

Philosophy of Science

June, 1971

SIMILAR SYSTEMS AND DIMENSIONALLY INVARIANT LAWS*

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Using H. Whitney's algebra of physical quantities and his definition of a similarity transformation, a family of similar systems (R. L. Causey [3] and [4]) is any maximal collection of subsets of a Cartesian product of dimensions for which every pair of subsets is related by a similarity transformation. We show that such families are characterized by dimensionally invariant laws (in Whitney's sense, [10], not Causey's). Dimensional constants play a crucial role in the formulation of such laws. They are represented as a function g , known as a system measure, from the family into a certain Cartesian product of dimensions and having the property $g\phi = \phi g$ for every similarity ϕ . The dimensions involved in g are related to the family by means of certain stability groups of similarities. A one-to-one system measure is a proportional representing function, which plays an analogous role in Causey's theory, but not conversely. The present results simplify and clarify those of Causey.

1. Introduction. Dimensional analysis rests upon the assumption that most, if not all, physical laws can be stated in terms of dimensionally invariant equations. Although this notion is defined later (Def. 2), perhaps a simple example of dimensional methods is not amiss. Consider a projectile of mass m (and dimension M) fired at velocity v (dimension LT^{-1}) at an angle α to the ground. Neglecting air resistance, at what distance d (dimension L) will it return to the ground. A key variable in the problem is the force of gravity g (dimension LT^{-2}). So, we anticipate some function F such that $d = F(m, v, \alpha, g)$, and if we require that F not change when we alter the units of these variables (which, roughly, is what we mean by dimensionally invariant), it is not difficult to show that the only possibility is $d = (v^2/g)G(\alpha)$, where G is an unspecified function. Note that this equation, like all physical laws, is dimensionally consistent: on the left the dimension is L and on the right it is $L^2T^{-2}L^{-1}T^2 = L$.

All of this is so familiar as to seem banal. Yet the question as to what dimensional invariance really means qualitatively and why we should assume that numerical physical laws exhibit it has never been fully answered. Some of the difficulty stems from the fact that dimensional invariance, as it is usually stated, is a feature of

* Received April, 1970.

¹ In working out these ideas, I have benefited greatly from several conversations and an extended correspondence with Robert Causey. He pointed out difficulties and ambiguities with my earlier formulations, and he has aided greatly in my understanding of the issues. I believe that he agrees that the present theorems are correct, but we have never been able to reach complete agreement as to their relation to his results. I also wish to thank a referee for a number of useful comments which helped to clarify the exact relation of this work to Causey's and eliminated an error.

numerical functions, and so there is a temptation to formulate an answer within that domain. Since, however, the claim is about physical laws, not about mathematics, it seems much more pertinent to discover exactly what kinds of qualitative, observable statements can be restated as dimensionally invariant numerical laws. A successful attack in this direction should go a long way to clarifying just what we mean by a physical law.

The idea of finding a qualitative equivalent to dimensionally invariant laws was suggested by Tolman [8], but the qualitative concepts remained elusive in his work. Recently, Causey [3, 4] made an important contribution to clarifying Tolman's notion of physically similar systems; however, he left the problem incompletely resolved and, as I discuss in Section 7, apparently his definition of dimensional invariance is not the standard one. Nonetheless, it should be realized that Causey was the first to take a truly major step in understanding just what qualitative assumptions underlie the existence of dimensionally invariant laws. His general approach and much of his formulation is accepted in this paper, which has as its goal the clarification and improvement of his results.

2. Physical Quantities. In formulating dimensional analysis, either of two somewhat different approaches can be taken. We can begin with a real multiplicative vector space, where a fixed set of coordinates are identified with the basic dimensions (e.g. mass, length, and time in dynamics) and other dimensions are products of powers of these basic ones. Thus, the dependence of a set of dimensions on the base dimensions can be described by the matrix of exponents. This tack, which is rather close to the way physicists work, has rather severe drawbacks from our point of view. One is purely notational: it can be tricky to keep track of a given dimension as it appears in different matrices. The other is deeper: if we are trying to find out the qualitative meaning of numerical laws, it is not especially helpful to have our basic language inherently numerical. Causey used this latter approach. Despite his skill in handling it, I, at least, have found it confusing (see Section 7).

The alternative is to axiomatize, as a primitive, the notion of physical quantity. As you would expect, the axiomatization leads to something much like a vector space, but it has many advantages for formulating dimensional analysis. Among the attempts at such an axiomatization are: Brand [1], Drobot [5], Kurth [6], Quade [7], Thun [9], Whitney [10]. Of these, I think that Whitney's is the most satisfactory.

