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AXIOMATIC THERMODYNAMICS AND EXTENSIVE MEASUREMENT

I. INTRODUCTION

A great deal of the theory of measurement concerns the numerical representation of systems $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ where $\mathscr{S}$ is a nonempty set, $\preccurlyeq$ is a binary (ordering) relation on $\mathscr{S}$, and $\circ$ is a binary (concatenation) operation on $\mathscr{S}$. The oldest and most central theorem is Hölder's[3], which treats the case where $\preccurlyeq$ is a simple order, $\circ$ is a group operation, and the two are related by the property of monotonicity: for any $c \in \mathscr{S}$, $a \preccurlyeq b$ if and only if $a \circ c \preccurlyeq b \circ c$. The theorem gives conditions for such a system to be isomorphic to a subgroup of the additive reals.

Hölder's theorem underlies, in one way or another, the various theories of extensive measurement that are designed to account for the existence of such numerical measures as mass, length, and time. In order to study more complicated numerical measures such as those which arise in thermodynamics, it is necessary to generalize Hölder's theorem. We shall present a result which improves on one of Giles[4], gives a natural quantitative representation of thermodynamics, and has as a special case a very nice formulation of necessary and sufficient conditions for traditional extensive measurement. It may be hoped that this and other generalizations of Hölder's theorem will be of interest in a wide variety of areas in both the physical and the behavioral sciences.

The major existing generalizations go in two directions. One major direction which has been studied is the case where $\circ$ is an abelian (commutative) semigroup operation (inverses may not exist)[5]. Other generalizations involving $\circ$ include cases when it is not closed, i.e., $a \circ b$ does not necessarily exist for $a, b \in \mathscr{S}$[6], and when it is not commutative or associative[7]. The only changes we make in this direction are to assume just *weak commutativity* $(a \circ b \sim b \circ a)$ and *weak associativity* $(a \circ (b \circ c) \sim (a \circ b) \circ c)$. $(x \sim y$ means $x \preccurlyeq y$ and $y \preccurlyeq x.)$

Another line of generalization involves less restrictive assumptions about $\preccurlyeq$. If $\preccurlyeq$ is taken to be a weak order (reflexive, transitive, and con-

Synthese **18** (1968) 311–326. © *D. Reidel Publishing Co., Dordrecht - Holland*

nected), the situation is not much different from a simple order except that if $a \sim b$, then a and b map into the same number. Retaining connectedness but weakening transitivity – in particular, so that $\sim$ is no longer transitive – results in considerable complications. Only the case where $\preccurlyeq$ is a semiorder has been analyzed in any detail.[8]

An alternative is to retain reflexivity and transitivity and to weaken connectedness; the resulting relations are called quasi-orders. In his excellent book on systems of the type $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$, Fuchs[9] includes a number of results about partially ordered groups and semigroups when the partial order satisfies the requirement that it can be embedded in a simple order on the same group or semigroup. These results easily generalize to quasi-orders. The restrictions imposed on $\preccurlyeq$ seem to be suggested by pure mathematical curiosity, not by any empirical problem. By contrast, Giles' axiomatic treatment of quasi-ordered semigroups is conspicuously motivated by physical – thermodynamical – considerations. It is on this approach that we base our work. Giles' avowed goal is to formulate axioms for $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ such that a natural quantitative representation of thermodynamics arises as the representation when $\mathscr{S}$ is interpreted as the possible states of a class of isolated physical systems, $a \preccurlyeq b$ means that it is physically possible for a system in state a to pass in time into state b, and $a \circ b$ denotes[10] the state of the system that obtains when one considers the noninteracting union of systems in states a and b. His axioms, therefore, are motivated by physical considerations. Perhaps the single most interesting one in his system is the weak form of connectedness, namely: if one state can pass into each of two others, then at least one of these two can pass into the other. Formally, if $a \preccurlyeq b$ and $a \preccurlyeq c$, then either $b \preccurlyeq c$ or $c \preccurlyeq b$. Giles credits Buchdahl and Falk and Jung[11] with first recognizing the importance of this property in the thermodynamic context. As it strikes us as an important generalization of connectedness – one which may prove of interest in a variety of fields –, we suggest that it be given a name and adopt the term *conditional connectedness*.

