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CONDITIONAL EXPECTED, EXTENSIVE UTILITY*

ABSTRACT Luce and Krantz (1971) presented an axiom system for conditional expected utility. In this theory a conditional decision is a function whose domain is a non-null subevent and whose range is a subset of a set of consequences. Given a family of conditional decisions that is closed under unions of decisions whose domains are disjoint and under restrictions to non-null subevents, the second major primitive is an ordering of the family. Axioms were given that are adequate to construct a numerical utility function over decisions and a probability function over events for which the conditional expectation of the utility is order preserving. Several of the axioms are quite complex and seem a bit artificial, and the proof is very long. Here the structure is modified by adding to the set of outcomes a concatenation operation, and the representation theorem is modified by requiring that the utility function be additive over this binary operation as well as exhibiting the expected utility property. The advantages of this pair of changes are, first, it exploits the obvious fact that the union of consequences is itself a consequence; second, it reduces the mathematical burden carried by the set theoretic structure of conditional decisions and, as a result, the axioms can be made much easier to understand; and third, it permits a considerably shorter proof because one can draw more readily on known results. The major drawback of this approach is, of course, that it is inconsistent with the evidence that utility is not additive over consequences – at least, not over increasing amounts of a single good (diminishing marginal utility).

Most theory about expected utility imposes considerable structure on the space of events but very little on the outcomes. This is certainly true of the two most general theories, Savage (1954) and Luce and Krantz (1971). It is clear that if we are willing to assume more about the outcomes, less need be required elsewhere in the system in order to arrive at the same result, and with luck the proofs might be simpler.

The only attempt I am aware of to introduce such added structure is that of Fishburn (references and a summary may be found in Chapter 11 of Fishburn, 1970, and in Section 8.4.2 of Krantz, Luce, Suppes, and Tversky, 1971). In this work, the outcomes are assumed to be commodity bundles and the aim is to show that they exhibit an additive conjoint structure (see Chapters 4 and 5 of Fishburn or Chapters 6 and 9 of Krantz *et al.*). This work does not quite achieve the goal stated above since, instead of weakening a very general system of expected utility such as Savage's, the usual expected utility representation is accepted and one

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additional axiom is shown to be sufficient to establish the additivity of utility over the components of the outcome n-tuple.

In this paper, I examine one way of rewriting the axioms for a general structure, namely the conditional expected utility one of Luce and Krantz, when the outcome space is assumed to have not a conjoint structure, but an extensive one. The idea is that two outcomes can be concatenated – interpretation: both are received – and such an operation generates an algebraic structure which we can try to represent by an additive utility. For general background about extensive measurement and about conditional expected utility see, respectively, Chapters 3 and 8 of Krantz *et al.*

As in the previous work on conditional expected utility, X and $\mathscr{C}$ are sets, $\mathscr{E}$ is an algebra of subsets of X, $\mathscr{N} \subset \mathscr{E}$, $\mathscr{D}$ is a family of functions whose domains are in $\mathscr{E} - \mathscr{N}$ and whose co-domains are subsets of $\mathscr{C}$, and $\succsim$ is a binary relation on $\mathscr{D}$. The interpretations are: $\mathscr{C}$ is the set of possible outcomes of decisions, $\mathscr{E}$ is an algebra of events on which decisions are conditional, X is the universal event, $\mathscr{N}$ is the set of null events, $\mathscr{D}$ is a set of conditional decisions, and $\succsim$ a preference ordering over $\mathscr{D}$. The new primitive is a binary relation o which we may take as defined either over $\mathscr{D} \times \mathscr{D}$ or over $\mathscr{C} \times \mathscr{C}$, in which case coc' is interpreted as the consequence of receiving both c and c' and o is extended to $\mathscr{D} \times \mathscr{D}$ as follows: If $f_A, g_B \in \mathscr{D}$, where A and B are the domains, define

$$f_A o g_B(x) = \begin{cases} f_A(x), & \text{if } x \in A - B \\ f_A(x) o g_B(x), & \text{if } x \in A \cap B \\ g_B(x), & \text{if } x \in B - A. \end{cases}$$

Note that for $A \cap B = \emptyset, f_A o g_B = f_A \cup g_B$. Either way, the axioms are stated for o over $\mathscr{D} \times \mathscr{D}$.

