The Annals of Mathematical Statistics, 38, 1967, pp. 780-786.

SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A FINITELY ADDITIVE PROBABILITY MEASURE[1]

By R. Duncan Luce

University of Pennsylvania

1. Introduction. Suppose that we have given a qualitative relation, which is to be interpreted as "at least as probable as," over a family of events, then under what conditions can we construct an order preserving, additive probability measure? A counter-example due to Kraft, Pratt, and Seidenberg [9] shows that certain obvious necessary conditions, which define what has been called a qualitative probability (see, e.g., p. 295 of [10] or p. 32 of [11]), are insufficient to guarantee the existence of such a measure. Sufficient conditions for a finitely additive measure have been presented by de Finetti [4], Koopman [5], [6], [7], and Savage [11]. They all involve, either explicitly or implicitly, the strong property that for each integer n the universal event can be partitioned into n events that are equally probable under the given ordering. Among other things, this forces the family of events to be infinite. The existence of countably additive measures has been studied by Villegas [14].

This paper provides still another axiomatization for the finitely additive representation; it is of interest because it does not demand arbitrarily fine partitions into equally likely events—in fact, finite models of the axioms exist. The proof is also of some inherent interest because it depends upon a theorem from the theory of extensive measurement. The parallel between the additivity of probability and of extensive measures has always been apparent, but to my knowledge no intimate connection between them has been previously shown. The reason is that the additivity of probability only holds for certain pairs of events—disjoint ones—whereas in the classical theory of extensive measurement (see [2], [3], [12], and [13]) additivity holds over all pairs of entities without any restriction. However, Behrend [1] and Luce and Marley [9], in attempts to make extensive measurement more realistic, have shown that an additive representation can still be constructed when only certain pairs of entities can be concatenated. Their result, which is stated fully in Section 3, is used to prove the present probability theorem.

2. The axioms and representation theorem. A preliminary definition is needed. It states, in essence, what we shall mean by an event having, qualitatively, a probability equal to that of n disjoint copies of a given event.

Definition 1. Let X be a non-empty set, A a family of subsets of X that includes $\varnothing$ and that is closed under complementation and union, and $\sim$ an

Received 12 December 1966.

[1] This work was carried out while the author was a National Science Foundation Senior Postdoctoral Fellow at the Center for Advanced Study in the Behavioral Sciences. The comments of David Krantz, Patrick Suppes, and Amos Tversky have been most helpful.

equivalence relation on A. Let $a \varepsilon A$. A sequence $a_1, a_2, \cdots, a_i, \cdots$, where $a_i \varepsilon A$, is a *standard series relative to a* if for $i = 1, 2, \cdots$ there exist b_i and c_i εA such that:

(i) $a_1 = b_1$ and $b_1 \sim a$,

(ii) $b_i \cap c_i = \varnothing$,

(iii) $a_{i+1} = b_i \cup c_i$,

(iv) $b_i \sim a_i$,

(v) $c_i \sim a$.

To gain an intuitive idea of the meaning of a standard series and the role it will play, note that

$$a_1 \sim a,$$

$$a_2 = b_1 \cup c_1, \quad \text{where} \quad b_1 \sim a, c_1 \sim a, \text{ and } b_1 \cap c_1 = \varnothing,$$

$$\cdots$$

$$a_{i+1} = b_i \cup c_i, \quad \text{where} \quad b_i \sim a_i, c_i \sim a, \text{ and } b_i \cap c_i = \varnothing.$$

Thus, a_{i+1} is equal to an event that is the disjoint union of one that is indifferent to a_i and another that is indifferent to a. So, crudely, a_i is an event that acts like the union of i mutually disjoint events each of which is indifferent to a; however, since i such events may not exist, the definition has to be somewhat indirect. If a finitely additive, order preserving probability measure p exists, then by induction it is easily seen that $p(a_i) = ip(a)$. Therefore, if $p(a) > 0$, the standard series must be finite. The qualitative restatement of this is one of the two new axioms in the following definition.

DEFINITION 2. Let X be a non-empty set, A family of subsets of X that includes $\varnothing$ and that is closed under complementation and union, and $\gtrsim$ a binary relation on A. The triple $\langle X, A, \gtrsim\rangle$ is called a *regular system of qualitative probability* if, for all $a, b, c, d \varepsilon A$, the following five axioms hold:[2]

(1) $\gtrsim$ is a weak ordering of A.

(2) $a \gtrsim \varnothing$ and $X > \varnothing$.

(3) If $a \cap c = b \cap c = \varnothing$, then $a \gtrsim b$ if and only if $a \cup c \gtrsim b \cup c$.

