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ON THE NUMERICAL REPRESENTATION OF QUALITATIVE CONDITIONAL PROBABILITY[1]

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1. Introduction. Let A, B, C, and D be events from some algebra $\mathcal{E}$, and let $A \mid B \succsim C \mid D$ signify that "the occurrence of event A conditional on B having occurred is judged as qualitatively at least as probable as the occurrence of C conditional on D having occurred." Were the rest of the ordering uniquely definable in terms of its unconditional part (obtained by restricting B and D to the universal event X), just as conditional probability is uniquely defined in terms of unconditional probability, then we could simply attend to unconditional orderings. That no such definition can be given is shown by the following example of two probability measures that induce the same unconditional ordering but different conditional ones:

	$\{1, 2, 3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1\}$	$\{2, 3\}$	$\{2\}$	$\{3\}$
P	9/9	8/9	6/9	5/9	4/9	3/9	1/9
P^*	9/9	8/9	7/9	6/9	3/9	2/9	1/9

$$P(\{3\} \mid \{2, 3\}) = \tfrac{1}{4} < \tfrac{3}{8} = P(\{2\} \mid \{1, 2\})$$
$$P^*(\{3\} \mid \{2, 3\}) = \tfrac{1}{3} > \tfrac{2}{8} = P^*(\{2\} \mid \{1, 2\}).$$

Thus, one is led to ask: what properties of the events and of $\succsim$ are necessary and/or sufficient for there to be a unique, finitely additive probability measure P that is order preserving in the following sense:

$$A \mid B \succsim C \mid D \quad \text{if and only if} \quad P(A \cap B)/P(B) \geqq P(C \cap D)/P(D)?$$

The question is surely of logical interest for a personalistic theory of probability, especially since the most easily elicited judgments of qualitative probability are conditional ones: is red on a roulette wheel (given that the wheel is spun) more likely than heads on a coin (given that the coin is flipped)? To formulate such choices in unconditional form demands some artifice.

The only published work on this problem of which I am aware is Koopman [4], [5], and [6]. The related work of Copeland [1], Császár [2] and Rényi [10] is also of interest. Those who have examined Koopman's axioms seem agreed that they are somewhat awkward, that several are not as intuitively compelling as one would like, and that their relation to those for qualitative (unconditional) probability (de Finetti [3], Luce [8], and Savage [11]) is not transparent. Although, to a considerable degree, these are really criticisms of his exposition,

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nonetheless, the system given here seems somewhat more satisfactory on all three counts. A detailed comparison of the two systems is given in Section 2.

More striking is the difference in the proofs. As was done for unconditional probability in [8], the problem is reduced to one in the theory of extensive measurement. The relevant results, which are proved in Luce and Marley [9], are summarized in [8], to which the reader is assumed to have access. Matters are, however, considerably more complicated for conditional than for unconditional probability because we must contend both with the additivity of P and with the division structure of the representation. The axioms are sufficiently strong so that an unconditional probability P can be constructed from the unconditional qualitative probability on $\mathcal{E}$. The main task then is to show that the remainder of $\succsim$ is compatible with the numerical conditional probability that is induced by P.

2. Axioms and representation theorem for conditional probability. In formulating the axioms, we begin with an algebra $\mathcal{E}$ of subsets of a given set X (i.e., $\mathcal{E}$ is closed under complementation and union), a subset $\mathfrak{N}$ of $\mathcal{E}$ that identifies the null events (i.e., those that will have probability 0), and a relation $\succsim$ on $\mathcal{E} \times (\mathcal{E} - \mathfrak{N})$. (Clearly, the relation cannot be on $\mathcal{E} \times \mathcal{E}$ if we are to avoid conditioning on null events.) As usual, $\succ$ means $\succsim$ but not $\precsim$, and $\sim$ means both $\succsim$ and $\precsim$. It will prove convenient to denote a typical element of $\mathcal{E} \times (\mathcal{E} - \mathfrak{N})$ by $A \mid B$, where $A \in \mathcal{E}$ and $B \in \mathcal{E} - \mathfrak{N}$, and to make the convention that symbols appearing just to the right of $\mid$ are always from $\mathcal{E} - \mathfrak{N}$.

Definition 1. Suppose that X is a non-empty set, $\mathcal{E}$ an algebra of subsets of X, $\mathfrak{N} \subseteq \mathcal{E}$, and $\succsim$ a relation on $\mathcal{E} \times (\mathcal{E} - \mathfrak{N})$. The quadruple $\langle X, \mathcal{E}, \mathfrak{N}, \succsim\rangle$ is called a *system of qualitative conditional probability* if the following six axioms hold:

Axiom 1. *$\succsim$ is a weak ordering of $\mathcal{E} \times (\mathcal{E} - \mathfrak{N})$*, i.e., *it is reflexive, transitive, and connected.*

Axiom 2. *$X \notin \mathfrak{N}$, and $A \in \mathfrak{N}$ if and only if $A \mid X \sim \varnothing \mid X$.*

Axiom 3. *$X \mid X \succsim A \mid B$ and $X \mid X \sim A \mid A$.*

Axiom 4. *$A \mid B \sim A \cap B \mid B$.*

Axiom 5. *Suppose that $A \cap B = A' \cap B' = \varnothing$. If $A \mid C \succsim A' \mid C'$ and $B \mid C \succsim B' \mid C'$, then $A \cup B \mid C \succsim A' \cup B' \mid C'$; moreover, if either hypothesis is $\succ$, then the conclusion is $\succ$.*

Axiom 6. *Suppose that $A \subset B \subset C$ and $A' \subset B' \subset C'$. If either $A \mid B \succsim A' \mid B'$ and $B \mid C \succsim B' \mid C'$ or $A \mid B \succsim B' \mid C'$ and $B \mid C \succsim A' \mid B'$, then $A \mid C \succsim A' \mid C$; moreover, if any of the hypotheses is $\succ$, then the conclusion is $\succ$.*

