiorder produces a system in which all axioms are existential. This is not acceptable. So one goal of research in this area is to achieve a sharp distinction between true existential (or structural) axioms and other universal ones. As we shall see, we do not fully meet this goal here.

Third, working with the induced weak order automatically leads to a representation which is not necessarily the most appropriate one, although it is surely the most obvious one. The typical axioms for extensive measurement lead to a real-valued function ϕ having two properties:

(i) Order preserving: $a\,R\,b$ iff $\phi(a) \geqq \phi(b)$;

(ii) Additive: $\phi(a \circ b) = \phi(a) + \phi(b)$.

With a semiorder, the representation is necessarily more complicated than order preserving (see Roberts' theorem above). In combining the extensive structure with the semiorder, the most obvious thing to expect is that in the threshold representation $\langle \phi, \delta \rangle$ the ϕ part will also be additive. This is the sort of result we shall establish below. However, it is unclear why the structure should exhibit additivity any more closely than within the threshold, i.e.,

$$\phi(a \circ b) \leqq \phi(a) + \phi(b) + \delta(a) + \delta(b)$$

and

$$\phi(a) + \phi(b) \leqq \phi(a \circ b) + \delta(a \circ b).$$

To the best of my knowledge, no theorems of this character are known.

The literature on semiordered additive structures includes only two distinct results. Independently Domotor and Stelzer (1971)[1] and Fishburn (1969) axiomatized finite, semiordered probability structures. Although this result meets the criterion of being stated wholly in terms of the semiorder, it is unsatisfactory in three other respects. First, it does not apply to the infinite case, which often is of interest. Second, as in all axiomatizations of finite ordered structures, one of the axioms is actually an infinite axiom schema. And third, the representation is not as precise as it might be because it fails to be tight.

Krantz (1967) presented an axiomatization of extensive measurement in which concatenation is represented by set theoretic union, as in probability measurement. In fact, he assumed $\mathscr{A}$ to be the set of all finite subsets of a given set. His axioms, which are not easy to assimilate, are mostly stated in terms of the induced weak order, and so are not really satisfactory from our point of view.

3. Discussion of monotonicity. A key property of structures with additive representations is what Krantz et al. (1971) call *monotonicity*, which in an ordinary extensive structure we write as

$$a\,R\,b \quad \text{iff} \quad a \circ c\,R\,b \circ c,$$

and in a conjoint one as

$$ap\,R\,bp \quad \text{iff} \quad aq\,R\,bq.$$

One major task is to try to reformulate this in the semiordered system.

For the present, we consider it only in extensive structures. It will prove convenient for us, as it did for Krantz (1967), to treat ordinary extensive systems in much the same way as probability ones, using the union of sets as the basic operation (the restriction to finite subsets is not needed, although it is not excluded). So we let $\mathscr{A}$ be a collection of subsets of X that includes $\varnothing$ and is closed under union, and we denote typical elements of $\mathscr{A}$ by A, B, etc. The ordering $\succsim$ is on $\mathscr{A}$, where $\succ$ and $\sim$ (indifference) are defined observationally and so $\sim$ need not be

[1] This paper is an amalgam of their dissertations: Domotor (1969) and Stelzer (1967).

the symmetric complement of $\succ$. Observe that if $A, B \in \mathscr{A}$ overlap, it may well be impossible to order them empirically—think of objects on a pan balance. There are two ways to proceed. First, we could accept that $\succsim$ is not connected and build a theory in terms of it. This seems the most desirable route, but it has not been successfully explored. Second, we can attempt to extend $\succsim$ to overlapping pairs by some sort of definition. The most obvious one is

$$(1) \qquad A \succsim B \quad \text{iff} \quad A - B \succsim B - A,$$

where we have simply dropped the common part, $A \cap B$. This will not do, however, because it has strong, unacceptable implications.

Consider mutually disjoint sets $A', B', C, C' \in \mathscr{A}$ which satisfy

$$A' \approx A, \quad B' \approx B \quad \text{and} \quad C \approx C' \approx A \cap B,$$

where $\approx$ is the equivalence relation defined in § 2. Then, (1) implies

$$(2) \qquad A' \succsim B' \quad \text{iff} \quad A' \cup C \succsim B' \cup C',$$

which is an empirical statement, not a definition. One might very well expect (2) to be false for orderings of weight determined by using a pan balance whose sensitivity decreases with the total weight on the pans. This could happen because of increased pressure and friction at the knife edge. In that case, we would expect that

$$(3) \qquad A' \cup C \succ B' \cup C' \quad \text{implies} \quad A' \succ B',$$

but we would not necessarily expect its converse,

$$(4) \qquad A' \succ B' \quad \text{implies} \quad A' \cup C \succ B' \cup C',$$

to hold.