Denote by $\mathscr{D}(A)$, $A \in \mathscr{E} - \mathscr{N}$, all functions in $\mathscr{D}$ with domain A. Let o_A and $\succsim_A$ be the restrictions of o and $\succsim$, respectively, to $\mathscr{D}(A)$. If $f_A o e_A \sim e_A o f_A \sim f_A$ for all $f_A \in \mathscr{D}(A)$, then e_A is an *identity* of $\mathscr{D}(A)$.

Consider the following axioms where, implicitly, f_A, g_B, etc., are in $\mathscr{D}$.

(1) $\mathscr{D}$ is closed under o and under restriction of functions to subdomains in $\mathscr{E} - \mathscr{N}$.

(2) $\succsim$ is a weak order.

(3) o is communitative and associative with respect to $\sim$.

(4) If $A, B \in \mathscr{E} - \mathscr{N}$, $A \cap B = \emptyset$, and $f_A \sim g_B$, then $f_A o g_B \sim f_A$.

(5) For $A, -A \in \mathscr{E} - \mathscr{N}, f_A \succsim f'_A$ iff $f_A o g_{-A} \succsim f'_A o g_{-A}$.

(6) Suppose $A, B \in \mathscr{E} - \mathscr{N}$ and $f_A \sim g_B$. Then

$$f'_A \gtrsim g'_B \quad \text{iff} \quad f_A o f'_A \gtrsim g_B o g'_B.$$

(7) (i) $R \in \mathscr{N}$ iff for all $A \in \mathscr{E} - \mathscr{N}$, $A \cap R = \emptyset$, and $f_{A \cup R} \in \mathscr{D}$, $f_{A \cup R} \sim (f_{A \cup R})_A$.

(ii) $R \in \mathscr{N}$ and $S \subset R$ implies $S \in \mathscr{N}$.

(8) For $A \in \mathscr{E} - \mathscr{N}$, $\langle \mathscr{D}(A), o_A, \gtrsim_A \rangle$ is an Archimedean, regular, ordered semigroup with identity e_A.

(9) For $A, B \in \mathscr{E} - \mathscr{N}$, if $f_A \gtrsim g_B \gtrsim h_A$, then there exists $g'_A \in \mathscr{D}(A)$ such that $g'_A \sim g_B$.

(10) For A, $B \in \mathscr{E} - \mathscr{N}$, there exists f_A, $g_B \in \mathscr{D}$ such that $f_A \sim g_B$ and not $f_A \sim e_A$.

In the presence of the other axioms, Axiom 8 probably can be weakened considerably without altering the conclusion embodied in the following theorem. Note that Axiom 10 says, in effect, that for each $A, B \in \mathscr{E} - \mathscr{N}$, the hypothesis of Axiom 9 is true for some decisions in $\mathscr{D}$.

THEOREM. *If Axioms* 1–10 *hold, there exist real-valued P on $\mathscr{E}$ and u on $\mathscr{D}$ such that, for all f_A, $g_B \in \mathscr{D}$,*

(i) $f_A \gtrsim g_B$ *iff* $u(f_A) \geqslant u(g_B)$;

(ii) $\langle X, \mathscr{E}, P \rangle$ *is a finitely additive probability space and* $P(R) = 0$ *iff* $R \in \mathscr{N}$;

(iii) $$u(f_A o g_B) = u(f_{A-B}) P(A - B \mid A \cup B) + u(g_{B-A}) P(B - A \mid A \cup B) + [u(f_{A \cap B}) + u(g_{A \cap B})] P(A \cap B \mid A \cup B),$$
where for $C \subset A$, $f_C = (f_A)_C$ *and* $u(f_R) P(R \mid A) = 0$ *if* $R \in \mathscr{N}$.

(iv) *P is unique and u is a ratio scale.*

LEMMA 1. *For* $A, B \in \mathscr{E} - \mathscr{N}$, $e_A \sim e_B$.

Proof. Suppose $e_A \succ e_B$. By Axiom 10, there exist $f_A \sim g_B$. By Axiom 6,

$$f_A \sim f_A o e_A \succ g_B o e_B \sim g_B,$$

contrary to Axiom 2.