To gain some feel for its meaning, let us examine the type of representation suggested by thermodynamic theory and (partially) axiomatized by Giles. In classical thermodynamic theory, there are assigned to each state of a system certain parameters such as volume, internal energy, and the number of molecules of each of several chemically pure components; the values of these parameters are preserved under state transition.[12]

Giles calls these parameters *components of content*. The motivation is, in a given thermodynamic situation, to find sufficiently many of these components of content so that state transition between states a and b can occur if and only if a and b have the same value on all components of content. Then, one more parameter is needed to describe whether the transition a to b or the transition b to a is the naturally occurring one. This is the entropy S, which by convention never decreases. Thus, the idea is that a system can pass from one state to another if and only if the components of content are all conserved and the entropy does not decrease. Formally, one would like to conclude that for some integer $n \geq 1$, there are real-valued functions of state $Q_1, Q_2, \ldots, Q_n, S$, which are nonnegative and additive (extensive) in the sense that for all $a, b \in \mathcal{S}$,

$$
\begin{aligned}
&Q_i(a) \geq 0, \quad i = 1, 2, \ldots, n \\
&Q_i(a \circ b) = Q_i(a) + Q_i(b), \quad i = 1, 2, \ldots, n \\
&S(a) \geq 0 \\
&S(a \circ b) = S(a) + S(b),
\end{aligned}
$$

and moreover satisfy the following condition: $a \preccurlyeq b$ if and only if $Q_i(a) = Q_i(b)$, $i = 1, 2, \ldots, n$, and $S(a) \leq S(b)$.

Where there is only one component of content (a scientifically important example is given in Section 3), the representation is easily graphed as in Figure 1. Transitions can occur only from left to right (increasing entropy) on a horizontal line (conservation of the component of content). The right-hand boundary – the states that can only pass into themselves – is the locus of equilibrium states; its properties are a major focus of interest in thermodynamic theory. The main feature of the conditional connectedness property is clear from the figure: it partitions $\mathcal{S}$ into subclasses (the horizontal lines in this special case) of states having identical values on all components of content, and each subclass is weakly ordered by entropy. In the special case of a weak order (connected quasi-order) there is just one subclass.

Among the several desirable properties listed for S and Q_i, perhaps the most innocent seeming is that they be nonnegative. In fact, it is the source of considerable trouble. Giles argues at length that the functions must be nonnegative (or nonpositive) in order that they be sufficiently well behaved to be physically acceptable. We do not repeat his arguments.

It may help to recall, however, that the non-pathological solutions of the real functional equation $f(x+y)=f(x)+f(y)$, with $x, y \geq 0$, are non-negative and non-positive. Additivity of real-valued functions on a semi-group is, of course, a natural generalization of this equation.

To prove the non-negativeness of Q_i and S and, indeed, to construct S, Giles is forced to invoke two rather complicated axioms.[13] In part, they embody an Archimedean property, which is undoubtedly needed in the construction of the order-preserving S; and in part, they embody

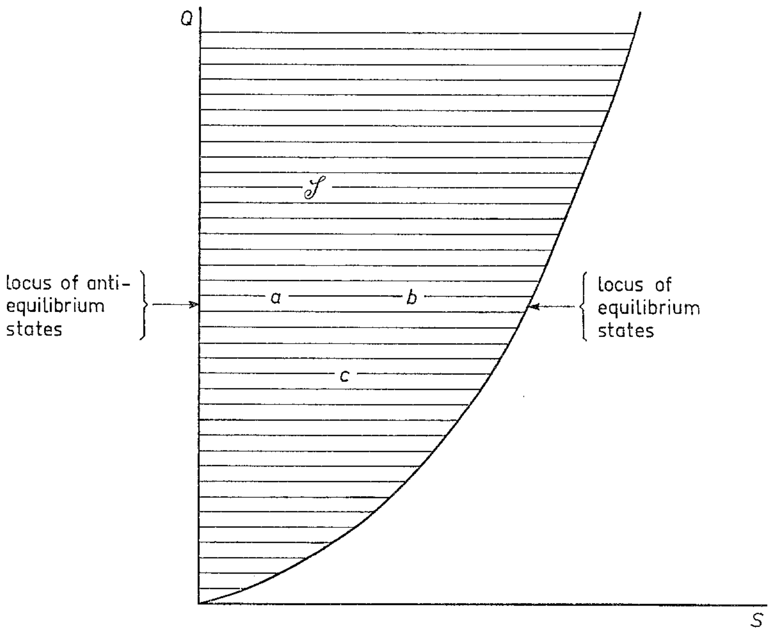

Fig. 1. Representation when there is exactly one component of content and both anti-equilibrium and equilibrium states exist. The shaded area represents $\mathscr{S}$. Notice that $a \prec b$ because $Q(a)=Q(b)$ and $S(a)<S(b)$, and that neither $a \preccurlyeq c$ nor $c \preccurlyeq a$ because $Q(a) \neq Q(c)$.

some topological constraints. There is little doubt that they blemish his system because of their complexity and their obscure physical meaning.