The system is *Archimedean* if, in addition, the following axiom holds:

Axiom 7. *Any sequence $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$, $A_i \in \mathcal{E} - \mathfrak{N}$, for which $A_i \mid A_{i+1} \sim A_1 \mid A_2 \prec X \mid X$, $i = 1, 2, \cdots$, is finite.*

Given the desired respresentation, it is not difficult to see that all seven axioms are necessary properties. For example, the last one follows from the fact that, since $A_1 \in \mathcal{E} - \mathfrak{N}$, $P(A_n) \leqq 1$, and $A_i \mid A_{i+1} \sim A_1 \mid A_2$,

$$0 < P(A_1) \leqq P(A_1)/P(A_n) = [P(A_1)/P(A_2)][P(A_2)/P(A_3)] \cdots [P(A_{n-1})/P(A_n)] = [P(A_1)/P(A_2)]^{n-1}.$$

Since $A_1 \mid A_2 < X \mid X$, $P(A_1)/P(A_2) < 1$, and so n must be bounded.

The counter-example of Kraft, Pratt, and Seidenberg [7] for unconditional probability leads one to suspect that these seven axioms are insufficient to prove the desired representation; however, no counterexample yet exists. We show that the representation follows when the following non-necessary property is added to the system.

AXIOM 8. *Suppose that $C \subset D$ and $C \varepsilon \mathcal{E} - \mathfrak{N}$. If $A \mid B > C \mid D$, then there exist C' and D' such that $C \subset C'$, $D' \subset D$ and $A \mid B \sim C' \mid D \sim C \mid D'$.*

We say that a system of qualitative conditional probability is *regular* when Axiom 8 holds. It simply states that the events are sufficiently finely graded so that an inequality of the form $A \mid B > C \mid D$ can be converted into an indifference either by augmenting C or by diminishing D. Note that C must be non-null for the diminishing of D to work. Intuitively, this seems closely related to the assumption made by Koopman and others that, for any integer n, some event can be partitioned into n equally probable events, and, as we shall see from the proof, in the presence of the other axioms, regularity almost implies this property.

At first glance, Koopman's [4], [5] system and this one appear rather different, but in fact they have much in common. Perhaps the simplest way to see the relations is to state what in his system corresponds to each of the above axioms.

Axiom 1: He postulates a quasi-order rather than a weak order, but he points out that were he to add the connectivity assumption then his Axioms P and S, which have no counterpart in this system, would be theorems.

Axiom 2: Since he assumes that $\varnothing$ is the only null event, he has no need for this axiom.

Axiom 3: This is given, in a somewhat different and stronger form, in his Axioms V and I.

Axiom 4: Although he uses this property throughout, he treats it simply as a "notational convention" that does not need to be listed explicitly as an axiom.

Axiom 5: This is not stated as an axiom, but is proved as a theorem. Its role is played by his Axiom A, namely that if $A \mid B \succsim C \mid D$, then $\bar{C} \mid D \succsim \bar{A} \mid B$, which we derive as Lemma 7.

Axiom 6: His Axiom C is the same as the first part of our axiom. (In [5] the statement is as given here; in [4] it appears to be more complex since he does not assume $A \subset B \subset C$ and $A' \subset B' \subset C'$.) His Axiom D, which he calls a quasi-converse of C, corresponds to our statement that if any of the hypotheses is $>$, then the conclusion is $>$.

Axiom 7: There is no counterpart to this axiom in his system, which was intentional since he only wanted to prove that $A \mid B \succsim C \mid D$ implies $P(A \cap B)/P(B) \geqq P(C \cap D)/P(D)$ and not the converse.

Axiom 8: As I noted earlier, the closest counterpart is his "assumption" (it is not listed among the axioms) that, for each positive integer n, some non-null event can be partitioned into n equi-probable events, but they are by no means the same assumption.

As was mentioned, he has two further axioms which become theorems when $\succsim$ is connected.

It is clear that his system is not a special case of this one since he does not require $\succsim$ to be a connected relation. From our proof, it will become clear that this system is not a special case of his because two examples, corresponding to probabilities of just 0 and 1 and just 0, $\frac{1}{2}$, and 1, fulfill our axioms but fail to have the property that some non-null event can be partitioned into n equi-probable events.

THEOREM 1. *Suppose that* $\langle X, \mathcal{E}, \mathcal{N}, \succsim \rangle$ *is a regular, Archimedean system of qualitative conditional probability, then there exists a unique real-valued function* P *on* $\mathcal{E}$ *such that*

(i) $\langle X, \mathcal{E}, P \rangle$ *is a finitely additive probability space;*

(ii) $A \in \mathcal{N}$ *if and only if* $P(A) = 0$;

(iii) $A \mid B \succsim C \mid D$ *if and only if* $P(A \cap B)/P(B) \geqq P(C \cap D)/P(D)$;

(iv) *the range of* P *is either* $\{0, 1\}$, $\{0, \frac{1}{2}, 1\}$, *or includes all rationals in* $[0, 1]$.

COROLLARY. *The following properties are equivalent*: $A \mid B \sim A \mid X$, $B \mid A \sim B \mid X$, $A \mid B \sim A \mid \bar{B}$, $B \mid A \sim B \mid \bar{A}$, $P(A \cap B) = P(A)P(B)$.

Obviously, the corollary formulates the concept that A and B are independent events.

As our proof of Theorem 1 depends upon a concept and a theorem (from the theory of extensive measurement) which, apparently, have not been published previously, the next section is devoted to this preliminary work. These results are of interest beyond the fact they are used to prove Theorem 1.