LEMMA 2. *For* $A_i \in \mathscr{E} - \mathscr{N}$, $i = 1, \ldots, n$, *let*

$$\mathscr{D}(A_1, \ldots, A_n) = \{ f_1 \mid f_1 \in \mathscr{D}(A_1) \text{ and for each } i = 2, \ldots, n \text{ there exists } f_i \in \mathscr{D}(A_i) \text{ such that } f_i \sim f_1 \}.$$

Then, $\langle \mathscr{D}(A_1, \ldots, A_n), \gtrsim, o \rangle$ *is an Archimedean, regular, ordered semigroup with identity.*

Proof. By Lemma 1, $\mathscr{D}(A_1,\ldots,A_n)$ includes the identity. Closure under o is obvious from Axiom 6.

To show regularity, suppose $f,\ g\in\mathscr{D}(A_1,\ldots,A_n)$ and $f\succ g$. By Axiom 8, there exist $h_i'\succ e_{A_i}$ such that $f\succsim h_i' o g$. Let the smallest be h, then by Lemma 1 and Axiom 9 there exists $h_i\sim h$, $h_i\in\mathscr{D}(A_1)$, and so $h\in\mathscr{D}(A_1,\ldots,A_n)$ and $f\succsim hog$.

The remaining axioms are immediate consequences of Axiom 8.

In the following, we let bold face denote equivalence classes under $\sim$.

LEMMA 3. *For* $A,\ -A\in\mathscr{E}-\mathscr{N}$, *the mapping* $\psi_A(\mathbf{f_A})=\mathbf{f_A}o\mathbf{e_{-A}}$ *from* $\mathscr{D}(A)/\sim_A$ *into* $\mathscr{D}(X)/\sim_X$ *is an ordered semigroup isomorphism into.*

Proof. By Axiom 5 ψ_A is 1:1 and

$$\mathbf{f_A}\succsim\mathbf{g_A} \quad \text{iff} \quad \psi_A(\mathbf{f_A})=\mathbf{f_A}o\mathbf{e_{-A}}\succsim\mathbf{g_A}o\mathbf{e_{-A}}=\psi_A(\mathbf{g_A}).$$

By Axioms 3, 4, and 6,

$$\begin{aligned}\psi_A(\mathbf{f_A})o\psi_A(\mathbf{g_A})&=\mathbf{f_A}o\mathbf{e_{-A}}o\mathbf{g_A}o\mathbf{e_{-A}}\\&\sim\mathbf{f_A}o\mathbf{g_A}o\mathbf{e_{-A}}\\&\sim\psi_A(\mathbf{f_A}o\mathbf{g_A}).\end{aligned}$$

Proof of Theorem. By Axiom 8, each $\mathscr{D}(A)$ has a ratio scale additive representation. Choose one of these, u_X, on $\mathscr{D}(X)$ and select the one u_A on $\mathscr{D}(A)$ that agrees with u_X over $\mathscr{D}(A,X)$, which by Axiom 10 is nontrivial. Observe that u_A and u_B agree over $\mathscr{D}(A,B)$ because of uniqueness and the fact that, by choice, they agree on $\mathscr{D}(A,B,X)$, which by Axioms 9 and 10 is nontrivial. Thus a unique function u is defined on $\mathscr{D}$.

(i) We show that u is order preserving. Suppose $f_A\succsim g_B$. If either f_A or $g_B\in\mathscr{D}(A,B)$, then by the choice of u, $u(f_A)\geqslant u(g_B)$, and conversely. So, suppose neither $f_A,\ g_B\in\mathscr{D}(A,B)$. Then by Axiom 9, for all $h\in\mathscr{D}(A,B)$, $f_A\succ h\succ g_B$, and so

$$u(f_A)>u_A(h)=u_B(h)>u(g_B).$$

(ii) For $A,\ -A\in\mathscr{E}-\mathscr{N}$, ψ_A is an order isomorphism into $\mathscr{D}(X)/\sim_X$ (Lemma 2) and so by the uniqueness of u (Axiom 8), $u\psi_A/u$ is a constant over A. So we define

$$P(A)=\begin{cases}0, & A\in\mathscr{N}\\ u(f_Aoe_{-A})/u(f_A), & A,\ -A\in\mathscr{E}-\mathscr{N}\\ 1, & -A\in\mathscr{N}.\end{cases}$$