The major purpose of this paper is to begin to attack this blemish. We have succeeded for the entropy, but not at all for the components of content. Specifically, by replacing his two axioms by one that is clearly Archimedean (although different in an interesting way from the usual one for extensive systems), we construct a non-negative S. The proof, which is non-constructive and much simpler than Giles', uses Hölder's theorem and the extension theorem for additive functions over groups that Giles proves and uses to construct the components of content.

There remains, therefore, the apparently difficult task of finding acceptable axioms that lead to non-negative components of content. In addition, as the theory stands, we cannot prove that finitely many components of content $Q_1, Q_2, \ldots, Q_n$ suffice to give our representation. Instead, the best we can do (as does Giles) is to prove that the collection of *all* components of content suffices. (See Theorem 2.) No physicist is likely to accept the theory as an axiomatization of thermodynamics unless this set can be taken as finite.

We have two secondary purposes. First, as we remarked earlier, we show that our theory for thermodynamics is a natural generalization of extensive measurement and, in fact, provides one of the most satisfactory formulations of extensive measurement in the literature. The proof here is constructive. All of the axioms are necessary given that the usual representation holds; the major novelty is in the Archimedean axiom. Second, by writing in this journal, we hope to bring these ideas on how to represent non-connected relations to those, especially behavioral scientists, who do not customarily browse in the thermodynamic literature.

We begin in the next section by discussing the axioms we shall study. Then, we show that statistical entropy gives rise to a system satisfying the axioms. Finally, we go on in Sections 4 and 5 to state the main representation theorems and in Sections 6 and 7 to present the proofs.

II. AXIOMS

We begin by listing and, when necessary, discussing the several axioms we shall study. Throughout, we are dealing with the system $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$; the set $\mathscr{S}$ is assumed to have at least two elements, and the operation $\circ$ is assumed closed. We should emphasize that closure is a substantive assumption.

AXIOM 1. *(Quasi-ordering.) The relation $\preccurlyeq$ on $\mathscr{S}$ is reflexive and transitive.*

AXIOM 2. *(Conditional connectedness.) For all $a, b, c \in \mathscr{S}$, if $a \precsim b$ and $a \precsim c$, then either $b \precsim c$ or $c \precsim b$.*

AXIOM 3. *(Abelian semigroup.) The operation $\circ$ on $\mathscr{S}$ is weakly associative and weakly commutative.*

AXIOM 4. *(Monotonicity.) For all $a, b, c \in \mathscr{S}$, $a \precsim b$ if and only if $a \circ c \precsim b \circ c$.*

These four axioms[14] define what may be called a *conditionally connected, ordered semigroup.*

As usual in systems having a closed operation, for any $a \in \mathscr{S}$ and any positive integer n, we define na inductively: $1a = a$; $na = [(n-1)\, a] \circ a$.

AXIOM 5. *For all $a, b \in \mathscr{S}$, if n is a positive integer and $na \precsim nb$, then $a \precsim b$.*

This axiom is similar to, but slightly simpler than, a property listed by Giles[15] and, inappropriately, called Archimedean. It is easy to derive the converse of Axiom 5 from Axioms 1, 3, and 4, but 5 itself does not follow from 1–4 as the following example shows. Let $\mathscr{S}$ be the integers, $\circ$ be $+$, and let $a \precsim b$ if and only if $a - b$ is even. It is routine to verify Axioms 1–4. Axiom 5 is false since $2 \precsim 4$ because $-2 = 2 - 4$ is even, whereas it is not true that $1 \precsim 2$ because $-1 = 1 - 2$ is odd.

AXIOM 6. *(Archimedean.) For all $a, b, c, d \in \mathscr{S}$, if $a \precsim b$ and not $b \precsim a$ and if either $c \precsim d$ or $d \precsim c$, then there exists a positive integer n such that $na \circ c \precsim nb \circ d$.*

This axiom, which generalizes the Archimedean axioms of simpler measurement systems, is apparently new. It turns out to be exactly the usual Archimedean condition on a subgroup of $\mathscr{S} \times \mathscr{S}$ after a certain equivalence relation is cancelled out. We verify that it follows from our desired representation by Archimedean properties of the real numbers. From the desired representation, for any positive integer n, at least one of $na \circ c \precsim nb \circ d$ and $nb \circ d \precsim na \circ c$ holds. For, if Q is any component of content, then the hypotheses imply that $Q(a) = Q(b)$ and $Q(c) = Q(d)$. Thus, $Q(na \circ c) = nQ(a) + Q(c) = nQ(b) + Q(d) = Q(nb \circ d)$. Now, in terms of the desired representation, $S(a) < S(b)$. Thus, no matter how much larger $S(c)$ is than $S(d)$, by the Archimedean properties of the real numbers, there is an integer n such that $n[S(b) - S(a)] \geq S(c) - S(d)$. Rewriting and using the additivity of S, $S(na \circ c) = nS(a) + S(c) \leq nS(b) + S(d) = S(nb \circ d)$, whence the assertion of the axiom.