3. Systems of positive differences. Perhaps the simplest interpretation to motivate the following system of axioms is the comparison of lengths of intervals on a line when the intervals are identified by their end points. Let $\mathcal{A}$ denote a set of points lying on a straight line. Each pair of points $a, b \in \mathcal{A}$ identify an interval, which we may denote either as ab or ba. If, however, we view the points as ordered, then by convention one of the two identifications will be positive and the other negative. Let $\mathcal{A}^* \subset \mathcal{A} \times \mathcal{A}$ denote the set of positive intervals. One task of the axiomatization is to capture just what we mean by "positive." In addition, we suppose that $\succsim$ is an ordering of $\mathcal{A}^*$ such that $ab \succsim cd$ means that the positive interval ab is at least as long as the positive interval cd. A very natural notion of concatenation exists for "adjacent" intervals. Suppose that ab and bc are both positive, then it is clear that ac should be treated as positive and that $ac = ab \circ bc$. Slightly generalized, this is the definition of concatenation.

DEFINITION 2. Let $\mathcal{A}$ be a non-empty set, $\mathcal{A}^*$ a non-empty subset of $\mathcal{A} \times \mathcal{A}$, and $\succsim$ a binary relation on $\mathcal{A}^*$. The triple $\langle \mathcal{A}, \mathcal{A}^*, \succsim \rangle$ is called a *system of positive differences* if, for all $a, b, c, d, a', b', c', d' \in \mathcal{A}$ the following six axioms hold:

AXIOM D1. $\succsim$ *is a weak ordering of* $\mathcal{A}^*$.

AXIOM D2. *If* $ab, a'b', bc, b'c' \in \mathcal{A}^*$, $ab \succsim a'b'$, *and* $bc \succsim b'c'$, *then* $ac, a'c' \in \mathcal{A}^*$ *and* $ac \succsim a'c'$.

AXIOM D3. *If* $ab, bc \in \mathcal{A}^*$, *then* $ac \succ ab$ *and* $ac \succ bc$.

AXIOM D4. *If ab, ac, bd, $cd \varepsilon \mathcal{A}^*$ and $ab \sim cd$, then $ac \sim bd$.*

AXIOM D5. *If ab, $cd \varepsilon \mathcal{A}^*$ and $ab \succ cd$, then there exist c', $d' \varepsilon \mathcal{A}$ such that ac', $c'b$, ad', $d'b \varepsilon \mathcal{A}^*$ and $ad' \sim c'b \sim cd$.*

AXIOM D6. *Suppose that ab, $cd \varepsilon \mathcal{A}^*$. If $a_i \varepsilon \mathcal{A}$, $i = 1, 2, \cdots$, are such that $a_{i+1}a_i \varepsilon \mathcal{A}^*$ and $a_{i+1}a_i \sim ab$, then the set $\{n \mid n \varepsilon I$ and $cd \succsim a_n a_1\}$ is finite.*

Keeping in mind the interpretation of a "positive difference" as an interval of length on a straight line, these axioms need but little comment. The first is the usual ordering assumption. The second embodies both the concatenation of adjacent intervals and the preservation of inequalities under that concatenation. The third says that $\mathcal{A}^*$ really does include only positive differences. The fourth is a form of commutativity of concatenation. The fifth is a solvability requirement that imposes a certain density of end points. And the sixth is an Archimedean condition.

Note that if $ab \varepsilon \mathcal{A}^*$, then $ba \not\varepsilon \mathcal{A}^*$. For suppose that $ba \varepsilon \mathcal{A}^*$, then by Axiom D2, $aa \varepsilon \mathcal{A}^*$, which with $ab \varepsilon \mathcal{A}^*$ implies, by Axiom D3, that $ab \succ ab$, violating Axiom D1.

In order to reduce this system to an extensive one, we must define what is to be meant by concatenation. This we do next.

DEFINITION 3. Suppose that $\langle \mathcal{A}, \mathcal{A}^*, \succsim \rangle$ is a system of positive differences, then

$$\mathcal{B} = \{(ab, cd) \mid ab, cd \varepsilon \mathcal{A}^*, \text{ and there exist } a', b', d' \varepsilon \mathcal{A} \text{ such that } a'b', b'c' \varepsilon \mathcal{A}^*, a'b' \sim ab, \text{ and } b'd' \sim cd\}.$$

$\circ$ on $\mathcal{B}$: if $(ab, cd) \varepsilon \mathcal{B}$, then $ab \circ cd \sim a'd'$.

It is evident from the definition of $\mathcal{B}$ and Axioms D1 and D2 that $\circ$ is well defined. Moreover, the definitions of $\mathcal{B}$ and $\circ$ seem plausible since $a'd' = a'b' \circ b'd'$ is beyond question, and if $ab \sim a'b'$ and $cd \sim b'd'$, simple substitution analogous to Axiom D2 suggests our definition, $ab \circ cd \sim a'd'$.

THEOREM 2. *If $\langle \mathcal{A}, \mathcal{A}^*, \succsim \rangle$ is a system of positive differences and if there exist ab, $cd \varepsilon \mathcal{A}^*$ such that $ab \succ cd$, then $\langle \mathcal{A}^*, \mathcal{B}, \succsim, \circ \rangle$ is an extensive system with no maximal element* (Definition 3, [8]).

COROLLARY. *If, in addition to the hypothesis of the theorem, for $a, b \varepsilon \mathcal{A}$, $a \neq b$, either ab or $ba \varepsilon \mathcal{A}^*$, then there exists a real-valued function ψ on $\mathcal{A}$ such that $\varphi(ab) = \psi(a) - \psi(b)$ is an extensive representation of $\langle \mathcal{A}^*, \mathcal{B}, \succsim, \circ \rangle$ and ψ is unique up to a positive linear transformation.*

4. Proof of Theorem 2. Throughout this section we assume that $\langle \mathcal{A}, \mathcal{A}^*, \succsim \rangle$ is a system of positive differences and that there exist ab, $cd \varepsilon \mathcal{A}^*$ for which $ab \succ cd$. Moreover, whenever we write $(ab, cd) \varepsilon \mathcal{B}$ we implicitly assume that ab, $cd \varepsilon \mathcal{A}^*$. Many of the simpler proofs are omitted.