A more subtle and satisfactory way to extend $\succsim$ is to find $A', B' \in \mathscr{A}$ for which

$$(5) \qquad A' \cap B' = \varnothing, \quad A' \approx A \quad \text{and} \quad B' \approx B,$$

and then define

$$(6) \qquad A \succsim B \quad \text{iff} \quad A' \succsim B'.$$

The difficulty with this, of course, is that it depends upon knowing $\approx$, which is not empirically possible in an infinite system. So, although we shall assume $\succsim$ is a connected relation in what follows, we must be aware that our axiomatization remains flawed by our implicit use of $\approx$ here. A later explicit use will only worsen matters.

Ideally, we would like to formulate a system of axioms that includes (3), but not (4). In fact, I have been unable to do so, and instead of (3) I invoke[2] the property that, for $A \cap C = B \cap C = \varnothing$,

$$(7) \qquad A \approx B \quad \text{iff} \quad A \cup C \approx B \cup C.$$

It should be noted that this property is implicit in the intuition lying behind the definition of $\approx$. For if it were false that would mean that two sets, which themselves

[2] The axioms are written in terms of P and I, but I is the same as $\approx$ for a semiorder.

are empirically indistinguishable when compared with all other sets, are indirectly distinguishable by either adding or deleting a common set.

As an axiom, (7) is unsatisfactory when translated back into the primitive relation. It reads

$$\text{(For all } D,\ A \sim D \text{ iff } B \sim D) \quad \text{is equivalent to}$$
$$\text{(for all } D,\ A \cup C \sim D \text{ iff } B \cup C \sim D).$$

Obviously, for a fixed A, B and C, this cannot be verified in a finite number of observations unless $\mathscr{A}$ is finite. The only reason in practice why it may not be an impossible axiom is that the region of potential indifference about a given element is relatively limited, and so it can be sampled rather representatively.

In the corollaries to the first two theorems, we show that (7) can be dropped if we are willing to invoke both (3) and (4); however, the resulting structures are too restrictive for many purposes since they have thresholds of constant size. One does not anticipate that, in general, additivity and constancy of threshold can both be satisfied simultaneously.

4. A system of extensive structures.

THEOREM 1. *Suppose X is a nonempty set, $\mathscr{A}$ is a collection of subsets of X that includes $\varnothing$ and is closed under $\cup$, and $\succ$ is a binary relation on $\mathscr{A}$. Let the following axioms hold for all $A, B, C, A_i \in \mathscr{A}$.*

Axiom 1. *$\langle \mathscr{A}, \succ \rangle$ is a semiorder.*

Axiom 2. *If $A \cap C = B \cap C = \varnothing$, then*

$$A\, I\, B \quad \textit{iff} \quad A \cup C\, I\, B \cup C.$$

Axiom 3. *If either $A \supset B$ and $B \succ C$ or $A \succ B$ and $B \supset C$, then $A \succ C$.*

Axiom 4. *If $A\, P\, B$, then there exists $C \in \mathscr{A}$ such that $C \cap B = \varnothing$ and $A\, I\, B \cup C$.*

Axiom 5. *If $A, B \in \mathscr{A}$, then there exists $C \in \mathscr{A}$ such that $A \cap C = \varnothing$ and $C\, I\, B$.*

Axiom 6. *Every strictly bounded standard sequence is finite, where $\{A_i | i \in N\}$ is a standard sequence iff there exists $A, B_i \in \mathscr{A}$, $i \in N$, such that not $(A\, I\, \varnothing)$, $B_i \cap B_j = \varnothing$ for $i, j \in N$ with $i \neq j$, $B_i \sim A$, and $A_i = \bigcup_{j \leq i} B_j$.*

Then there exists an additive, tight threshold representation $\langle \varphi, \delta \rangle$; i.e., for all $A, B \in \mathscr{A}$,

(i) *$A\, I\, B$ iff $\phi(A) = \phi(B)$;*

(ii) *$A \succ B$ implies $\phi(A) \geqq \phi(B) + \delta(B)$, and $\phi(A) > \phi(B) + \delta(B)$ implies $A \succ B$;*

(iii) $\delta(A) = \sup\limits_{\substack{B \\ B \sim A}} [\phi(B) - \phi(A)]$;

(iv) *if $A \cap B = \varnothing$, then $\phi(A \cup B) = \phi(A) + \phi(B)$.*

The representation is unique up to multiplication by a positive constant.

COROLLARY 1. *Suppose, in addition, the structure satisfies (3), i.e., for $A \cap C = B \cap C = \varnothing$,*

$$A \cup C \succ B \cup C \quad \textit{implies} \quad A \succ B.$$

If $A \supset B$ and $A - B \in \mathscr{A}$, then $\delta(A) \geqq \delta(B)$.