(4) If $a \cap b = \varnothing$, $a > c$, and $b \gtrsim d$, then there exist $c', d', e \varepsilon A$ such that $e \sim a \cup b$, $c' \sim c$, $d' \sim d$, $e \supseteq c' \cup d'$, and $c' \cap d' = \varnothing$.

(5) If $a > \varnothing$, then any standard series relative to a is finite.

The first three axioms, which are necessary whenever a non-trivial finitely additive representation exists, define what is called a *qualitative probability structure*. The fourth axiom is a somewhat weaker, and so more acceptable, version of the assertion that if a and b are disjoint and dominate c and d, respectively, then there are disjoint subsets of $a \cup b$ that are equivalent in probability to c and d. It is not a necessary condition. The last axiom, which is really an Archimedian property, states in essence that any event that is strictly more probable than the null event behaves as if it has non-zero probability. It is a

[2] We define $>$ and $\sim$ in terms of $\gtrsim$ in the usual way.

necessary condition when an order preserving, finitely additive probability measure exists.

The following is to be proved.

THEOREM 1. *If* $\langle X, A, \gtrsim \rangle$ *is a regular system of qualitative probability, then there exists a unique, finitely additive probability measure* p *over* A *that preserves the order of* $\gtrsim$, i.e., *for all* $a, b \in A$:

(i) $a \gtrsim b$ *if and only if* $p(a) \geqq p(b)$.

(ii) $0 \leqq p(a) \leqq 1$.

(iii) $p(\varnothing) = 0$ *and* $p(X) = 1$.

(iv) *If* $a \cap b = \varnothing$, *then* $p(a \cup b) = p(a) + p(b)$.

As was indicated in the introduction, the proof involves reducing this assertion to a result in the theory of estensive measurement. This theorem is stated next.

3. A result from the theory of extensive measurement.

DEFINITION 3. Let A be a non-empty set, B a non-empty subset of $A \times A$, R a binary relation on A, *and* $\circ$ a binary function from B into A. The quadruple $\langle A, B, R, \circ \rangle$ is called an *extensive system without a maximal element*[3] if, for all x, y, z, εA,

(1) R is a weak ordering of A.

(2) If $(x, y) \in B$ and $(x \circ y, z) \in B$, then $(y, z) \in B$, $(x, y \circ z) \in B$, and $(x \circ y) \circ zRx \circ (y \circ z)$.

(3) If $(x, z) \in B$ and xRy, then $(z, y) \in B$ *and* $x \circ zRz \circ y$.

(4) If not xRy, then there exists $y - x \in A$ such that $(x, y - x) \in B$, $yRx \circ (y - x)$, and $x \circ (y - x)Ry$.

(5) If $(x, y) \in B$, then not $xRx \circ y$.

(6) Let n be a positive integer. For $n = 1$, define $1x = x$. For $n > 1$, if $(n - 1)x$ is defined and $((n - 1)x, x) \in B$, then define $nx = (n - 1)x \circ x$. For all $x, y \in A$, the set

$$\{n \mid n \quad \text{is a positive integer and} \quad yRnx\}$$

is finite.

In this system, Axiom 2 captures associativity; Axiom 3 both insures commutativity and that inequalities are preserved when the same element is concatenated with both terms; Axiom 4 asserts that the system is complete in the sense that certain equations can be solved; Axiom 5 excludes both zero and negative elements; and Axiom 6 is a suitable formulation of the Archimedean property when only some pairs of elements can be concatenated.

THEOREM 2. *If* $\langle A, B, R, \circ \rangle$ *is an extensive system without a maximal element, then there exists a positive real-valued function* φ *on* A *such that*

[3] The theory in [9] also deals with extensive systems that have maximal elements in the following special sense: $a \in A$ is maximal relative to R and $\circ$ if for all $x \in A$, aRx, and if for some $x \in A$, $(a, x) \in B$. As the proof of Theorem 1 does not require the results for the case when there is a maximal element, I do not state them here.

(i) *xRy if and only if* $\varphi(x) \geqq \varphi(y)$.

(ii) *If* $(x, y) \varepsilon B$, *then* $\varphi(x \circ y) = \varphi(x) + \varphi(y)$.

(iii) *If* φ' *also satisfies* (i) *and* (ii), *then there exists a number* $\alpha > 0$ *such that* $\varphi' = \alpha\varphi$.

This theorem is proved in [9]; a closely related result is given in [1].

4. Preliminary lemmas. Throughout this section, the implicit hypothesis is that the axioms of a regular system of qualitative probability are satisfied.