LEMMA 1. *$\mathcal{B}$ is non-empty.*

LEMMA 2. *If ab, $bc \varepsilon \mathcal{A}^*$, then $ac \varepsilon \mathcal{A}^*$, $(ab, bc) \varepsilon \mathcal{B}$, and $ac \sim ab \circ bc$.*

LEMMA 3. *If $(ab, cd) \varepsilon \mathcal{B}$, $ab \sim a'b'$, and $cd \sim c'd'$, then $(a'b', c'd') \varepsilon \mathcal{B}$ and $ab \circ cd \sim a'b' \circ c'd'$.*

LEMMA 4. *If* $(ab, ef) \varepsilon \mathcal{B}$ *and* $ab \gtrsim cd$, *then* $(cd, ef) \varepsilon \mathcal{B}$ *and* $ab \circ ef \gtrsim cd \circ ef$.

PROOF. There exist a', b', f' such that $ab \sim a'b' \gtrsim cd$ and $ef \sim b'f'$. By Lemma 3, we may suppose $a'b' > cd$, and so by Axiom D5 there exists c' such that $c'b' \sim cd$ and $ac' \varepsilon \mathcal{A}^*$. By Lemmas 2 and 3, $(c'b', b'f') \varepsilon \mathcal{B}$ and $(cd, ef) \varepsilon \mathcal{B}$. Moreover, using Axiom D3 and Lemma 3, $ab \circ ef \sim a'c' \circ c'f' > c'f' \sim c'b' \circ b'f' \sim c\, d \circ ef$. QED

LEMMA 5. *If* (ab, cd), $(a'b', b'd') \varepsilon \mathcal{B}$, *and* $ab \circ cd \sim a'b' \circ b'd'$, *then* $ab \sim a'b'$ *if and only if* $cd \sim b'd'$.

PROOF. Suppose that $ab \sim a'b'$ and $b'd' > cd$, then there exists d'' such that $b'd'' \sim cd$ and $d''d' \varepsilon \mathcal{A}^*$. By Axiom D2 and Lemma 3, $a'd' \sim a'b' \circ b'd' \sim ab \circ cd \sim a'b' \circ b'd'' \sim a'd''$. By Axiom D3, $a'd' \sim a'd'' \circ d''d' > a'd''$, a contradiction. The other cases are similar. QED

LEMMA 6. *If* $(ab, cd) \varepsilon \mathcal{B}$, *then* $(cd, ab) \varepsilon \mathcal{B}$ *and* $ab \circ cd \sim cd \circ ab$.

PROOF. Since $ab \sim a'b'$, $cd \sim b'd'$, then $a'd' \sim a'b' \circ b'd' > b'd' \sim cd$. There exists c' such that $a'c' \sim b'd' \sim cd$, and $c'd' \varepsilon \mathcal{A}^*$. Thus, by Axiom D4, $c'd' \sim a'b' \sim ab$. So by Lemmas 2 and 3, $(cd, ab) \varepsilon \mathcal{B}$, and $ab \circ cd \sim a'd' \sim a'c' \circ c'd' \sim cd \circ ab$. QED

We turn to a proof of Theorem 2. By hypothesis, $\mathcal{A}^*$ is non-empty and, by Lemma 1, so is $\mathcal{B}$. So we need only check the axioms of Definition 3 in [8]. Of these, only 2 and 6 cause any problems.

2. Suppose that (ab, cd) and $(ab \circ cd, ef) \varepsilon \mathcal{B}$. By definition of $\mathcal{B}$, there exist $a', b', d', a'', d'', f''$ such that $ab \sim a'b'$, $cd \sim b'd'$, $a''d'' \sim a'd' \sim ab \circ cd$, $d''f'' \sim ef$. By Axiom D3, $a''d'' \sim a'b' \circ b'd' > b'd' \sim cd$, so by Axiom D5 there exists c' such that $c'd'' \sim cd$ and $a''c' \varepsilon \mathcal{A}^*$. Since $ab \circ cd \sim a''d'' \sim a''c' \circ c'd''$, Lemma 5 implies $ab \sim a''c'$. By Lemmas 2 and 3, $(cd, ef) \varepsilon \mathcal{B}$. Similarly, $(ab, cd \circ ef) \varepsilon \mathcal{B}$. Moreover, $(ab \circ cd) \circ ef \sim a''f'' \sim a''c' \circ c'f'' \sim ab \circ (cd \circ ef)$.

6. Suppose $n > 1$ and $cd \gtrsim n(ab) \sim ef$. Then exists a_1 such that $ca_1 \sim ef > ab$. So there exists a_2 such that $a_2a_1 \varepsilon \mathcal{A}^*$, $a_2a_1 \sim ab$, and $ca_2 \varepsilon \mathcal{A}^*$. Proceeding inductively, we may construct a sequence that, by Axiom 6, must terminate. Therefore, $\{n \mid n \varepsilon I \text{ and } cd \gtrsim n(ab)\}$ is finite.

Turning to the corollary, we know by Theorem 2 of [8] that there exists an extensive representation φ of $\langle \mathcal{A}^*, \mathcal{B}, \gtrsim, \circ \rangle$. Fix $a_0 \varepsilon \mathcal{A}$ and define

$$\begin{aligned} \psi(a) &= \varphi(aa_0) && \text{if } aa_0 \varepsilon \mathcal{A}^* \\ &= 0 && \text{if } a = a_0 \\ &= \varphi(a_0a) && \text{if } a_0a \varepsilon \mathcal{A}^* \end{aligned}$$

This can easily be shown to satisfy the properties listed. QED

5. Preliminary lemmas about qualitative conditional probability. The common hypothesis of the following lemmas is that $\langle X, \mathcal{E}, \mathfrak{N}, \gtrsim \rangle$ is a system of qualitative conditional probability; Axioms 7 and 8 are not used. Some of the simpler proofs are omitted.