To formulate the last two axioms, we define a special class $\mathscr{S}_0$ of states, called *anti-equilibrium states* by Giles.

$$\mathscr{S}_0 = \{x \mid x \in \mathscr{S} \text{ and, for all } a \in \mathscr{S}, \text{ if } a \preccurlyeq x, \text{ then } x \preccurlyeq a\}.$$

These are the elements of $\mathscr{S}$ minimal under $\preccurlyeq$. Axiom 7 says that the concatenation of two minimal elements is again minimal, and Axiom 8 says that each state can arise from some minimal state. These two properties were first formulated by Giles.[16] Formally:

AXIOM 7. *(Closure of $\mathscr{S}_0$.) For all $x, y \in \mathscr{S}$, if $x, y \in \mathscr{S}_0$, then $x \circ y \in \mathscr{S}_0$.*
AXIOM 8. *For all $a \in \mathscr{S}$, there exists $x \in \mathscr{S}_0$ such that $x \preccurlyeq a$.*

III. AN EXAMPLE: STATISTICAL ENTROPY

Before going on to the statement of our theorem, we give an example which satisfies the axioms. In statistical mechanics and in Shannon's theory of information[17], the state of a system is described by a probability distribution over a finite set. It is convenient to think of a state p as the equivalence class of vectors obtained by permutations from a vector $(p_1, \ldots, p_m)$, where, for $i = 1, \ldots, m$, $p_i \geq 0$ and $\sum p_i = 1$. Let $N(p)$ denote the number of components of any of the vectors in p. If p and q are states, let $p \circ q$ be the equivalence class of vectors of the form

$$(p_1 q_1, \ldots, p_1 q_n, p_2 q_1, \ldots, p_{m-1} q_n, p_m q_1, \ldots, p_m q_n).$$

Observe that $N(p \circ q) = N(p)\, N(q)$. Let $\mathscr{S}_0$ denote the set of equivalence classes of vectors of the form $(1, 0, \ldots, 0)$.

Suppose that H is any real-valued function defined over the set $\mathscr{S}$ of these equivalence classes for which

(i) $H(p) \geq 0$, and $H(p) = 0$ if and only if $p \in \mathscr{S}_0$;
(ii) H is continuous;
(iii) $H(p \circ q) = H(p) + H(q)$.

For such a function, define

$$p \preccurlyeq q \quad \text{if and only if } N(p) = N(q) \quad \text{and} \quad H(p) \leq H(q).$$

Axioms 1–8 are easily verified for this $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$.

It is well known that Shannon's information measure, i.e., the entropy of statistical mechanics

$$H_1(p) = -\sum_{i=1}^{m} p_i \log_2 \frac{1}{p_i},$$

satisfies requirements (i)–(iii), and so it is consistent with the axioms. It does not follow, however, that a system in which $\mathscr{S}$ and $\circ$ are defined as above and for which there is a relation $\preccurlyeq$ that satisfies Axioms 1–8 necessarily leads to the H_1 measure. As Rényi[18] points out, the functions

$$H_\alpha(p) = \frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{m} p_i^\alpha \right), \quad \alpha \neq 1$$

also have the same three properties. (Note that $\lim_{\alpha \to 1} H_\alpha(p) = H_1(p)$.) This means that Axioms 1–8 do not uniquely characterize statistical entropy, although they do include it among the possibilities. A complete physical theory needs further axioms in order to be categorical.[19]

IV. STATEMENT OF THE PRINCIPAL RESULT

In this section, we state the principal result, leaving proofs for Sections 6 and 7. To describe the numerical representation, we formulate precisely what is meant by a component of content.

DEFINITION 1. *Suppose that* $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ *satisfies* Axioms 1, 3 and 4. *A real valued function* Q *on* $\mathscr{S}$ *is called a component of content if, for all* $a, b \in \mathscr{S}$,

(i) $Q(a \circ b) = Q(a) + Q(b)$,
(ii) if $a \preccurlyeq b$, then $Q(a) = Q(b)$.

It is called non-trivial if $Q \not\equiv 0$.

For completeness, we state without proof a major result of Giles.[20] Let us say that a and b in $\mathscr{S}$ are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.

THEOREM 1 (Giles). *Suppose that* $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ *satisfies Axioms* 1–5. *If* $\preccurlyeq$ *is connected, then there is no non-trivial component of content. If* $\preccurlyeq$ *is not connected, then for each incomparable pair* $a, b \in \mathscr{S}$ *there exists a component of content* Q *for which* $Q(a) \neq Q(b)$.