For A, $B \in \mathscr{E}$, $A \cap B = \emptyset$, we show $P(A \cup B) = P(A) + P(B)$. Consider first $A, B \in \mathscr{E} - \mathscr{N}$. Since $A \cap B = \emptyset$, $-A$, $-B \in \mathscr{E} - \mathscr{N}$. Suppose $-(A \cup B) \in \in \mathscr{E} - \mathscr{N}$, then for any f_A and g_B,

$$f_A o g_B o e_{-(A \cup B)} \sim f_A o e_B o e_{-(A \cup B)} o g_B o e_A o e_{-(A \cup B)} \quad \text{(Axioms 3, 4, and 5)}$$
$$\sim f_A o e_{-A} o g_B o e_{-B} \quad (-A = B \cup -(A \cup B) \text{ and Axiom 4}).$$

So

$$\begin{aligned} u(f_A o g_B) P(A \cup B) &= u(f_A o g_B o e_{-(A \cup B)}) \\ &= u(f_A o e_{-A} o g_B o e_{-B}) \\ &= u(f_A o e_{-A}) + u(g_B o e_{-B}) \quad \text{(additivity of } u \text{ on } \mathscr{D}(X)) \\ &= u(f_A) P(A) + u(g_B) P(B). \end{aligned}$$

We next show the same thing for $-(A \cup B) \in \mathscr{N}$. Since $-B = A \cup -(A \cup B)$, we may extend f_A to f_{-B} and by Axiom 7, $f_{-B} \sim f_A$. There is a similar extension $g_{-A} \sim g_B$. Observe,

$$f_A o g_B \sim f_{-B} o g_A \quad \text{(they agree on } A \cup B\text{, Axiom 7)}$$
$$\sim f_{-B} o e_{-A} o g_{-A} o e_{-B} \quad \text{(Axioms 3 and 5).}$$

Since $X = -A \cup -B$, the additivity of u on $\mathscr{D}(X)$ yields,

$$\begin{aligned} u(f_A o g_B) P(A \cup B) &= u(f_A o g_B) \\ &= u(f_{-B} o e_{-A}) + u(g_{-A} o e_{-B}). \end{aligned}$$

By Axiom 7, $f_{-B} o e_{-A} \sim f_A o e_{-A}$ and $g_{-A} o e_{-B} \sim g_B o e_{-B}$, and so the conclusion follows. By Axiom 10, we may choose $f_A \sim g_B \nsim e_A \sim e_B$, and so by Axiom 4, we can cancel $u(f_A o g_B) = u(f_A) = u(g_B) \neq 0$, proving the additivity of P when A, $B \in \mathscr{E} - \mathscr{N}$.

If A, $B \in \mathscr{N}$, then by Axiom 7 (see Lemma 1, p. 382, Krantz *et al.*, 1971) $A \cup B \in \mathscr{N}$, and so

$$P(A \cup B) = 0 = P(A) + P(B).$$

If $A \in \mathscr{E} - \mathscr{N}$ and $B \in \mathscr{N}$, then by Axiom 7 $A \cup B \in \mathscr{E} - \mathscr{N}$ and by repeated uses of Axiom 7

$$\begin{aligned} P(A \cup B) &= u(f_{A \cup B} o e_{-(A \cup B)}) / u(f_{A \cup B}) \\ &= u(f_{A \cup B} o e_A o e_B o e_{-(A \cup B)}) / u(f_{A \cup B}) \\ & u(f_A o e_{-A}) / u(f_A) \quad (f_A \text{ is the restriction of } f_{A \cup B}) \\ &= P(A) + P(B). \end{aligned}$$

(iii) By definition of o,

$$f_A o g_B \sim f_{A-B} o f_{A\cap B} o g_{A\cap B} o g_{B-A},$$

provided we make the convention of writing $(f_A)_C = f_{A\cap C}$ and of omitting any term whose domain is in $\mathcal{N}$, which by Axiom 7 does not affect $\sim$. By what we have shown for disjoint sets,

$$u(f_A o g_B)\, P\,(A \cup B) = u(f_{A-B})\, P\,(A-B) + [u(f_{A\cap B}) + \\ + u(g_{A\cap B})]\, P\,(A \cap B) + u(g_{B-A})\, P\,(B-A)$$

(iv) The uniqueness of u follows from Axiom 8 and Lemma 2 and that of P from its definition.

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