The key ideas are these. First expressions referring to *physical* quantities, such as those that refer to particular masses, are primitives of the system; they are not numbers in somebody's representation of mass, length, duration, etc. For example, the standard masses of a pan balance are useful physical quantities, and the numerals printed on them need only serve to identify them. Nonetheless, there is a key numerical idea involved, namely, that two physical quantities having the same dimension can be said to differ by a (dimensionless) numerical ratio. This is an invariant of the quantities and has nothing to do with any particular additive numerical representation of them. So the axiomatization should capture this inherent numerical feature without, however, supposing that the quantities are re-

presented numerically, as is done when we suppose they are described as points in a real vector space. Second, two physical quantities of different dimensions can be composed “multiplicatively” to form a new physical quantity. We use ‘*’ to denote this operation. An object can be given a velocity, in which case there is associated with it a dimension of the form mass * velocity. So, of course, is mass * velocity * velocity. The former has something to do with momentum; the latter, with kinetic energy. Clearly, we sometimes want to compose dimensions with themselves. Indeed, at a minimum we want to formulate rational powers of dimensions, which we can do without difficulty. Although apparently not needed for classical physics, some results seem to be simpler to state if, at the onset, we include the possibility of irrational powers. This requires an additional primitive. Finally, although it is convenient to have both positive and negative physical quantities, it is important to be able to distinguish between them.

In what follows, let ‘Re’ denote the real numbers and ‘Re⁺’ the positive reals.

DEFINITION 1. (Whitney) *Suppose that A is a nonempty set, A^+ a nonempty subset of A , * a closed operation from $A \times A$ into A , and Θ a function from $A \times Re$ into A . Then $\langle A, A^+, *, \Theta \rangle$ is a structure of physical quantities if, for all $a, b \in A$ and $\alpha, \beta \in Re$,*

(Relation to real numbers)

1. $Re \subset A$.
2. $Re \cap A^+ = Re^+$.
3. $\alpha * \beta = \alpha\beta$.
4. $\Theta(\alpha, \beta) = \alpha^\beta$.

(Properties of multiplication)

5. * is associative and commutative.
6. $1 * a = a$ and $0 * a = 0$.

(Properties of A^+)

7. $a, b \in A^+$ implies $a * b \in A^+$.
8. If $a \neq 0$, then exactly one of a and $(-1) * a \in A^+$.

(Properties of powers)

9. $\Theta(a, 1) = a$ and $\Theta(a, 0) = 1$.
10. $\Theta(a * b, \alpha) = \Theta(a, \alpha) * \Theta(b, \alpha)$.
11. $\Theta(a, \alpha + \beta) = \Theta(a, \alpha) * \Theta(a, \beta)$.
12. $\Theta[\Theta(a, \alpha), \beta] = \Theta(a, \alpha\beta)$.

Because of the assumptions we have made about Θ , it is clearly reasonable to write a^α for $\Theta(a, \alpha)$, and we do so.

For any $a \neq 0$, define

$$[a] = \{\alpha * a \mid \alpha \in Re\} \quad \text{and} \quad [a]^+ = [a] \cap A^+.$$

Note that we can interpret $[a]$ as the dimension of a since it includes all of the

physical quantities generated from a simply by multiplying it by a (dimensionless) real number. On the set of these dimensions we can define

$$[a] * [b] = [a * b] \quad \text{and} \quad [a]^\alpha = [a^\alpha].$$

It can be shown that both definitions are well defined, and Whitney has shown that under the assumptions of Def. 1, the resulting structure is a multiplicative vector space over the reals, with $[1] = Re$ the identity and $[a]^{-1}$ the inverse of $[a]$. Thus, the usual notions of *span*, *independence*, and *base* are defined in the vector space and have their immediate analogues in the structure of physical quantities. When a base exists, we say that the structure is of *finite dimension*. It is not difficult to show that $a_1, \dots, a_n \in A^+$ form a base of a structure iff for every $a \in A$, $a \neq 0$, there exist unique α and $\rho_1, \dots, \rho_n \in Re$ such that

$$(1) \quad a = \alpha * a_1^{\rho_1} * \dots * a_n^{\rho_n}.$$

Moreover, the ρ_i depend only on $[a]$ not on a .

In a finite dimensional structure, a *similarity* ϕ is any function of A into A with the following property. For a_i in the base, there exists $\phi_i \in Re^+$ such that if a is of the form given in Eq. 1, then

$$(2) \quad \phi(a) = (\prod_{i=1}^n \phi_i^{\rho_i}) * a.$$

3. Dimensionally Invariant Functions. Within the above framework, a dimensionally invariant function is defined as follows, which corresponds to the usual definition in vector space notation.

DEFINITION 2. Suppose that $\langle A, A^+, *, \Theta \rangle$ is a finite dimensional structure of physical quantities and that $P_i, i = 1, \dots, s$, are positive dimensions (i.e. of the form $[a] \cap A^+$) in the structure. A function f from $\times_{i=1}^s P_i$ into Re is called dimensionally invariant iff, for all $p_i \in P_i$,

$$f(p_1, \dots, p_s) = 0$$

is equivalent to: for all similarities ϕ ,

$$f[\phi(p_1), \dots, \phi(p_s)] = 0.$$

Although we do not need it in the rest of the paper, for completeness we include Whitney's version of the working basis of dimensional analysis: Buckingham's classical π -Theorem, [2].