COROLLARY 2. *Suppose the structure is uniform in the sense that Axiom* 2 *is replaced by* (3) *and* (4); *i.e., for* $A \cap C = B \cap C = \varnothing$,

$$A \succ B \quad \textit{iff} \quad A \cup C \succ B \cup C;$$

then Axiom 2 *holds and* δ *is a constant function.*

This theory is not suitable for qualitative probability because Axiom 5 cannot hold with $X \in \mathscr{A}$ (set $A = B = X$). A suitable theory is given in the next section.

Axioms 1 and 2 have been discussed at length. Axiom 3 simply says that a strict inequality is not changed either by deleting elements from the lesser set or by augmenting the greater one, and it is hardly controversial. Axioms 4 and 5 are strong structural conditions and, as such, are stated in terms of the induced weak order. The former is a solvability condition, and the latter states that an exact copy always exists that does not involve any element of a prescribed set. It is perhaps worth noting that Axiom 5 is implicit in all axiomatizations of extensive measurement in which concatenation is taken as a closed operation, for what are we to mean empirically by $a \circ a$ if we cannot find an exact copy of a distinct from a? These are the axioms we must either accept or reject on a priori grounds; they seem acceptable for weight measurement. The last axiom is a typical Archimedean one.

The common hypothesis of the first four lemmas is Axioms 1–4 of Theorem 1.

LEMMA 1. $A \succsim \varnothing$.

Proof. Suppose $\varnothing \succ A$, then since $A \supset \varnothing$, Axiom 3 implies $\varnothing \succ \varnothing$, contrary to Axiom 1.

COROLLARY. $A\,R\,\varnothing$.

Proof. Suppose not, then since R is a weak order, $\varnothing\,P\,A$. Since $\varnothing \succ D$ is impossible by the lemma, we conclude there exists a D such that $\varnothing \sim D$ and $D \succ A$. But $D \succ A$ and $A \supset \varnothing$ implies by Axiom 3 the contradiction $D \succ \varnothing$.

LEMMA 2. *In Axiom* 4, $C\,P\,\varnothing$.

Proof. If $C\,I\,\varnothing$, then by Axiom 2, $A\,I\,B \cup C\,I\,B \cup \varnothing = B$, contrary to $A\,P\,B$. So by the Corollary to Lemma 1, $C\,P\,\varnothing$.

LEMMA 3. *If* $A \cap B = \varnothing$ *and* $A\,P\,\varnothing$, *then* $A \cup B\,P\,B$.

Proof. Suppose, on the contrary, $B\,R\,A \cup B$. If $B\,I\,A \cup B$, then by Axiom 2, $A\,I\,\varnothing$, contrary to hypothesis. If $B\,P\,A \cup B$ and there exists C such that $B \sim C$ and $C \succ A \cup B$, then since $A \cup B \supset B$, Axiom 3 implies $C \succ B$, which yields the contradiction $B\,P\,B$. Alternatively, $B \succ C$ and $C \sim A \cup B$, in which case $A \cup B \supset B$ and $B \succ C$ yield, via Axiom 3, the contradiction $A \cup B \succ C$.

LEMMA 4. *Suppose* $A \cap C = B \cap C = \varnothing$. *Then* $A\,R\,B$ *iff* $A \cup C\,R\,B \cup C$.

Proof. If $A\,I\,B$, the result follows from Axiom 2. Suppose $A\,P\,B$ and $B \cup C\,P\,A \cup C$. By Lemma 2, there exists D such that $D\,P\,\varnothing$ and $B \cup C\,I\,A \cup C \cup D$. By Axiom 2, $B\,I\,A \cup D$, and by Lemma 3, $B\,P\,A$, which is impossible. The converse is similar.

LEMMA 5. *Suppose Axioms* 1, 2 *and* 5 *hold. For* $A \cap C = B \cap D = \varnothing$, *if* $A\,I\,B$ *and* $C\,I\,D$, *then* $A \cup C\,I\,B \cup D$.

Proof. By Axiom 5, there exists C' such that $C' \cap (A \cup B) = \varnothing$ and $C'\,I\,C\,I\,D$. By repeated uses of Axiom 2,

$$A \cup C = C \cup A\,I\,C' \cup A = A \cup C'\,I\,B \cup C' = C' \cup B\,I\,D \cup B = B \cup D.$$

The conclusion follows since I is an equivalence relation by Axiom 1.

Proof of Theorem 1. Define

$$\mathscr{A}' = \mathscr{A} - \{B | B\, I\, \varnothing\},$$

$$\mathscr{B} = \mathscr{A}' \times \mathscr{A}',$$

$$A \circ B = A \cup B', \quad \text{where } B' \cap A = \varnothing \text{ and } B'\, I\, B.$$

By Axiom 5, $A \circ B$ is defined for all $A, B \in \mathscr{A}'$, and by Axiom 2 it is unique up to I.