Thus, there exists a class $\mathscr{Q}$ of non-trivial components of content such that a and b are comparable if and only if $Q(a)=Q(b)$ for all $Q\in\mathscr{Q}$. As we mentioned earlier, it is important to discover acceptable axioms such that $\mathscr{Q}$ is finite and each $Q\in\mathscr{Q}$ is non-negative.

Our main contribution concerns the entropy function, and it is formulated as:

THEOREM 2. *Suppose that* $\langle\mathscr{S},\preccurlyeq,\circ\rangle$ *satisfies Axioms* 1–4 *and* 6. *Then there exists a real-valued function S on* $\mathscr{S}$ *such that, for all* $a, b\in\mathscr{S}$,

(i) $S(a\circ b)=S(a)+S(b)$,

(ii) *if a and b are comparable, then* $a\preccurlyeq b$ *if and only if* $S(a)\leq S(b)$.

If S' *is another function satisfying* (i) *and* (ii), *then for some* $\rho>0$ *and some component of content* Q, $S=\rho S'+Q$. *If, in addition, Axioms* 7 *and* 8 *hold, then S can be chosen to satisfy not only* (i) *and* (ii), *but also*

(iii) $S\geq 0$,

(iv) $S(x)=0$ *if and only if* $x\in\mathscr{S}_0$.

And if S' *now also satisfies* (i)–(iv), *then for some* $\rho>0$, $S=\rho S'$.

COROLLARY. *Suppose that* $\langle\mathscr{S},\preccurlyeq,\circ\rangle$ *satisfies Axioms* 1–6. *Then* (ii) *of the Theorem can be written*:

(ii′) $a\preccurlyeq b$ *if and only if* $S(a)\leq S(b)$ *and, for every component of content* Q, $Q(a)=Q(b)$.

Conversely, if there is an S satisfying (i) *and* (ii′), *then* $\langle\mathscr{S},\preccurlyeq,\circ\rangle$ *satisfies Axioms* 1–6.

V. EXTENSIVE MEASUREMENT

Before going on to the proofs, we note that with appropriate specialization, Theorem 2 provides an interesting axiomatization for extensive measurement. If $\preccurlyeq$ is connected, as in the usual setting for extensive measurement, part (ii) of Theorem 2 can be simplified by dropping the condition 'if a and b are comparable'. Furthermore, the statements of Axioms 1–4 and 6 can be simplified. This suggests:

DEFINITION 2. $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ *is an* extensive system *if*

(i) $\preccurlyeq$ *is a weak ordering of* $\mathscr{S}$;
(ii) *Axiom* 3 *holds;*
(iii) *Axiom* 4 *holds;*

and

(iv) *for all* $a, b \in \mathscr{S}$, *if not* $b \preccurlyeq a$, *then for every* $c, d \in \mathscr{S}$ *there exists a positive integer* n *such that* $na \circ c \preccurlyeq nb \circ d$.

The system is called positive *if, in addition,*

(v) *for all* $a, b \in \mathscr{S}$, *not* $a \circ b \preccurlyeq a$.

THEOREM 3. $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ *is an extensive system* (Definition 2) *if and only if there is a real-valued function* S *on* $\mathscr{S}$ *such that, for all* $a, b \in \mathscr{S}$,

(i) $S(a \circ b) = S(a) + S(b)$,
(ii) $a \preccurlyeq b$ *if and only if* $S(a) \leq S(b)$.

If S' *is another function satisfying* (i) *and* (ii), *then for some* $\rho > 0$, $S = \rho S'$. *Such a system is positive if and only if, for all* $a \in \mathscr{S}$, $S(a) > 0$.

A routine verification proves the 'if' statement, and the 'only if' follows almost immediately from Theorem 2. Uniqueness also follows from the uniqueness part of Theorem 2 and the first part of Theorem 1.

A careful analysis of the proof of Theorem 2, as we shall note in Section 7, shows that the only non-constructive step can be avoided if $\preccurlyeq$ is connected. Thus, our proof of Theorem 3 is entirely constructive.

The only other known necessary and sufficient conditions for extensive measurement are due to Alimov and Holman [21]. Other axiom systems include a simpler Archimedean axiom and some form of solvability (Fuchs calls such solvable semigroups 'naturally ordered', but that seems quite a misleading term). Typical of an Archimedean axiom in a positive system, i.e., a system satisfying (i), (ii), (iii) and (v) of Definition 2, is: if not $b \preccurlyeq a$, i.e., $a \prec b$, then there exists an integer n such that $b \prec na$. Typical of a (weak) solvability axiom in a positive system is: if $a \prec b$, then there exists $c \in \mathscr{S}$ such that $a \circ c \preccurlyeq b$.