THEOREM 1. Suppose that $\langle A, A^+, *, \Theta \rangle$ is a finite dimensional structure of physical quantities, that $P_i, i = 1, \dots, s$, are positive dimensions which are indexed so that the first $r < s$ form a maximal independent subset of the subspace spanned by the P_i , and that $f: \times_{i=1}^s P_i \rightarrow Re$ is a dimensionally invariant function. Then there exists a real-valued function F on Re^{s-r} and $\rho_{ij} \in Re, i = r + 1, \dots, s, j = 1, \dots, r$, such that for all $p_i \in P_i$,

$$\pi_{i-r} = p_i * p_1^{-\rho_{i1}} * \dots * p_r^{-\rho_{ir}}, \quad i = r + 1, \dots, s,$$

are real numbers and

$$\begin{aligned} & f(p_1, \dots, p_s) = 0 \\ \text{iff} & \\ & F(\pi_1, \dots, \pi_{s-r}) = 0. \end{aligned}$$

Conversely, any function of the π 's is dimensionally invariant.

Our task is to find a formulation of dimensionally invariant functions that goes in the opposite direction from the π -Theorem, one that gets away from numerical functions and states something about observables.

4. Families of Similar Systems and Their Stability Groups. The key idea, suggested by Tolman and made precise by Causey, is that of (physically) similar systems. To get at this we need the notion of a configuration² of a system, which intuitively is simply a listing of the observable physical quantities that characterize the system at the time of observation. To use Causey's example, consider a spring. If we apply a force to it then we observe two physical quantities, the applied force and the length of the spring. That particular combination of force and length is one configuration of the spring. If we apply a different force, we obtain a different length, and that is another configuration of the system. By force and length are meant physical quantities, not numerical representations of them.

Generalizing, we may suppose that a physical system is specified in terms of some r positive physical dimensions P_1, \dots, P_r , from which we can generate a finite dimensional substructure of the structure of all physical quantities. Let $\mathbf{P} = \times_{i=1}^r P_i$. A configuration of the system is some point $p \in \mathbf{P}$, where we simultaneously observe the system has the values p_i on dimensions P_i , $i = 1, \dots, r$. The set of all possible configurations of a particular system forms a subset S of \mathbf{P} . We shall speak of the system and its set of possible configurations interchangeably.

We say that two systems (subsets) S and S' of \mathbf{P} are similar if and only if there exists a similarity ϕ such that

$$S' = \phi(S) = \{p' \mid \text{there exist } p \in S \text{ such that } p' = \phi(p)\}$$

where by $\phi(p)$ we simply mean that if $p = (p_1, \dots, p_r)$, then $\phi(p) = (\phi(p_1), \dots, \phi(p_r))$. Since, as is easily shown, the class of all similarities forms an abelian group, being similar is an equivalence relation. One equivalence class of this relation, i.e. the set \mathbf{S} of all sets similar to a given set, is called a family of similar systems.

A salient fact about a family of similar systems is the set of similarities that do not alter any of its members. We call this the stability group of \mathbf{S} and define it as:

$$SG(\mathbf{S}) = \{\psi \mid \text{for all } S \in \mathbf{S}, \psi(S) = S\}.$$

² Causey used the term 'state' for what I have called 'configuration'. As P. Suppes pointed out to me, this use of state sometimes conflicts with conventional physical usage. For example, in a dynamical system of classical particles, physicists say that the state of the system at any time is given by the location and velocity of each of the particles. If we are attempting to formulate Newton's second law, then a configuration of the system would consist of the force acting on and the acceleration of each of the particles. Calling this the state of the system would only generate confusion.

It should be noted that to show $\psi \in SG(\mathbf{S})$ it is sufficient to show $\psi(S) = S$ for some $S \in \mathbf{S}$ because for any other $S' \in \mathbf{S}$ we can find a similarity ϕ such that $S' = \phi(S)$ and since similarities are commutative,

$$\psi(S') = \psi\phi(S) = \phi\psi(S) = \phi(S) = S'.$$

5. System Measures and Numerical Laws. Physical systems are not usually described just in terms of their observable configurations. Various dimensional constants play a role. Some of these are so-called universal constants, such as the gravitational constant, whereas others depend only on the system. For example, in a spring that satisfies Hooke's law, the usual formulation is that there is associated with each spring a constant k (with dimensions MT^{-2}) such that if F is a numerical measure of force and l a numerical measure of length, k can be chosen so that all physically realizable combinations of F and l satisfy $F = kl$. Presumably, however, the numerical measures have nothing to do with the matter. The essence of the statement is that if S is the set of configurations of the spring, there is a point $g(S)$ in some dimension Q that somehow captures something about the spring. Indeed, only by finding this assignment of a quantity to the spring are we able to formulate the usual physical law for *all* springs satisfying Hooke's law.

It seems intuitively clear, however, that knowing all of the systems that are similar must, to a considerable degree, limit both our choice of Q and of the possible functions g . For example, suppose S is (the set of configurations of) a spring and $g(S)$ is its spring constant. Let ϕ be a similarity. Then $\phi(S)$ is a realizable spring—the ratio of forces is $\phi_M\phi_L\phi_T^{-2}$ and of lengths is ϕ_L —and it has the spring constant

$$g[\phi(S)] = g(S) \phi_M\phi_L\phi_T^{-2} = \phi[g(S)],$$

i.e. $g\phi = \phi g$. This appears to be the basic property of derived system measures; our evidence for this statement is Theorem 2.