If $\mathscr{A}' = \varnothing$, set $\phi = \delta = 0$ and the result is trivial; otherwise we show that $\langle \mathscr{A}', R, \mathscr{B}, \circ \rangle$ satisfies the axioms of an extensive structure with no essential maximum (Krantz et al (1971), Definition 3.3, p. 84).

1. $\langle \mathscr{A}', R \rangle$ is a weak order as was noted above.

2. $(A \circ B) \circ C\, I\, A \circ (B \circ C)$ follows from Axiom 5, Lemma 5, and the associativity of $\cup$.

3. Suppose $A\, R\, B$. By Axiom 5 there exists C' such that $C'\, I\, C$ and $C' \cap (A \cup B) = \varnothing$. So by Lemma 4,

$$A \circ C = A \cup C'\, R\, B \cup C' = C' \cup B.$$

But $C \circ B = C \cup B'$, where $B'\, I\, B$ and $B' \cap C = \varnothing$. By Lemma 5, $C' \cup B\, I\, C \cup B'$, whence $A \circ C\, R\, C \circ B$.

4. If $A\, P\, B$, then there exists $C \in \mathscr{A}'$ such that $A\, R\, B \circ C$ by Lemma 2.

5. $A \circ B\, P\, A$ follows from Lemma 3.

6. The Archimedean property is an immediate consequence of Lemma 1 and Axiom 6.

By Theorem 3.3, p. 85 of Krantz et al. (1971) an additive representation ϕ of R exists. Extend it to $\mathscr{A}$ by defining $\phi(B) = 0$ if $B\, I\, \varnothing$. Define δ by property (iii) (tightness). We show property (ii). Suppose $A \succ B$ and $\phi(A) < \phi(B) + \delta(B)$. By definition of δ, there exists $C \sim B$ and $\phi(C) \geqq \phi(A)$. But $A \succ B$ and $B \sim C$ imply $A\, P\, C$, whence by (i), $\phi(A) > \phi(C)$, which is a contradiction. Conversely, suppose $\phi(A) > \phi(B) + \delta(B)$ and $B \succsim A$. If $B \succ A$, then by what we have just shown $\phi(B) \geqq \phi(A) + \delta(A)$, and so $\phi(A) > \phi(A)$, which is impossible. If $B \sim A$, then by definition $\delta(B) \geqq \phi(A) - \phi(B) > \delta(B)$, which again is impossible.

Proof of Corollary 1. Suppose $A \supset B$ and $A - B \in \mathscr{A}$. For any $C \sim B$, by Axiom 5 choose D such that $D\, I\, A - B$ and $D \cap (B \cup C) = \varnothing$. By (3),

$$C \cup D \sim B \cup D\, I\, B \cup (A - B) = A.$$

Thus, using properties (iii) and (iv),

$$\delta(A) = \sup_{\substack{E \\ E \sim A}} [\phi(E) - \phi(A)] \geqq \sup_{\substack{C \\ C \sim B}} \{\phi(C \cup D) - \phi[B \cup (A - B)]\}$$

$$= \sup_{\substack{C \\ C \sim B}} [\phi(C) - \phi(B)] = \delta(B).$$

Proof of Corollary 2. We first show Axiom 2. Suppose, on the contrary, $A\, I\, B$ and not $A \cup C\, I\, B \cup C$. With no loss of generality, $A \cup C\, P\, B \cup C$. If there exists $D \in \mathscr{A}$ such that $A \cup C \sim D$ and $D \succ B \cup C$, then Axioms 4 and 5 imply there exists $E \in \mathscr{A}$ such that $E \cap (B \cup C) = \varnothing$ and $A \cup C \sim D\, I\, B \cup C \cup E \succ B \cup C$. By the hypothesis of the corollary, $A \sim B \cup E \succ B$, contrary to $A\, I\, B$.

If there exists D such that $A \cup C \succ D$ and $D \sim B \cup C$, then by Axiom 4 there exists E such that $E \cap (B \cup C \cup D) = \emptyset$ and $A \cup C\, I\, D \cup E$. By the hypothesis, $D \cup E \sim B \cup C \cup E$ which, in turn, implies $A \sim B \cup E$. We show $B \cup E \succ B$, which establishes the contradiction. If $B \cup \emptyset = B \succsim B \cup E$, then by the hypothesis and Lemma 1 (which does not depend on Axiom 2), $\emptyset \sim E$, and so by the hypothesis, $A \cup C\, I\, D \cup E \sim D$, which is contrary to $A \cup C \succ D$. So $B \cup E \succ B$.

Conversely, suppose $A\, P\, B$ and $A \cup C\, I\, B \cup C$. There exists D such that either $A \sim D \succ B$ or $A \succ D \sim B$. By Axiom 5, we may assume D is disjoint from C, so by the hypothesis of the corollary,

$$A \cup C \sim D \cup C \succ B \cup C \quad \text{or} \quad A \cup C \succ \mathrm{D} \cup C \sim B \cup C,$$

contrary to $A \cup C\, I\, B \cup C$.