LEMMA 7. *If* $A \mid B \gtrsim C \mid D$, *then* $\bar{C} \mid D \gtrsim \bar{A} \mid B$.

COROLLARY 1. $A \mid B \gtrsim \varnothing \mid X$.

COROLLARY 2. $\varnothing \mid A \sim \varnothing \mid X$.

COROLLARY 3. *If* $A \mid X \sim \bar{A} \mid X$ *and* $B \mid X \sim \bar{B} \mid X$, *then* $A \mid X \sim B \mid X$.

LEMMA 8. *If* $A \supset B$, *then* $A \mid C \succsim B \mid C$.

LEMMA 9. (i) $\varnothing \, \varepsilon \, \mathfrak{N}$.

Suppose that $A \, \varepsilon \, \mathfrak{N}$, *then*

(ii) $\bar{A} \not\varepsilon \, \mathfrak{N}$;

(iii) *if* $A \supset B$, *then* $B \, \varepsilon \, \mathfrak{N}$;

(iv) *if* $B \, \varepsilon \, \mathfrak{N}$, *then* $A \cup B \, \varepsilon \, \mathfrak{N}$;

(v) *if* $B \not\varepsilon \, \mathfrak{N}$, *then* $A \mid B \sim \varnothing \mid X \sim \varnothing \mid B$.

LEMMA 10. *Suppose that* $A \supset B$, *then* $A \mid C \sim B \mid C$ *if and only if* $(A - B) \cap C \, \varepsilon \, \mathfrak{N}$.

PROOF. Since $A \supset B$, Axiom 4 yields $A \mid C \sim A \cap C \mid C \sim [(A - B) \cap C] \cup (B \cap C) \mid C$. If $(A - B) \cap C \, \varepsilon \, \mathfrak{N}$, then by Lemma 9(v), $(A - B) \cap C \mid C \sim \varnothing \mid C$. By Axioms 4 and 5, $A \mid C \sim [(A - B) \cap C] \cup (B \cap C) \mid C \sim B \cap C \mid C \sim B \mid C$. Conversely, if $(A - B) \cap C \not\varepsilon \, \mathfrak{N}$, then by Axiom 6 and Corollaries 1 and 2 of Lemma 7, $(A - B) \cap C \mid C \succ \varnothing \mid C$. By Axioms 4 and 5, $A \mid C \sim A \cap C \mid C \succ B \cap C \mid C \sim B \mid C$.

LEMMA 11. *Suppose that* $A \supset B$ *and that* $A \not\varepsilon \, \mathfrak{N}$, *then* $A - B \, \varepsilon \, \mathfrak{N}$ *if and only if* $B \mid A \sim X \mid X$.

PROOF. Suppose that $A - B \, \varepsilon \, \mathfrak{N}$. By Lemma 9(v) and Corollary 2 of Lemma 7, $(A - B) \mid A \sim \varnothing \mid A$. So, by Axiom 5, $A \mid A \sim (A - B) \cup B \mid A \sim B \mid A$, and the conclusion follows from Axiom 3. Conversely, suppose that $X \mid X \sim B \mid A$. By Axiom 3, $A \mid A \sim B \mid A$, and so by Lemma 10, $A - B = (A - B) \cap A \, \varepsilon \, \mathfrak{N}$. QED

LEMMA 12. $A \mid B \succsim A \cap B \mid X$.

PROOF. Assume the contrary and use Axioms 4 and 6 to show $\varnothing \mid X \succ \varnothing \mid B$, which is impossible by Corollary 2 of Lemma 7. QED

6. Proof of a non-additive representation. Define

$$\mathfrak{A}^* = \{AB \mid A, B \, \varepsilon \, \mathcal{E} - \mathfrak{N}, A \supset B, \text{and } A - B \, \varepsilon \, \mathcal{E} - \mathfrak{N}\}$$

$$\succsim^* \text{ on } \mathfrak{A}^*: AB \succsim^* CD \quad \text{if and only if} \quad D \mid C \succsim B \mid A.$$

THEOREM 3. *If* $\langle X, \mathcal{E}, \mathfrak{N}, \succsim \rangle$ *is a regular Archimedean system of qualitative conditional probability, then* $\langle \mathcal{E} - \mathfrak{N}, \mathfrak{A}^*, \succsim^* \rangle$ *is a system of positive differences* (Definition 2).

PROOF. We verify the six Axioms of Definition 2.

D1. Obvious.

D2. Suppose that AB, BC, $A'B'$, $B'C' \, \varepsilon \, \mathfrak{A}^*$, that $AB \succsim^* A'B'$, and that $BC \succsim^* B'C'$. Clearly, $A \supset C$. $A - C \not\varepsilon \, \mathfrak{N}$ since if not then, by Lemma 9(iii), $A - B \, \varepsilon \, \mathfrak{N}$, which is impossible. Similarly, $A' \supset C'$ and $A' - C' \not\varepsilon \, \mathfrak{N}$. Thus, AC and $A'C' \, \varepsilon \, \mathfrak{A}^*$. Using Axiom 6, $AC \succsim^* A'C'$.

D3. Suppose that AB and $BC \, \varepsilon \, \mathfrak{A}^*$. As in part 2, $AC \, \varepsilon \, \mathfrak{A}^*$. Because $A \supset B \supset C$, $B \mid A \succsim C \mid A$ (Lemma 8). Since $B - C \not\varepsilon \, \mathfrak{N}$, Lemma 10 implies $B \mid A \succ C \mid A$, and so $AC \succ^* AB$. Suppose that $BC \succsim^* AC$, then $C \mid A \succsim C \mid B$. Since $A \mid X$

$\succsim B \mid X$ and $A - B \not\varepsilon \mathfrak{N}$, Lemma 10 yields $A \mid X \succ B \mid X$. Therefore, by Axiom 6, $C \mid X \succ C \mid X$, which is impossible.