As a result, we make the following definition. Suppose that \mathbf{S} is a family of similar systems and Q_1, \dots, Q_t are positive dimensions. A function g from \mathbf{S} into $\mathbf{Q} = \times_{j=1}^t Q_j$ is called a *system measure* of \mathbf{S} iff for all similarities ϕ , $g\phi = \phi g$.

Again the notion of a stability group plays a role. We define

$$SG(\mathbf{Q}) = \{\psi \mid \text{for all } q \in \mathbf{Q}, \psi(q) = q\}.$$

As with $SG(\mathbf{S})$, to show $\psi \in SG(\mathbf{Q})$ it is sufficient to show $\psi(q) = q$ for some $q \in \mathbf{Q}$.

As was noted, system measures seem to play an essential role in formulating numerical laws to describe families of systems. In fact, they appear to enter in the following way. Suppose \mathbf{S} is a family of similar systems defined in \mathbf{P} , that g is a function from \mathbf{S} into \mathbf{Q} , and that f is a real-valued function on $\mathbf{P} \times \mathbf{Q}$. We then say \mathbf{S} satisfies the law (f, g) iff for all $p \in \mathbf{P}$ and $q \in \mathbf{Q}$,

$$f(p, q) = 0 \text{ iff there is some } S \in \mathbf{S} \text{ for which } p \in S \text{ and } g(S) = q.$$

Such a law is said to be *dimensionally invariant* iff f is a dimensionally invariant function in the sense of Def. 2. Note that we have formulated the statement that \mathbf{S} satisfies a law (f, g) without requiring g to be a system measure, although we

anticipate that the two should have something to do with one another. Theorem 2 shows this to be so.

In the discussion so far we have assumed that the relevant products of dimensions, \mathbf{P} and \mathbf{Q} , were given. When we begin with a dimensionally invariant function, we must find them; Theorem 1 suggests a natural way. Suppose that f from $\times_{i=1}^s P_i \rightarrow Re$ is a dimensionally invariant function. Choose the indices so that the first r dimensions form a maximal independent subset of the space spanned by all s P_i . We then set $\mathbf{P} = \times_{i=1}^r P_i$ and $\mathbf{Q} = \times_{i=r+1}^s P_i$.

If f is any real-valued function on $\mathbf{P} \times \mathbf{Q}$, then for each $q \in \mathbf{Q}$ we may define a set S_q as follows:

$$S_q = \{p \mid f(p, q) = 0\}.$$

Relative to \mathbf{P} and \mathbf{Q} , let \mathbf{S}_f denote the collection of all nonempty S_q . In general \mathbf{S}_f is not a family of similar systems, but it is when f is dimensionally invariant. For if ϕ is a similarity, then

$$\begin{aligned} \phi(S_q) &= \phi\{p \mid f(p, q) = 0\} \\ &= \{\phi(p) \mid f[\phi(p), \phi(q)] = 0\} \\ &= S_{\phi(q)}, \end{aligned}$$

and so any two members of \mathbf{S}_f are related by a similarity transformation.

6. The Principal Results.

THEOREM 2. *Suppose that \mathbf{S} is a family of similar systems defined over $\mathbf{P} = \times_{i=1}^r P_i$, where P_1, \dots, P_r are positive dimensions; that Q_1, \dots, Q_t are positive dimensions of the structure spanned by P_1, \dots, P_r ; and let $\mathbf{Q} = \times_{j=1}^t Q_j$. The following three properties are equivalent:*

- (i) *There exists a system measure g from \mathbf{S} into \mathbf{Q} .*
- (ii) *There exists a function f from $\mathbf{P} \times \mathbf{Q}$ into Re and a function g from \mathbf{S} into \mathbf{Q} such that \mathbf{S} satisfies the dimensionally invariant law (f, g) .*
- (iii) $SG(\mathbf{S}) \subset SG(\mathbf{Q})$.

Assuming the above properties, the following three properties are equivalent:

- (iv) *The system measure g is one-to-one.*
- (v) $\mathbf{S}_f = \mathbf{S}$.
- (vi) $SG(\mathbf{S}) \supset SG(\mathbf{Q})$.