Next we show that δ is a constant function by showing $\delta(A) = \delta(\emptyset)$ for all $A \in \mathscr{A}$. By Corollary 1, $\delta(A) \geqq \delta(\emptyset)$. Suppose for some A, $\delta(A) > \delta(\emptyset)$. Select $\varepsilon > 0$ such that $\delta(A) - \varepsilon > \delta(\emptyset)$. Then by tightness (property (iii)), there exists $B \sim A$ such that

$$\phi(B) - \phi(A) \geqq \delta(\mathrm{A}) - \varepsilon > \delta(\emptyset) \geqq 0.$$

By properties (i) and (iii) of Theorem 1, $B\, P\, A$. By Axiom 4 there exists C such that $B\, I\, A \cup C$. By property (iv),

$$\phi(B) - \phi(A) = \phi(A \cup C) - \phi(A) = \phi(C) > \delta(\emptyset).$$

Thus, $C \succ \emptyset$. But by (4), this implies $B\, I\, A \cup C \succ A$, which is contrary to choice. So $\delta(A) = \delta(\emptyset)$.

5. A system of probability structures.

THEOREM 2. *Suppose X is a nonempty set, $\mathscr{A}$ is an algebra of subsets of X (i.e., $\emptyset \in \mathscr{A}$ and $\mathscr{A}$ is closed under union and complementation), and $\succ$ is a binary relation on $\mathscr{A}$. Let the following axioms hold for all $A, B, C \in \mathscr{A}$.*

Axiom 1. *$\langle \mathscr{A}, \succ \rangle$ is a semiorder.*

Axiom 2. *For $A \cap C = B \cap C = \emptyset$,*

$$A\, I\, B \quad \textit{iff} \quad A \cup C\, I\, B \cup C.$$

Axiom 3. *If either $A \supset B$ and $B \succ C$ or $A \succ B$ and $B \supset C$, then $A \succ C$.*

Axiom 4. *If $A\, P\, B$, then there exists $B' \in \mathscr{A}$ such that $B' \subset A$ and $B'\, I\, B$.*

Axiom 5. *$X \succ \emptyset$.*

Axiom 6. *Every strictly bounded standard sequence is finite.*

Then there exist unique real-valued functions ϕ and $\delta \geqq 0$ on $\mathscr{A}$ such that $\langle X, \mathscr{A}, \phi \rangle$ is a finitely additive probability space and $\langle \phi, \delta \rangle$ is a tight threshold representation, i.e., properties (i), (ii), *and* (iii) *of Theorem 1 hold.*

COROLLARY. *Suppose that the structure is uniform in the sense that Axiom 2 is replaced by the following two conditions:*

(α) *For $A \cap C = B \cap C' = \emptyset$ and $C\, I\, C'$,*

$$A \succ B \quad \textit{iff} \quad A \cup C \succ B \cup C'.$$

(β) *If $A\, I\, B \sim \emptyset$, then there does not exist a C such that*

$$A \cap C = B \cap C = \emptyset \quad \textit{and} \quad A \cup C \succ \emptyset \sim B \cup C.$$

Then Axiom 2 holds and δ is a constant function over the set $\{A|X \succ A\}$.

The relation of this Corollary to Theorem 2 is much like that of Corollary 2 to Theorem 1; however, there are differences. First, the version of monotonicity postulated is slightly stronger ($C\ I\ C'$ versus just C). Second, I have not been able to prove Axiom 2 from this alone without also invoking Axiom 2 for elements indifferent to $\varnothing$. The existence of the upper bound X in the probability structure, but not in the extensive one, makes it possible to fail to discriminate differences near the lower bound. Condition (β) simply rules the trouble out.

It is easy to see that this corollary cannot be strengthened to say that δ is tight and constant over all of $\mathscr{A}$. The reason is that X is an upper bound in the sense that $A\ P\ X$ is false for all A (this is proved in Lemma 6) and so for a tight δ we have $\delta(X) = 0$, whereas it is not generally true that $\delta(\varnothing) = 0$.

Axioms 1–3 are the same as in the semiordered extensive structure. Axiom 4 is a structural condition, and as such we do not object to stating it in terms of R. It is, however, stronger than the one used for weakly ordered probability structures in Chapter 5 of Krantz et al (1971). Axiom 5 is the usual nontrivialness postulate for probability structures. To complete Axiom 6, we must define a standard sequence in such a way that, in the presence of the other axioms, it implies the Archimedean condition in Definition 5.3, p. 204 of Krantz et al (1971). As this is easy to do, I do not make it explicit.