Since, evidently, our more complicated Archimedean axiom substitutes for these two assumptions, it may not be amiss to show directly that it follows from them using the axioms of a positive system and also to show that the usual Archimedean axiom follows in a positive system from our axiom. First, suppose that $a \prec b$ and that $c, d \in \mathscr{S}$. By solvability, there exists $e \in \mathscr{S}$ such that $a \circ e \preccurlyeq b$. By positiveness, $c \prec d \circ c$. By the usual Archimedean axiom, there is a positive integer n such that $c \prec ne$. So, using the other axioms freely, $na \circ c \prec na \circ d \circ c \prec na \circ d \circ ne \preccurlyeq nb \circ d$, which proves our Archimedean axiom. Second, suppose that $a \prec b$ in a positive extensive system. By positiveness, $a \prec 2a$, whence by our Archimedean axiom there is a positive integer n such that $na \circ b \preccurlyeq n(2a) \circ a \sim na \circ (n+1)\, a$. So, by Axiom 4 and positiveness, $b \prec (n+2)\, a$, which proves the usual Archimedean axiom.

VI. PRELIMINARY LEMMAS

To prove Theorem 2, we shall need several lemmas. Proofs of Lemmas 1, 3 and 4 are omitted because they involve only routine use of the axioms and are for the most part proved by Giles.

LEMMA 1. *Suppose that Axioms* 1–4 *hold for* $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$. *If* $a, b, c \in \mathscr{S}$, $b \preccurlyeq a$ *and* $c \preccurlyeq a$, *then either* $b \preccurlyeq c$ *or* $c \preccurlyeq b$.

LEMMA 2. *If Axioms* 1–4 *hold and* $x \preccurlyeq a$, $y \preccurlyeq a$ *for* $x, y \in \mathscr{S}_0$, *then* $x \sim y$.

PROOF: Lemma 1 and definition of $\mathscr{S}_0$. Q.E.D.

We define the relation $\approx$ on $\mathscr{S} \times \mathscr{S}$ as follows: for $a, b, c, d \in \mathscr{S}$, $(a, b) \approx (c, d)$ if and only if $a \circ d \sim b \circ c$. The definition of $\approx$ corresponds to that of the equivalence relation used in passing from the integers to the rational numbers.

LEMMA 3. *Suppose that Axioms* 1–4 *hold. The relation* $\approx$ *on* $\mathscr{S} \times \mathscr{S}$ *is an equivalence relation.*

Denote by $(a, b)^*$ the equivalence class under $\approx$ that contains (a, b), and define

$$\mathscr{P} = (\mathscr{S} \times \mathscr{S}) / \approx$$
$$\mathscr{P}_P = \{(a, b)^* \mid a \preccurlyeq b \text{ or } b \preccurlyeq a\}$$
$$\mathscr{P}_N = \{(a, b)^* \mid a \preccurlyeq b\}.$$

Giles refers to these as, respectively, the sets of formal, possible, and natural processes. Following Giles, we may define on $\mathscr{P}$ an operation, identity element, inverse, and ordering as follows: for all $\alpha=(a, b)^*$, $\beta=(c, d)^* \in \mathscr{P}$,

$$\begin{aligned} \alpha \circ \beta &= (a \circ c, b \circ d)^* \\ 0 &= (x, x)^*, \text{ any } x \\ \alpha^{-1} &= (b, a)^* \\ \alpha \preccurlyeq \beta & \quad \text{if and only if } \alpha \circ \beta^{-1} \in \mathscr{P}_N. \end{aligned}$$

LEMMA 4. *Suppose that Axioms* 1–4 *hold.*

(i) *The operation* $\circ$, *the identity* 0 *and the relation* $\preccurlyeq$ *on* $\mathscr{P}$ *are well defined*

(ii) $\langle \mathscr{P}, \circ \rangle$ *and* $\langle \mathscr{P}_p, \circ \rangle$ *are abelian groups.*

(iii) *The relation* $\preccurlyeq$ *on* $\mathscr{P}_p$ *is a simple order.*

Remark: Axiom 2 is crucial in the proof that $\mathscr{P}_p$ is closed under $\circ$. (Cf. Theorem 2.4.2, Giles)

LEMMA 5. *If Axioms* 1–4 *and* 6 *hold, then* $\langle \mathscr{P}_p, \preccurlyeq, \circ \rangle$ *is an Archimedean simply ordered group.*[22]

PROOF. By Lemma 4 it is a simply ordered abelian group. To show that it is Archimedean, suppose that $\alpha, \beta \in \mathscr{P}_p$ and $0 \prec \alpha$, where $\alpha=(a, b)^*$ and $\beta=(c, d)^*$. Since $0 \prec \alpha$, by definition $0 \circ \alpha^{-1}=\alpha^{-1}=(b, a)^* \in \mathscr{P}_N$ and not

$$[\alpha \circ 0^{-1}=\alpha=(a, b)^* \in \mathscr{P}_N].$$

Thus, $b \prec a$. By Axiom 6, there exists a positive integer n such that $nb \circ c \preccurlyeq na \circ d$. By definition,

$$\beta \circ (n\alpha)^{-1}=(c, d)^* \circ (nb, na)^*=(nb \circ c, na \circ d)^* \in \mathscr{P}_N,$$

whence $\beta \preccurlyeq n\alpha$. Since $0 \prec \alpha$, $\beta \prec (n+1)\,\alpha$. Q.E.D.