Uniqueness. *Suppose that g is a system measure into \mathbf{Q} , then g' is a system measure into \mathbf{Q} iff there exists a similarity ϕ such that $g' = \phi g$. If $g' = \phi g$ and $f'(p, q) = f[\phi(p), q]$, then \mathbf{S} satisfies the dimensionally invariant law (f, g) iff \mathbf{S} satisfies the dimensionally invariant law (f', g') .*

Proof. (i) implies (ii). Assume g is a system measure and define f as follows:

$$f(p, q) = \begin{cases} 0 & \text{if there exists } S \in \mathbf{S} \text{ such that } p \in S \text{ and } g(S) = q, \\ 1 & \text{otherwise.} \end{cases}$$

By definition, \mathbf{S} satisfies the law (f, g) . We show that f is dimensionally invariant. If ϕ is a similarity, then $\phi(p) \in \phi(S) \in \mathbf{S}$ and, by definition, $g\phi(S) = \phi g(S) = \phi(q)$, and so

$$f(p, q) = 0 \text{ iff } f[\phi(p), \phi(q)] = 0.$$

(ii) implies (iii). Suppose $\psi \in SG(\mathbf{S})$ and consider any $S \in \mathbf{S}$. If $p \in S$ and $g(S) = q$, then $f(p, q) = 0$ and, by dimensional invariance, $f[\psi(p), \psi(q)] = 0$. So, by definition, there exists S' such that $\psi(p) \in S'$ and $g(S') = \psi(q)$. Since $\psi(S) = S$, it follows that $S \subset S'$. The converse argument implies $S' = S$ and so

$$\psi(q) = g(S') = g(S) = q,$$

proving $\psi \in SG(\mathbf{Q})$.

(iii) implies (i). Select any $q \in \mathbf{Q}$ and any $S_0 \in \mathbf{S}$. For $S \in \mathbf{S}$, there exists a similarity ϕ such that $S = \phi(S_0)$. Define g by $g(S) = \phi(q)$. First, we show that g is a function:

$$\begin{aligned} S = S' \text{ implies } \phi(S_0) &= \phi'(S_0) \\ &\text{implies } \phi^{-1}\phi' \in SG(\mathbf{S}) \subset SG(\mathbf{Q}) \quad (\text{by iii}) \\ &\text{implies } \phi(q) = \phi'(q) \\ &\text{implies } g(S) = g(S'). \end{aligned}$$

Second, we show $g\psi = \psi g$:

$$g\psi(S) = g\psi\phi(S_0) = \psi\phi(q) = \psi g\phi(S_0) = \psi g(S).$$

(iv) implies (v). First, suppose $S \in \mathbf{S}$. Since \mathbf{S} satisfies the law (f, g) , it follows that $S \subset S_{g(S)}$. Suppose $p \in S_{g(S)} - S$, then there exists $S' \in \mathbf{S}$ such that $p \in S'$ and $g(S') = g(S)$. Since g is 1:1, $S' = S$, so $S = S_{g(S)} \in \mathbf{S}_f$. Conversely, suppose $S_q \in \mathbf{S}_f$, then for $p \in S_q$ there exists $S \in \mathbf{S}$ such that $p \in S$ and $f[p, g(S)] = 0$. Since g is 1:1, this implies $S_q \subset S$. Of course, $S \subset S_q$, and so $S_q \in \mathbf{S}$.

(v) implies (vi). Since $\mathbf{S} = \mathbf{S}_f$, for any $S \in \mathbf{S}$ there exists $q \in \mathbf{Q}$ such that $S = S_q$. If $\psi \in SG(\mathbf{Q})$, then since f is dimensionally invariant,

$$\psi(S) = \psi(S_q) = S_{\psi(q)} = S_q = S,$$

and so $\psi \in SG(\mathbf{S})$.

(vi) implies (iv). Observe,

$$\begin{aligned} g(S) = g(S') \text{ implies } g\phi(S_0) &= g\phi'(S_0) \\ &\text{implies } \phi g(S_0) = \phi' g(S_0) \\ &\text{implies } \phi^{-1}\phi' \in SG(\mathbf{Q}) \subset SG(\mathbf{S}) \\ &\text{implies } S = \phi(S_0) = \phi'(S_0) = S', \end{aligned}$$

and so g is one-to-one.

Uniqueness. Suppose g is a system measure and $g' = \phi g$. Then

$$g'\psi(S) = \phi g\psi(S) = \phi\psi g(S) = \psi\phi g(S) = \psi g'(S).$$

Next, suppose g and g' are both system measures. Let $q = g(S_0)$ and $q' = g'(S_0)$, and let ϕ be any similarity such that $q' = \phi(q)$. Since if $S \in \mathbf{S}$, $S = \psi(S_0)$,

$$\begin{aligned}\phi g(S) &= \phi g\psi(S_0) = \phi\psi g(S_0) = \phi\psi(q) \\ &= \psi\phi(q) = \psi(q') = \psi g'(S_0) = g'\psi(S_0) = g'(S).\end{aligned}$$

Finally, suppose $g' = \phi g$ and $f'(p, q) = f[\phi(p), q]$, then \mathbf{S} satisfies the dimensionally invariant law (f, g)

$$\begin{aligned}&\text{iff \{there exists } p \in S \in \mathbf{S} \text{ iff } f[p, g(S)] = 0\}} \\ &\text{iff \{there exists } p \in S \in \mathbf{S} \text{ iff } f[\phi(p), \phi g(S)] = 0\}} \\ &\text{iff \{there exists } p \in S \in \mathbf{S} \text{ iff } f[p, g'(S)] = 0\}} \\ &\text{iff } \mathbf{S} \text{ satisfies the dimensionally invariant law } (f', g').\end{aligned}\quad \text{Q.E.D.}$$