It is worth noting that Axiom 4 of Theorem 2 is stronger than Axiom 4 of Theorem 1 in the sense that the former can be substituted in the statement of Theorem 1 and the latter axiom deduced. The argument is simple. Suppose $A\ P\ B$. By Axiom 4 of Theorem 2, there exists $B' \subset A$ with $B'\ I\ B$. By Axiom 5 of Theorem 1, there exists $C\ I\ A - B'$ and $C \cap B = \varnothing$. So, by Lemma 5, which depends only on Axioms 1, 2 and 5 of Theorem 1, $A = (A - B') \cup B'\ I\ C \cup B$, thereby proving Axiom 4 of Theorem 1.

LEMMA 6. *Suppose Axioms* 1 *and* 3 *of Theorem* 2 *hold. If* $A, B \in \mathscr{A}$ *and* $A \supset B$, *then* $A\ R\ B$.

Proof. Suppose, on the contrary, $B\ P\ A$. If $B \sim C \succ A \supset B$, then Axiom 3 implies $B \sim C \succ B$, so $B\ P\ B$, which is impossible. The other case is similar provided that $C \sim A \supset B$ implies $C \succsim B$. Suppose on the contrary, $B \succ C$, then by Axiom 3, $A \succ C$, contrary to assumption.

Proof of Theorem 2. We show that $\langle X, \mathscr{A}, R\rangle$ satisfies Axioms 1–4 of Definition 5.4 and Axiom 5, p. 207 of Krantz et al. (1971).

1. $\langle \mathscr{A}, R\rangle$ is a weak order.

2. $X\ P\ \varnothing$ and $A\ R\ \varnothing$ by Axiom 5 of Theorem 2 and the Corollary to Lemma 1 (which rests only on Axioms 1 and 3 which are common to Theorems 1 and 2).

3. Suppose $A \cap C = B \cap C = \varnothing$; then we show $A\ R\ B$ iff $A \cup C\ R\ B \cup C$. First, suppose $A\ R\ B$. If $A\ I\ B$, the conclusion follows by Axiom 2. So $A\ P\ B$. By Axiom 4, there exists $B' \subset A$ such that $B'\ I\ B$. Since $A \cup C \supset B' \cup C$, Lemma 6 says $A \cup C\ R\ B' \cup C$. By Axiom 2, $B' \cup C\ I\ B \cup C$, whence $A \cup C\ R\ B \cup C$.

Conversely, suppose $A \cup C\ R\ B \cup C$. The I case is handled by Axiom 2, so we suppose P holds. If $B\ P\ A$, then by what we have just shown, $B \cup C\ R\ A \cup C$, which is contrary to assumption, so $A\ R\ B$.

4. The Archimedean property follows from the assumed one.

5. Suppose $A \cap B = \varnothing$, $A\ P\ C$, $B\ R\ D$. By Axiom 4, choose $C' \subset A$, $D' \subset B$ such that $C'\ I\ C$, $D'\ I\ D$. Clearly $C' \cap D' = \varnothing$ and $C' \cup C' \subset E = A \cup B$.

By Theorem 5.2 of Krantz et al. (1971), a unique, finitely-additive probability space $\langle X, \mathscr{A}, \phi \rangle$ exists. Define δ by property (iii) and, as in Theorem 1, $\langle \phi, \delta \rangle$ is a tight representation.

LEMMA 7. *If the hypotheses of the Corollary to Theorem* 2 *hold, then* $A \succsim B$ *iff* $-B \succsim -A$.

Proof. By hypothesis (α),

$$-B \cup (A \cap B) = A \cup -(A \cup B) \succsim B \cup -(A \cup B) = -A \cup (A \cap B).$$

So, by (α), $-B\ R\ -A$.

COROLLARY 1. $A\ R\ B$ *iff* $-B\ R\ -A$.

Proof. Suppose $A\ P\ B$. If there exists D such that $A \sim D \succ B$, then by Lemma 7, $-A \sim -D \prec -B$, so $-B\ P\ -A$. The other case is similar.

COROLLARY 2. *If* $C\ I\ C' \sim X$, *then there does not exist* $A \subset C \cap C'$ *for which* $C - A \sim X \succ C' - A$.

Proof. If such an A exists, then by Lemma 7 and its first corollary, $-C\ I\ -C' \sim \varnothing$ and $-(C - A) = -C \cup A \sim \varnothing \prec -(C' - A) = -C' \cup A$, contrary to hypothesis (β).

Proof of the Corollary to Theorem 2. First, we prove Axiom 2. Suppose that $A\ I\ B$ and $A \cup C\ P\ B \cup C$. There are two cases:

1. There exists D such that $A \cup C \sim D \succ B \cup C$. Since $B \cup C \supset C$, Axiom 3 implies $D \succ C$ and so, by Axiom 4, there exists $C' \subset D$ such that $C'\ I\ C$. By (α), $A \cup C \sim D = (D - C') \cup C' \succ B \cup C$ implies $A \sim D - C' \succ B$, contrary to $A\ I\ B$.