D4. Suppose that AB, AC, BD, $CD \varepsilon \mathfrak{A}^*$ and $AB \sim^* CD$. Suppose that $C \mid A \not\sim D \mid B$. Since $D \mid C \sim B \mid A$, Axiom 6 yields $D \mid A \not\sim D \mid A$, which is impossible. So $AC \sim^* BD$.

D5. Suppose that AB and $CD \varepsilon \mathfrak{A}^*$ and that $AB \succ^* CD$. By Axiom 8 there exist C' and D' such that $B \subset D'$, $C' \subset A$ and $D' \mid A \sim B \mid C' \sim D \mid C$. To show that AC', $C'B$, AD', and $D'B \varepsilon \mathfrak{A}^*$, it suffices to show that $A - C'$, $C' - B$, $A - D'$, and $D' - B \not\varepsilon \mathfrak{N}$. If $A - C' \varepsilon \mathfrak{N}$, then by Lemma 10, $C' \mid X \sim A \mid X$. Since $B \mid C' \sim D \mid C \succ B \mid A$, Axiom 6 yields $B \mid X \succ B \mid X$, which is impossible. If $C' - B \varepsilon \mathfrak{N}$, then by Lemma 11, $X \mid X \sim B \mid C' \sim D \mid C$, and so $C - D \varepsilon \mathfrak{N}$, which is impossible. The remaining two cases are similar.

D6. Suppose that $AB, CD \varepsilon \mathfrak{A}^*$ and that $A_i \varepsilon \mathcal{E} - \mathfrak{N}$ are such that $A_{i+1}A_i \varepsilon \mathfrak{A}^*$ and $A_{i+1}A_i \sim^* AB$. By definition $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$, $A_i \mid A_{i+1} \sim B \mid A \sim A_1 \mid A_2$. Moreover, since $A_{i+1} - A_i \varepsilon \mathcal{E} - \mathfrak{N}$, Lemma 11 implies $X \mid X \succ A_i \mid A_{i+1}$. Therefore, by Axiom 7, $\{n \mid n \varepsilon I \text{ and } CD \succsim^* A_nA_1\}$ is finite. QED

Corollary. *If $\langle X, \mathcal{E}, \mathfrak{N}, \succsim \rangle$ is a regular Archimedean system of qualitative probability, then there exists a function Q from $\mathcal{E}$ into the real interval* $[0, 1]$ *such that*

(i) $Q(\varnothing) = 0$ *and* $Q(X) = 1$;

(ii) $A \varepsilon \mathfrak{N}$ *if and only if* $Q(A) = 0$;

(iii) $A \mid B \succsim C \mid D$ *if and only if* $Q(A \cap B)/Q(B) \geqq Q(C \cap D)/Q(D)$;

(iv) *If Q' is any other function satisfying* (i)–(iii), *then there exists $\alpha > 0$ such that $Q' = Q^\alpha$.*

Proof. If, for all $A \varepsilon \mathcal{E}$, either A or $\bar{A} \varepsilon \eta$, then define

$$Q(A) = 0 \quad \text{if } A \varepsilon \mathfrak{N}$$
$$= 1 \quad \text{if } \bar{A} \varepsilon \mathfrak{N}.$$

Clearly it fulfills the assertions.

Next, suppose that whenever both A and $\bar{A} \varepsilon \mathcal{E} - \mathfrak{N}$, then $A \mid X \sim \bar{A} \mid X$. If A and B are two such elements and $A \cap B \not\varepsilon \mathfrak{N}$, then since $\overline{A \cap B} \supset \bar{A}$ and $\bar{A} \not\varepsilon \mathfrak{N}$, Lemma 9(iii) yields $\overline{A \cap B} \not\varepsilon \mathfrak{N}$. Thus, $A \cap B \mid X \sim \overline{A \cap B} \mid X$. By Corollary 3 of Lemma 7, $B \mid X \sim A \cap B \mid X$. Since $B \supset A \cap B$, Lemma 10 implies $B - (A \cap B) \varepsilon \mathfrak{N}$, and so by Axiom 4 and Lemma 11, $A \mid B \sim A \cap B \mid B \sim X \mid X$. If $A \cap B \varepsilon \mathfrak{N}$, then $A \mid B \sim \varnothing \mid X$. This then shows that, for $0 < q < 1$,

$$Q(A) = 0 \quad \text{if } A \varepsilon \mathfrak{N}$$
$$= q \quad \text{if } A, \bar{A} \not\varepsilon \mathfrak{N}$$
$$= 1 \quad \text{if } \bar{A} \varepsilon \mathfrak{N}$$

fulfills the assertions.

Finally, suppose that there exists an A with $A, \bar{A} \varepsilon \mathcal{E} - \mathfrak{N}$ and $A \mid X \succ \bar{A} \mid X$. Thus, in $\mathfrak{A}^*$, $X\bar{A} \succ^* XA$. By Theorem 2, $\mathfrak{A}^*$ generates an extensive system with $XB = XA \circ AB$ when $AB \varepsilon \mathfrak{A},^*$. By Theorem 2 of [8] there exists a positive ratio scale φ on $\succsim^*$ such that $\varphi(AB) \geqq \varphi(CD)$ if and only if $AB \succsim^* CD$ and, for $AB \in \mathcal{A}^*$, $\varphi(XB) = \varphi(XA) + \varphi(AB)$. Define

$$\begin{aligned} Q(A) &= 0 && \text{if } A \in \mathfrak{N} \\ &= \exp[-\varphi(XA)] && \text{if } A, \bar{A} \notin \mathfrak{N} \\ &= 1 && \text{if } \bar{A} \in \mathfrak{N}. \end{aligned}$$

This function has the asserted properties, as can be shown without great difficulty. QED

7. Proof of an additive measure on $\mathcal{E}$.

LEMMA 13. *Suppose that $\langle X, \mathcal{E}, \mathfrak{N}, \succsim\rangle$ is a system of qualitative conditional probability for which Axiom 8 holds. If $A \in \mathcal{E}$ and $B \in \mathcal{E} - \mathfrak{N}$, then there exists $C \in \mathcal{E}$ such that $C \supset A \cap B$ and $A \mid B \sim C \mid X$. If $A \mid X \succsim B \mid X$, then there exists $B' \subset A$ such that $B' \mid X \sim B \mid X$.*

PROOF. Obvious.