The following useful result can be found on page 221 of Giles' book; its proof depends on the axiom of choice in much the same way as that of the Hahn-Banach theorem.

LEMMA 6 (Giles). *If* $\mathscr{K}$ *is a subgroup of an abelian group* $\mathscr{J}$ *and I is a real-*

valued, additive function over $\mathscr{K}$, then there exists a real-valued additive function I^* over $\mathscr{J}$ that coincides with I over $\mathscr{K}$.

VII. PROOF OF THEOREM 2 AND COROLLARY

By Lemma 5 and Hölder's theorem, there exists a real-valued function I on $\mathscr{P}_p$ such that, for all $\alpha, \beta \in \mathscr{P}_p$,

(a) $\quad I(\alpha \circ \beta) = I(\alpha) + I(\beta)$,

(b) $\quad I(\alpha) \geq 0$ if and only if $\alpha \in \mathscr{P}_N$;

and if I' is another function satisfying (a) and (b), then for some $\rho > 0$, $I = \rho I'$. By Lemmas 4 and 6, there is an additive extension I^* of I to $\mathscr{P}$. (Note that if $\preccurlyeq$ is connected, as in extensive measurement, $\mathscr{P}_p = \mathscr{P}$, and so this step, the only non-constructive step in the proof, is not needed.) Define S on $\mathscr{S}$ by: for all $a \in \mathscr{S}$, $S(a) = I^*(a, 2a)^*$. We verify that S has the two properties listed in Theorem 2.

(i)
$$\begin{aligned} S(a \circ b) &= I^*(a \circ b, 2(a \circ b))^* \\ &= I^*(a, 2a)^* + I^*(b, 2b)^* \\ &= S(a) + S(b). \end{aligned}$$

(ii) If a and b are comparable, then $(a, b)^* \in \mathscr{P}_p$,

and so $I^*(a, b)^* = I(a, b)^*$. Taking into account that $(a \circ b, a \circ b)^* = 0$, we have

$$\begin{aligned} S(b) - S(a) &= I^*(b, 2b)^* - I^*(a, 2a)^* \\ &= I^*(b, 2b)^* + I^*(2a, a)^* \\ &= I^*(b \circ 2a, a \circ 2b)^* \\ &= I^*(a \circ b, a \circ b)^* + I^*(a, b)^* \\ &= I^*(a, b)^* \\ &= I(a, b)^* \\ &\geq 0 \quad \text{if and only if } a \preccurlyeq b, \end{aligned}$$

where we used property (b) of I in the last step.

To establish uniqueness, observe that if S is given then we may define I on $\mathscr{P}_p$ by: $I(a, b)^* = S(b) - S(a)$. I is well defined since if $(a, b) \approx (c, d)$, then $a \circ d \sim b \circ c$ and so, by the additivity of S, $S(b) - S(a) = S(d) - S(c)$. Moreover, it is easily verified that I satisfies properties (a) and (b) at the beginning of the proof. If S' is another such function, then it similarly defines I', and by the uniqueness part of Hölder's theorem, $I=\rho I'$ for some $\rho>0$. If $Q=S-\rho S'$, it is trivial to show that Q is a component of content.

Suppose, next, that Axioms 7 and 8 hold. For $a\in\mathscr{S}$, denote by x_a some element of $\mathscr{S}_0$ for which $x_a\preccurlyeq a$. At least one exists by Axiom 8. Let S be any function satisfying properties (i) and (ii) of the theorem. Define S^* on $\mathscr{S}$ as follows: $S^*(a)=S(a)-S(x_a)$. By Lemma 2, S^* remains the same regardless of choice of x_a. Clearly, S^* satisfies (iii) and (iv). It suffices, therefore, to show that S^* satisfies (i) and (ii).

(i) By Axioms 1 and 4, $x_a\circ x_b\preccurlyeq a\circ b$, and by Axiom 7, $x_a\circ x_b\in\mathscr{S}_0$. Thus, $x_{a\circ b}\sim x_a\circ x_b$, and so

$$\begin{aligned}S^*(a\circ b) &= S(a\circ b)-S(x_{a\circ b})\\ &= S(a\circ b)-S(x_a\circ x_b)\\ &= S(a)+S(b)-S(x_a)-S(x_b)\\ &= S^*(a)+S^*(b).\end{aligned}$$

(ii) Suppose a and b are comparable. By Axiom 1 and Lemma 2, $x_a\sim x_b$, and so $S^*(a)\leq S^*(b)$ is equivalent to $S(a)\leq S(b)$ which, in turn, is equivalent to $a\preccurlyeq b$.