2. There exists D such that $A \cup C \succ D \sim B \cup C$. If $D\ R\ C$, the argument is as in 1. Suppose $C\ P\ D$. If $A \cup C \succ C \sim B \cup C$, hypothesis ($\alpha$) implies $A \succ \varnothing \sim B$, which is contrary to $A\ I\ B$. So by Lemma 6, we may assume $A \cup C \sim C \sim B \cup C$, so $A \sim \varnothing \sim B$. For $C\ P\ D$, Axiom 4 implies there exists $D' \subset C$ such that $D'\ I\ D$. So $A \cup (C - D') \cup D' \succ D \sim B \cup (C - D') \cup D'$ from which (α) implies $A \cup (C - D') \succ \varnothing \sim B \cup (C - D')$, which is impossible by (β).

Next, suppose $A\ P\ B$ and $A \cup C\ I\ B \cup C$.

1. If there exists D such that $A \succ D \sim B$, Axiom 4 implies the existence of D' such that $A \succ D' \sim B$ and $D' \subset A$. By (α), $A \cup C \succ D' \cup C \sim B \cup C$, contrary to $A \cup C\ I\ B \cup C$.

2. Suppose there exists D such that $A \sim D \succ B$. If $-D\ P\ C$, then there exists $C' \subset -D$ such that $C'\ I\ C$. By (α), $A \cup C \sim D \cup C' \succ B \cup C$, contrary to assumption. If $-D\ I\ C$, then by (α), $A \cup C \sim D \cup -D \succ B \cup C$, which is impossible. If $C\ P\ -D$, choose D' by Axiom 4 so that $-D' \subset C$ and $-D'\ I\ -D$. By Corollary 1 to Lemma 7, $D'\ I\ D$. Since $A \sim D'$, Lemma 6 and (α) yield

$$A \cup C = A \cup -D' \cup (C \cap D') \succsim A \cup -D' \sim D' \cup -D' = X.$$

Since by Lemmas 1 and 7, $X \succsim A \cup C$, we conclude $B \cup C\ I\ A \cup C \sim X$. But $A \sim D' \succ B$ implies $A \cup -D' \sim X \succ B \cup -D'$. Since $A \cup -D' = (A \cup C) - (C \cap D')$ and $B \cup -D' = (B \cup C) - (C \cap D')$, this is contrary to Corollary 2 of Lemma 7.

We show δ is constant over $\{A | X \succ A\}$. Suppose $X \succ A$ and $B \sim \varnothing$. By hypothesis (α), $X - A \succ \varnothing$, so $X - A\ P\ B$. By Axiom 4, there exists $B' \subset X - A$ such that $B'\ I\ B$. By (α), $B' \cup A \sim A$ and by additivity of ϕ, $\phi(B' \cup A) - \phi(A) = \phi(B') = \phi(B) - \phi(\varnothing)$. Thus, $\delta(A) \geqq \delta(\varnothing)$. The proof of $\delta(\varnothing) \geqq \delta(A)$ is as in

Corollary 2, Theorem 1, once we show that Axiom 4 of Theorem 1 holds in this structure. Suppose $A\ P\ B$; then by Corollary 1 to Lemma 7, $-B\ P\ -A$. By Axiom 4 of Theorem 2, there exists $D \subset -B$ such that $D\ I\ -A$. By Corollary 1 of Lemma 7, $-D\ I\ A$ and $-D \supset B$. Clearly $A\ I\ B \cup C$, where $C = -D - B$.

6. A system of uniform conjoint structures. We turn next to semiordered, additive conjoint structures. We state the theorem first and then discuss it.

THEOREM 3. *Suppose* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are nonempty sets and* $\succ$ *is a binary relation on* $\mathscr{A}_1 \times \mathscr{A}_2$. *For all* $a, b, f \in \mathscr{A}_1$ *and* $p, q, x \in \mathscr{A}_2$, *let the following axioms be satisfied.*

Axiom 1. $\langle \mathscr{A}_1 \times \mathscr{A}_2, \succ \rangle$ *is a semiorder.*

Axiom 2. $ap \succ bp$ *iff* $aq \succ bq$, *and* $ap \succ aq$ *iff* $bp \succ bq$.

Axiom 3. $ax\ I\ fq$ *and* $fp\ I\ bx$ *imply* $ap\ I\ bq$.

Axiom 4. *Given any three of* a, b, p, q, *the fourth exists so that* $ap\ I\ bq$.

Axiom 5. *Every strictly bounded standard sequence is finite.*

Axiom 6. *There exist* $a, b \in \mathscr{A}_1$ *and* $p, q \in \mathscr{A}_2$ *such that not* $ap\ I\ bq$.