THEOREM 4. *Suppose that $\langle X, \mathcal{E}, \mathfrak{N}, \succsim\rangle$ is a regular Archimedean system of qualitative probability. There exists a unique real-valued function P on $\mathcal{E}$ such that*

(i) *$\langle X, \mathcal{E}, P\rangle$ is a finitely additive probability space;*

(ii) *$A \in \mathfrak{N}$ if and only if $P(A) = 0$;*

(iii) *$A \mid X \succsim B \mid X$ if and only if $P(A) \geqq P(B)$.*

PROOF. By Lemma 9 and Theorem 1 of [8] it is sufficient to show that the order $\succsim$ on $\mathcal{E}$ defined by

$$A \succsim B \quad \text{if and only if} \quad A \mid X \succsim B \mid X$$

satisfies the axioms of a regular Archimedean system of qualitative probability. Axioms 1–3 are trivial, 4 is an immediate consequence of Lemma 13, and 5 is shown as follows:

Suppose that $A_1', A_2', \cdots$ is a standard series relative to $A \succ \emptyset$. Let $A_1 = A_1' \sim A$. Since $A_2' \mid X \succsim A_1 \mid X$, there exists by Axiom 8, $A_2 \supset A_1$ such that $A_2 \mid X \sim A_2' \mid X$. By induction, we may construct the standard series $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ such that $A_i \sim A_i'$ and $A_{i+1} - A_i \sim A \in \mathcal{E} - \mathfrak{N}$. Consider, $A_1 \mid A_2, A_2 \mid A_4, \cdots, A_{2^i} \mid A_{2^{i+1}}, \cdots$. Observe that $A_{2^{i+1}} = A_{2^i} \cup (A_{2^{i+1}} - A_{2^i})$, that the two factors are disjoint, and that $A_{2^{i+1}} - A_{2^i} \sim A_{2^i}$. By Lemma 12, $A_{2^i} \mid A_{2^{i+1}} \succsim A_{2^i} \cap A_{2^{i+1}} \mid X$, and so by Axiom 8 there exists a C_i such that $A_{2^i} \mid A_{2^{i+1}} \sim C_i \mid X$. Since

$$\bar{C}_i \mid X \sim \bar{A}_{2^i} \mid A_{2^{i+1}} = (A_{2^{i+1}} - A_{2^i}) \mid A_{2^{i+1}} \sim A_{2^i} \mid A_{2^{i+1}} \sim C_i \mid X,$$

Lemma 8 yields $C_i \mid X \sim C_j \mid X$. Therefore, $A_{2^i} \mid A_{2^{i+1}} \sim A_1 \mid A_2$. Since $A_{2^{i+1}} - A_{2^i} \supset A_{2^{i+1}-1} \in \mathcal{E} - \mathfrak{N}$, Lemma 9 implies $A_{2^{i+1}} - A_{2^i} \in \mathcal{E} - \mathfrak{N}$. And since $A_2 - A_1 \in \mathcal{E} - \mathfrak{N}$, Lemma 11 implies $A_1 \mid A_2 \prec X \mid X$. Therefore, by Axiom 7, the sequence $A_1, A_2, \cdots, A_{2^i}, \cdots$ must be finite, hence the given standard series must be also.

8. Proof of Theorem 1. We begin with the following preliminary result.

LEMMA 14. *Suppose that $\langle X, \varepsilon, \mathfrak{N}, \succsim\rangle$ is a regular Archimedean system of qualita-*

tive conditional probability, that Q and P are the functions described, respectively, in the corollary to Theorem 3 and in Theorem 4, and that $\Pi \subset [0, 1]$ is the image of P. Then Π has the properties that

(i) $0, 1 \varepsilon \Pi$;

(ii) *if* $x, y \varepsilon \Pi$ *and* $x \geqq y$, *then* $x - y \varepsilon \Pi$;

(iii) *if* $x, y \varepsilon \Pi$ *and* $x + y \leqq 1$, *then* $x + y \varepsilon \Pi$.

The function f defined by $Q(A) = f[P(A)]$ for $A \varepsilon \mathcal{E}$ has the properties that

(iv) *f is strictly monotonic increasing;*

(v) $f(0) = 0$ *and* $f(1) = 1$;

(vi) *if* $x, y, z, x', y', z' \varepsilon \Pi$, $z, z' \neq 0$, $z \geqq x + y$, *and* $z' \geqq x' + y'$, *and if* $f(x)/f(z) \geqq f(x')/f(z')$ *and* $f(y)/f(z) \geqq f(y')/f(z')$, *then* $f(x + y)/f(z) \geqq f(x' + y')/f(z')$.

(vii) *if m and n are positive integers such that $m \leqq n$ and if $x \varepsilon \Pi$ is such that $0 < x \leqq 1/n$, then mx and $m/n \varepsilon \Pi$ and $f(mx) = f(nx)f(m/n)$.*