Theorem 2 shows equivalent ways of saying that \mathbf{S} satisfies a dimensionally invariant law, but it does not answer the question we are really interested in: to what extent are the notions of a family of similar systems and that of a dimensionally invariant function the same? An answer is formulated as:

THEOREM 3. *If \mathbf{S} is a family of similar systems, then \mathbf{S} satisfies a dimensionally invariant law (f, g) . Conversely, if f is a dimensionally invariant function, then \mathbf{S}_f is a family of similar systems. If $\mathbf{S} = \mathbf{S}_f$, then g defined by $g(S_q) = q$ is a one-to-one system measure.*

Proof. Suppose that \mathbf{S} is a family of similar systems on $\mathbf{P} = \times_{i=1}^r P_i$. For $x, y \in (Re^+)^r$ and $\alpha \in Re$, define

$$xy = (x_1y_1, \dots, x_ry_r) \quad \text{and} \quad x^\alpha = (x_1^\alpha, \dots, x_r^\alpha).$$

It is easy to see that under these definitions $(Re^+)^r$ is a multiplicative vector space over the reals. Any similarity is, by Eq. 2, a point in this space. Thus, $SG(\mathbf{S})$ is contained in a subspace of minimal dimension, say, $r - t$. If we select, as we may, the first $r - t$ components as independent, then there exist real ρ_{ij} , $i = r - t + 1, \dots, r, j = 1, \dots, r - t$, such that for $\psi \in SG(\mathbf{S})$,

$$\psi_i = \prod_{j=1}^{r-t} \psi_j^{\rho_{ij}}, \quad i = r - t + 1, \dots, r.$$

For $j = 1, \dots, t$, define

$$Q_j = P_{r-t+j} * P_1^{-\rho_{r-t+j,1}} * \dots * P_{r-t}^{-\rho_{r-t+j,r-t}}.$$

Now, we have, for each $j = 1, \dots, t$,

$$\begin{aligned}\psi \in SG(\mathbf{S}) \text{ implies } \psi_{r-t+j} &= \prod_{k=1}^{r-t} \psi_k^{\rho_{r-t+j,k}}, \quad j = 1, \dots, t \\ \text{iff } 1 &= \psi_{r-t+j} \prod_{k=1}^{r-t} \psi_k^{-\rho_{r-t+j,k}} \\ \text{iff } \psi &\in SG(\mathbf{Q}).\end{aligned}$$

The conclusion follows by Theorem 2.

Conversely, as we showed prior to Theorem 2, \mathbf{S}_f is a family of similar systems when f is dimensionally invariant.

It is trivial that g is one-to-one, and it is a system measure because

$$g\phi(S_q) = gS_{\phi(q)} = \phi(q) = \phi g(S_q).\quad \text{Q.E.D.}$$

I do not know of any interesting characterization of those similar systems that have one-to-one system measures. The question is of considerable importance because, in practice, the one-to-one case appears to arise in those laws that include no universal constants.

7. Relations to Causey's Theory. The ideas we have just presented modify those of Causey [3], [4]; I shall attempt to sketch the relations. The most obvious difference is the restriction imposed on the function g . It will be recalled that we have required $g\phi = \phi g$. To state Causey's condition, we must have a little notation. Suppose $q \in \mathbf{Q}$ and $\alpha, \beta \in (Re^+)^t$, then we define $\alpha q = (\alpha_1 * q_1, \dots, \alpha_t * q_t)$, $(\alpha q)(\beta q) = (\alpha\beta)q$ and $(\alpha q)^{-1} = \alpha^{-1}q$. Using these definitions, a one-to-one function $g: \mathbf{S} \rightarrow \mathbf{Q}$ is called a *proportional representing measure* provided that, for all $S_i \in \mathbf{S}$, $i = 1, 2, 3, 4$,

$$g(S_1)g(S_2)^{-1} = g(S_3)g(S_4)^{-1}$$

iff there exists a similarity such that $S_1 = \phi(S_2)$, $S_3 = \phi(S_4)$. Note that g is assumed to be one-to-one.

Our last result shows that the concept of a system measure is stronger than that of a proportional representing measure.

THEOREM 4. *A one-to-one system measure is a proportional representing measure, but not conversely.*

Proof. Let g be a one-to-one system measure on \mathbf{S} . Suppose that $S_i \in \mathbf{S}$ and $S_i = \psi_i(S_0)$, $i = 1, 2, 3, 4$. If there exists ϕ such that $S_1 = \phi(S_2)$ and $S_3 = \phi(S_4)$, then

$$\begin{aligned} g(S_1)g(S_2)^{-1} &= g\phi\psi_2(S_0)[g\psi_2(S_0)]^{-1} \\ &= \phi\psi_2g(S_0)\psi_2^{-1}g(S_0)^{-1} \\ &= \phi g(S_0)g(S_0)^{-1} \\ &= g\phi\psi_4(S_0)[g\psi_4(S_0)]^{-1} \\ &= g(S_3)g(S_4)^{-1}. \end{aligned}$$