Suppose S' also satisfies (i)–(iv). We already know that $S^*=\rho S'+Q$, where $\rho>0$ and Q is a component of content. Since $S^*(x)=S'(x)=0$ for $x\in\mathscr{S}_0$, $Q(x)=0$. But $x_a\preccurlyeq a$ implies, by definition of Q, that $Q(a)=Q(x_a)=0$. Thus $Q\equiv 0$ and that completes the proof of Theorem 2.

The first part of the corollary follows immediately from Theorems 1 and 2. Finally, the necessity of Axioms 1–5 is obvious and that of Axiom 6 is proved in Section 2.

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[1] Roberts began work on this problem when he was a postdoctoral fellow, NIH training grant in mathematical psychology, University of Pennsylvania, and completed it at The RAND Corporation.
[2] Luce began work on it at the Center for Advanced Study in the Behavioral Sciences, Stanford, California, and he was supported in part by NSF grant GB-6536 to the University of Pennsylvania.
[3] Hölder [9]; for a modern treatment see Fuchs [7], p. 45.
[4] Giles [8].
[5] Hölder [9]; Suppes [16].
[6] Behrend [3]; Luce and Marley [12].
[7] Pfanzagl [13].
[8] Krantz [11].
[9] Fuchs [7].
[10] Giles writes $a \to b$ for $a \preccurlyeq b$ and $a + b$ for $a \circ b$.
[11] Buchdahl [4]; Falk and Jung [6].
[12] Callen [5].

[13] Axioms 7.1.2, p. 62, and 7.2.1, p. 66, and restated as Axioms A.3.6, p. 196, and A.4.3, p. 200.
[14] Axioms 1–4 correspond to and are slightly weaker than Axioms 2.1.1 and 2.1.2, p. 30–31, which are restated as Axioms A.2.1 and A.2.2, p. 193, in Giles [8].
[15] Axiom 2.1.3, p. 31, and Theorem A.4.4, p. 200, Giles [8].
[16] Axiom 7.2.3, p. 69, and restated as Axiom A.9.4, p. 212, Giles [8].
[17] Shannon and Weaver [15].
[18] Rényi [14].
[19] Rényi [14] (also see Aczél [1], p. 153, and the references given there) studies additional properties for H that uniquely characterize H_1. He enlarges the space of distributions to include all those with $0 < \Sigma p_i \leq 1$ and he introduces a further axiom concerning a composition $p \cup q$ given by $(p_1, ..., p_m, q_1, ..., q_n)$ when this vector is a distribution. As these ideas do not seem to have any natural counterpart in our axiomatic system, we do not pursue them further.
[20] Theorem 3.2.1, p. 39, Giles [8]. The only major modification required in the proof is to use Axiom 5 when he uses Theorem 2.4.4.
[21] Alimov [2], Holman [10]. Alimov formulates the Archimedean property as the assertion that there are no anomalous pairs, where $a, b \in \mathscr{S}$ is *anomalous* if not $(a \sim b)$ and either

$$na \prec (n+1)b \text{ and } nb \prec (n+1)a \text{ for all positive integers } n,$$

or

$$(n+1)b \prec na \text{ and } (n+1)a \prec nb \text{ for all positive integers } n.$$

(Note: $x \prec y$ means $x \preccurlyeq y$ and not $x \sim y$.) By using Alimov's results, it is not hard to obtain a strengthening of Theorem 3, namely by showing that the weak commutativity assumption is not needed, i.e. that (ii) in the definition of extensive system can be modified to read

(ii)′ $\langle \mathscr{S}, \preccurlyeq, \circ \rangle$ is weakly associative.

For, one can prove from Axioms (i), (ii)′, (iii) and (iv) that the system has no anomalous pairs. Weak commutativity follows readily. (Cf. the argument on p. 168, Fuchs [7].) Thus (ii) is satisfied and so the conclusion of Theorem 3 follows. The reader will recall that Hölder's theorem also does not require a commutativity assumption.
[22] For the appropriate definition, see Fuchs [7], p. 12; note his fn. 8.

EDITOR'S NOTE: The authors use in their MS the symbol $\precsim$ instead of the symbol $\preccurlyeq$ employed above. We regret very much that our printer does not have the right symbol available at the present moment, and that our schedule does not permit us to wait for its being ordered. We hope that no confusion will result from the usage that has been adopted in the paper.