PROOF. Parts (i)–(vi) are simple to prove. To show (vii) we use induction. For $n = 1$, the result is obvious, so we assume $n > 1$. Let A be an event such that $P(A) = x \leqq 1/n$. Since

$$P(\bar{A}) = 1 - P(A) \geqq (n - 1)/n \geqq 1/n \geqq P(A),$$

$\bar{A} \mid X \succsim A \mid X$. By Lemma 13 there exists $A_2 \subset \bar{A}$ such that $A_2 \mid X \sim A \mid X$ and

$$P(\bar{A} - A_2) = P(\bar{A}) - P(A_2) \geqq (n - 2)/n.$$

We proceed inductively to construct $A = A_1, A_2, \cdots, A_n$ such that $A_i \mid X \sim A \mid X$ and, for $i \neq j$, $A_i \cap A_j = \varnothing$. Let $B = \bigcup_{i=1}^n A_i$. Clearly, $P(B) = nP(A) = nx > 0$. By Lemma 13, there exists C such that $A \mid B \sim C \mid X$. Since $A_1 \cup A_2 \mid B \succsim A \mid B \sim C \mid X$, Axiom 8 implies that there exists $C_2 \supset C$ such that $A_1 \cup A_2 \mid B \sim C_2 \mid X$. We claim that $(C_2 - C) \mid X \sim A_2 \mid B$, for otherwise $C \mid X \sim A \mid B$ implies, by Axiom 5, that $C_2 \mid X = (C_2 - C) \cup C \mid X \nsim A_2 \cup A_2 \mid B$, contrary to choice. So, $(C_2 - C) \mid X \sim C \mid X$. We proceed inductively to construct $C_n \supset C_{n-1} \supset \cdots \supset C_1 = C \supset C_0 = \varnothing$ such that for $i = 1, \cdots, n$, $(C_i - C_{i-1}) \mid X \sim C \mid X$. Note that

$$C_n \mid X \sim \bigcup_{i=1}^n A_i \mid B \sim B \mid B \sim X \mid X,$$

and so $C_n \sim X$. Thus

$$1 = P(X) = P(C_n) = \sum_{i=1}^n P(C_i - C_{i-1}) = nP(C),$$

i.e., $P(C) = 1/n$. Since $A \mid B \sim C \mid X$ and $A \subset B$, the Corollary to Theorem 3 yields

$$f(x)/f(nx) = f[P(A)]/f[P(B)] = Q(A)/Q(B) = Q(C)/Q(X)$$
$$= f[P(C)]/f[P(X)] = f(1/n)/f(1).$$

Since this holds for $n = 1$ and since $1 \leqq m \leqq n$, a finite induction on property (vi) yields

$$f(mx)/f(nx) = f(m/n)/f(1).$$

The result follows from property (v). QED

PROOF OF THEOREM 1. If for every positive integer n there exists an $x \varepsilon \Pi$ such that $0 < x \leqq 1/n$, then by Lemma 14 (vii), $m/n \varepsilon \Pi$ for $m \leqq n$. Thus, all rationals in [0, 1] are in Π. Let $r = m/n$ and s be any two rationals in [0, 1], then $x = s/n$ is rational and $x \leqq 1/n$, so by Lemma 14 (vii), $f(rs) = f(r)f(s)$. It is well known that the only strictly monotonic increasing (part (iv)) solution to this equation is $f(r) = r^\alpha$, where $\alpha > 0$. Furthermore, since the rationals are dense in Π and f is strictly increasing, it follows that $f(x) = x^\alpha$ for $x \varepsilon \Pi$. So the theorem is true in this case.

We therefore assume that there is some smallest n such that for all $x \varepsilon \Pi$ with $x > 0, x \geqq 1/n$. For some $x_0 \varepsilon \Pi$ with $x_0 > 0, x_0(n - 1) < 1$, since otherwise all $x \geqq 1/(n - 1)$, contrary to the choice of n. By Lemma 14 (i)–(iii), $1 - x_0(n - 1) \varepsilon \Pi$ and so $1 - x_0(n - 1) \geqq 1/n$, whence either $n = 1$, and so $\Pi = \{0, 1\}$, or $x_0 \leqq 1/n$. Clearly the theorem holds in the former case, and in the latter $x_0 = 1/n \varepsilon \Pi$. Let m be any integer $< n$, so $1/n < 1/m$. By Lemma 14 (vii), $1/m \varepsilon \Pi$ and so by part (ii), $1/m - 1/n = (n - m)/mn \varepsilon \Pi$. Thus $(n - m)/mn \geqq 1/n$, and so $n \geqq 2m$ which is possible for all $m < n$ only if $n = 2$. So by Lemma 14 (ii), $\Pi = \{0, \frac{1}{2}, 1\}$ and for some $\alpha, f(\frac{1}{2}) = (\frac{1}{2})^\alpha$. QED

REFERENCES

[1] COPELAND, A. H. (1956). Probabilities, observations, and predictions. *Proc. Third Berkeley Symp. Math. Statist. Prob.* 41–47. Univ. of California Press.

[2] CSÁSZÁR, A. (1955). Sur la structure des espaces de probabilité conditionnelle. *Acta Math. Acad. Sc. Hungar.* **6** 281–283.

[3] DE FINETTI, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Ann. Inst. Poincaré* **7** 1–68.

[4] KOOPMAN, B. O. (1940). The axioms and algebra of intuitive probability. *Ann. Math.* **41** 269–292.

[5] KOOPMAN, B. O. (1940). The bases of probability. *Bull. Amer. Math. Soc.* **46** 763–774.

[6] KOOPMAN, B. O. (1941). Intuitive probability and sequences. *Ann. Math.* **42** 169–187.

[7] KRAFT, C. H., PRATT, J. W. and SEIDENBERG, A. (1959). Intuitive probability on finite sets. *Ann. Math. Statist.* **30** 408–419.

[8] LUCE, R. D. (1967). Sufficient conditions for the existence of a finitely additive probability measure. *Ann. Math. Statist.* **38** 780–786.

[9] LUCE, R. D. and MARLEY, A. A. J. (1968). Extensive measurement when concatenation in restricted and maximal elements may exist. In S. Morgenbisser. P. Suppes, and M. G. White. *Essays in honor of Ernest Nagel.* St. Martins Press, New York.

[10] RÉNYI, A. (1955). On a new axiomatic theory of probability. *Acta Math. Acad. Sc. Hungar.* **6** 285–335.

[11] SAVAGE, L. J. (1954). *The Foundations of Statistics.* Wiley, New York.