Conversely, $g(S_1)g(S_2)^{-1} = g(S_3)g(S_4)^{-1}$ implies

$$\begin{aligned} g\psi_1(S_0)g\psi_2(S_0)^{-1} &= \psi_1\psi_2^{-1}g(S_0)g(S_0)^{-1} \\ &= \psi_3\psi_4^{-1}g(S_0)g(S_0)^{-1}, \end{aligned}$$

and so $[\psi_1\psi_2^{-1}]^{-1}\psi_3\psi_4^{-1} = \psi \in SG(\mathbf{Q})$. Therefore, $g\psi(S) = \psi g(S) = g(S)$. But g is one-to-one, so $\psi(S) = S$, i.e. $\psi \in SG(\mathbf{S})$. Let $\phi = \psi_1\psi_2^{-1}$, then $\psi_3\psi_4^{-1} = \phi\psi$, whence

$$\begin{aligned} S_1 = \psi_1(S_0) &= \phi\psi_2(S_0) = \phi(S_2), \\ S_3 = \psi_3(S_0) &= \phi\psi_4\psi(S_0) = \phi\psi_4(S_0) = \phi(S_4). \end{aligned}$$

To produce an example of a proportional representing measure that is not a system measure, consider any family of similar systems \mathbf{S} , defined in independent positive dimensions P_1, \dots, P_r , for which there is a one-to-one system measure g into $\times_{j=1}^t Q_j = \mathbf{Q}$. Let $P = [\rho_{ij}] \neq 0$ be the $r \times t$ matrix describing the dependence of the Q_j on the P_i (Eq. 2). Let T be any nonsingular $t \times t$ matrix for which

$T + I$ is also nonsingular. Let $\Sigma = [\sigma_{ij}] = P(T + I)$. For some $q \in \mathbf{Q}$ and $S_0 \in \mathbf{S}$, define h from \mathbf{S} into \mathbf{Q} as follows: for $S = \phi(S_0)$,

$$h(S) = (\dots, \prod_{i=1}^r \phi_i^{\sigma_{ij}} * q_j, \dots).$$

First, h is a well-defined function. For suppose $S = \phi(S_0) = \phi'(S_0)$, then $\psi = \phi^{-1}\phi' \in SG(\mathbf{S}) = SG(\mathbf{Q})$ (because, by assumption, \mathbf{S} has a one-to-one system measure and Theorem 2). Thus,

$$\prod_{i=1}^r \phi_i^{\sigma_{ij}} = \prod_{i=1}^r \psi_i^{\sigma_{ij}} \prod_{i=1}^r \phi_i^{\sigma_{ij}}$$

and so it is sufficient to show $y_j = \prod_{i=1}^r \psi_i^{\sigma_{ij}} = 1, j = 1, \dots, t$.

If we let $x = (\dots, \log \psi_i, \dots)$,

$$\begin{aligned} y_j = 1, j = 1, \dots, t, & \text{ iff } 0 = x\Sigma = xP(T + I) \\ & \text{ iff } 0 = xP \quad (\text{since } T + I \text{ is nonsingular}) \\ & \text{ iff } \prod_{i=1}^r \psi_i^{\rho_{ij}} = 1 \\ & \text{ iff } \psi \in SG(\mathbf{Q}) = SG(\mathbf{S}) \quad (\text{a one-to-one system} \\ & \text{measure exists}). \end{aligned}$$

Second, h is one-to-one. For suppose $S = \phi(S_0)$, $S' = \phi'(S_0)$, and $h(S) = h(S')$. Clearly, there exist real ψ_i such that $\phi'_i = \psi_i \phi_i$, and so from $h(S) = h(S')$ we conclude $\prod_{i=1}^r \psi_i^{\sigma_{ij}} = 1$ which, as we have just shown, is equivalent to $\psi \in SG(\mathbf{S})$, hence $S = S'$.

Third, h is not a system measure. If it were, then for any similarity ψ , $h\psi = \psi h$ implies

$$\prod_{i=1}^r \psi_i^{\rho_{ij}} = \prod_{i=1}^r \psi_i^{\sigma_{ij}}, \quad j = 1, \dots, t.$$

As above, $xP = x\Sigma = xP(T + I)$, and so $xPT = 0$. Since T is nonsingular and this holds for all x , it follows that $P = 0$, which is impossible. So h is not a system measure.