Then there exist real-valued functions ϕ_i *on* $\mathscr{A}_i$, $i = 1, 2$, *and a constant* $\delta \geqq 0$ *such that* $\langle \phi_1 + \phi_2, \delta \rangle$ *is a tight, constant threshold representation. Moreover,* $\langle \phi'_1 + \phi'_2, \delta' \rangle$ *is another tight, constant representation iff there exist real constants* $\alpha > 0$, β_1, *and* β_2 *such that*

$$\phi'_i = \alpha\phi_i + \beta_i \quad \text{and} \quad \delta' = \alpha\delta.$$

Axiom 2, which asserts independence in the observed relation $\succ$, is the reason that we call the structure uniform since it obviously says that the overall level of intensity does not affect these orderings. It is similar in spirit to the uniformity property of extensive and probability structures, and it leads to a constant threshold representation. Axiom 3 is the Thomsen condition for the induced equivalence relation I. As in the extensive and probability cases, such an axiom is marginally acceptable; one would prefer a simple axiom stated solely in terms of $\succ$. Lemma 8 derives versions of double cancellation stated in terms of $\succ$ and $\sim$; they are probably part of what is needed, but they do not appear to be sufficient. Axiom 4 is unrestricted solvability, and since it is a structural condition, we do not greatly object to stating it in terms of I. Of course, it would be desirable to assume only restricted solvability, but the resulting complications are surely considerable. We leave the precise formulation of the Archimedean axiom to the reader. Axiom 6 insures the essentialness of the components.

LEMMA 8. *Suppose Axioms* 1–4 *of Theorem* 3 *hold. If* $ax \succ fq$ *and* $fp \succ bx$, *then there exists* cr *such that* $ap \succ cr \succ bq$. *If either hypothesis is* $\sim$, *then the corresponding term of the conclusion is* $\sim$.

Proof. By Axiom 4, let a' solve $a'x\ I\ fq$ and b' solve $b'x\ I\ fp$. By Axiom 3, $a'p\ I\ b'q = cr$. By Axiom 2, $ap \succ a'p$ and $b'q \succ bq$. The second part follows since Axiom 2 holds with $\succ$ replaced by $\sim$.

Proof of Theorem 3. We establish that $\langle \mathscr{A}_1 \times \mathscr{A}_2, R \rangle$ satisfies those axioms of Definition 6.7, p. 256 required by Theorem 6.1, p. 262, of Krantz et al (1971).

1. $\langle \mathscr{A}_1 \times \mathscr{A}_2, R \rangle$ is a weak order.

2. To show double cancellation, suppose $ax\ R\ fq$ and $fp\ R\ bx$. If both R's are I, the result follows from Axiom 3. If one is P and the other is I, we show that the conclusion follows from the case where both are P. For example, suppose $ax\ I\ fq$,

$fp\ P\ bx$, and not $ap\ R\ bq$. From $bq\ P\ ap$ and $fp\ P\ bx$, we will show $fq\ P\ bx$. By definition, there are four possible cases. As they are all handled in essentially the same way, we illustrate one in detail and a second more briefly.

(i) There exist cy and dz such that $ax \sim cy \succ fq$ and $fp \sim dz \succ bx$. By Axiom 4, let a' and p' solve $a'x\ I\ cy$ and $fp'\ I\ dz$. Lemma 8 implies there is hu such that $a'p' \succ hu \succ bq$, and Axiom 2 implies $ap' \sim a'p'$ and $ap \sim ap'$. By Axiom 1, either $a'p' \succ ap'$, which is impossible, or $ap \sim ap' \succ bq$, whence $ap\ P\ bq$.

(ii) There exist cy and dz such that $ax \sim cy \succ fq$ and $fp \succ dz \sim bx$. By Axiom 4, let a' and b' solve $a'x\ I\ cy$ and $b'x\ I\ dz$. Then follow the pattern of (i).

3. Axiom 4 asserts unrestricted solvability.
4. The Archimedean property follows from Axiom 5.
5. Axiom 6 asserts essentialness.

By Theorems 6.1 and 6.2 of Krantz et al. (1971), there is an additive representation $\phi_1 + \phi_2$ of R which is unique up to positive linear transformations with a common unit. Define δ as in Theorem 1 so that $\langle \phi_1 + \phi_2, \delta \rangle$ is a tight representation. Obviously, δ is unique up to the unit of the transformation.

To complete the proof, we must show that δ is constant. Consider any $a'p' \sim bq$. By Axiom 4, there exists a'' such that $a''q\ I\ a'p'$. Applying Axiom 2 to $a''q \sim bq$, we obtain $a''p \sim bp$. Thus $\delta(bp) \geqq \delta(bq)$. Interchanging p and q, $\delta(bp) = \delta(bq)$. Similarly, $\delta(ap) = \delta(bp)$, hence δ is a constant.

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