Fourth, h is a proportional representing measure. Let $S_i = \psi^{(i)}(S_0)$, $i = 1, \dots, 4$ and suppose that there exists a similarity ϕ such that $S_1 = \phi(S_2)$ and $S_3 = \phi(S_4)$. Thus, $\psi^{(1)} = \phi\psi^{(2)}$ and $\psi^{(3)} = \phi\psi^{(4)}$. Observe that

$$\begin{aligned} h(S_1)h(S_2)^{-1} &= (\dots, \prod_{i=1}^r (\psi_i^{(1)}/\psi_i^{(2)})^{\sigma_{ij}} * q_j, \dots) \\ &= (\dots, \prod_{i=1}^r \phi_i^{\sigma_{ij}} * q_j, \dots) \\ &= (\dots, \prod_{i=1}^r (\psi_i^{(3)}/\psi_i^{(4)})^{\sigma_{ij}} * q_j, \dots) \\ &= h(S_3)h(S_4)^{-1}. \end{aligned}$$

Conversely, suppose this last property holds. Thus, $h(S_1)h(S_2)^{-1}h(S_3)^{-1}h(S_4) = q$, and so

$$\prod_{i=1}^r \psi_i^{\sigma_{ij}} = 1, \quad j = 1, \dots, t$$

where

$$\psi_i = \frac{\psi_i^{(1)}\psi_i^{(4)}}{\psi_i^{(2)}\psi_i^{(3)}}.$$

Taking logarithms,

$$\sum_{i=1}^r (\log \psi_i) \sigma_{ij} = 0.$$

Since $\Sigma = P(T + I)$ and $T + I$ is nonsingular, P and Σ have the same null space, so

$$\prod_{i=1}^r \psi_i^{\rho_{ij}} = 1.$$

So the similarity ψ defined by ψ_1, \dots, ψ_r is in $SG(\mathbf{Q}) = SG(\mathbf{S})$.

Define the similarity ϕ by $\phi_i = \psi_i^{(1)}/\psi_i^{(2)}$, $i = 1, \dots, r$, and note that $\phi_i \psi_i^{-1} = \psi_i^{(3)}/\psi_i^{(4)}$. Therefore

$$S_1 = \psi^{(1)}(S_0) = \phi \psi^{(2)}(S_0) = \phi(S_2)$$

and

$$S_3 = \psi^{(3)}(S_0) = \psi^{(4)} \phi \psi^{-1}(S_0) = \phi \psi^{(4)}(S_0) = \phi(S_4). \quad \text{Q.E.D.}$$

Given that a system measure is a more restrictive concept than that of a proportional representing measure (Theorem 4), given that they are both intended to play the same role in formulating physical laws (see Theorem 2), and assuming that the above proofs are correct, then one is led to conclude either that Causey's paper includes an error⁹ or that he is employing a definition of dimensional invariance different from Whitney's (Def. 2), which is widely conceded to correspond to the one used in applications of dimensional analysis (see Theorem 1).

Let me confess my great difficulties in understanding Causey on this and my uncertainty as to exactly what his paper says. Initially I was convinced that he intended the usual definition of dimensional invariance and that he had made an error. A referee has kindly pointed out that I was probably wrong and that Causey really intended a somewhat different notion of invariance. Specifically he suggests the following (in our notation). Causey proved that any proportional representing function g has the property that, for every similarity ϕ , there is a function ϕ' (called a dimensional function) from \mathcal{Q} into \mathcal{Q} such that $g\phi = \phi'g$. Note that ϕ' depends both on ϕ and on g . Now, f is a dimensionally invariant law of a family of similar systems with a proportional representing function g iff, for all $p \in \mathbf{P}$, $q \in \mathbf{Q}$,

$$f(p, q) = 0 \text{ iff, for all similarities } \phi, f[\phi(p), \phi'(q)] = 0.$$

Perhaps that is what Causey meant. It certainly does not seem obvious from the definition given in [4], p. 255, and by making the effect of the similarity on \mathbf{Q} depend on the family in question, it surely is an idiosyncratic concept of dimensional invariance.

⁹ If there is an error, it lies in the comment in [4], p. 268, "Thus, comparison of (8) with (2) shows that the law of C is indeed dimensionally invariant in the customary way". The difficulty is that the definition of dimensional invariance (*ibid*, p. 255) includes a matrix D , which describes the effect of a similarity on dependent dimensions. Since there is no comment to the contrary, one assumes that D is characteristic of the dimensions, and in no way is affected by the law in question. This is certainly the case in dimensional analysis. By contrast, the expression derived in Eq. 8 includes a matrix that explicitly depends upon the proportional representing measure (see text below).

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ANNOUNCEMENT

PHILOSOPHY OF SCIENCE ASSOCIATION

ELECTIONS — FEBRUARY 1971

<i>Office of President</i>	Elected: Wesley D. Salmon, University of Indiana.
<i>Office of Secretary-Treasurer</i>	Elected: Peter D. Asquith, Michigan State University.
<i>Governing Board</i>	Elected: Robert Cohen, Boston University Stephen Toulmin, Michigan State University.

Ten proposed revisions to the Association's constitution were also voted upon in the election. All proposed revisions passed. The revised constitution will be printed in the September issue of this Volume of Philosophy of Science.

MEMORANDUM

TO: Members of the Governing Board and Nominees
FROM: Peter D. Asquith, Secretary-Treasurer

On behalf of the Association, I wish to express thanks to all the nominees for having indicated their willingness to be of service to the Association. The services provided to the Association by those retiring from the Governing Board, Gerald Massey, Mary Hesse, and Richard Jeffrey, are deeply appreciated. Finally, a special note of thanks to Adolf Grünbaum for his services to the Association as President. Although retiring from the office of President he will continue to serve as a member of the Governing Board and the Editorial Board.