LEMMA 1. *If* $a \supseteq b$, *then* $a \gtrsim b$.

PROOF. By Axiom 2, $(a - b) \gtrsim \varnothing$ and $b \cap (a - b) = \varnothing = b \cap \varnothing$, whence, by Axiom 3, $a = (a - b) \cup b \gtrsim \varnothing \cup b = b$. Q.E.D.

LEMMA 2. *Suppose that* $a \cap c = b \cap c = \varnothing$. *Then* $a \sim b$ *if and only if* $a \cup c \sim b \cup c$, *and* $a > b$ *if and only if* $a \cup c > b \cup c$.

PROOF. Two applications of Axiom 3 in each direction prove the first assertion, and the second follows immediately from it. Q.E.D.

LEMMA 3. *If* $a \cap b = \varnothing$, $a \gtrsim c$, *and* $b \gtrsim d$, *then* $a \cup b \gtrsim c \cup d$.

PROOF. It is sufficient to prove the result for disjoint c and d, since if they are not disjoint let $c' = c - d$. Since $c \supseteq c'$, Lemma 1 implies $a \gtrsim c \gtrsim c'$. So if the lemma is true in the disjoint case, $a \cup b \gtrsim c' \cup d = c \cup d$.

Let $a' = a - d$, $d' = d - a$, and $e = a \cap d$, then using Axiom 3 twice,

$$a' \cup b \gtrsim a' \cup d = a \cup d' \gtrsim c \cup d'.$$

But $e \cap (a' \cup b) = \varnothing$ and $e \cap (c \cup d') = \varnothing$, so by Axiom 3,

$$a \cup b = e \cup a' \cup b \gtrsim e \cup c \cup d' = c \cup d. \qquad \text{Q.E.D.}$$

COROLLARY 1. *If* $a \cap b = c \cap d = \varnothing$, $a \sim c$, *and* $b \sim d$, *then* $a \cup b \sim c \cup d$.

PROOF. Apply the Lemma twice. Q.E.D.

COROLLARY 2. *If* $a \cap b = \varnothing$, $a > c$, *and* $b \gtrsim d$, *then* $a \cup b > c \cup d$.

PROOF. Use the same proof as in the Lemma, but replace Axiom 3 by the second part of Lemma 2 whenever a strict inequality occurs. Q.E.D.

LEMMA 4. *If* $a > b$, *then there exist* $a', b' \varepsilon A$ *such that* $a' \supseteq b'$, $a' \sim a$, *and* $b' \sim b$.

PROOF. Apply Axiom 4 to $a > b$ and $\varnothing \sim \varnothing$. Q.E.D.

LEMMA 5. *Suppose that* $a \supseteq b$. *If* $a - b > \varnothing$, *then* $a > b$; *and if* $a - b \sim \varnothing$, *then* $a \sim b$.

PROOF. Since $(a - b) \cap b = \varnothing$ and $b \sim b$, then if $a - b > \varnothing$ Corollary 2 of Lemma 3 implies $a = (a - b) \cup b > \varnothing \cup b = b$; and if $a - b \sim \varnothing$, Corollary 1 implies $a \sim b$. Q.E.D.

LEMMA 6. *For all* $a \varepsilon A$, $X \gtrsim a$.

PROOF. Suppose, on the contrary, $a > X$. By Axiom 2, $\bar{a} \gtrsim \varnothing$. Since $a \cap \bar{a} = \varnothing$, Corollary 2 of Lemma 3 implies $X = a \cup \bar{a} > X \cup \varnothing = X$, which is impossible by Axiom 1. Q.E.D.

5. Proof of Theorem 1. Let $\mathbf{a}$ denote the element (equivalence class) of $A/\sim$ that includes a.

Suppose that there is no a for which both $a > \varnothing$ and $\bar{a} > \varnothing$. Then $a \,\varepsilon\, A$ implies either $a \,\varepsilon\, \boldsymbol{\phi}$ or $a \,\varepsilon\, \mathbf{X}$, and

$$\begin{aligned} p(a) &= 1 \text{ if } a \,\varepsilon\, \mathbf{X} \\ &= 0 \text{ if } a \,\varepsilon\, \boldsymbol{\phi} \end{aligned}$$

clearly fulfills the assertions of the theorem.

Henceforth we assume that $a > \varnothing$ and $\bar{a} > \varnothing$ for some $a \,\varepsilon\, A$ and define:

$$B = \{(\mathbf{a}, \mathbf{b}) \mid a > \varnothing, b > \varnothing, \text{and there exist } a' \,\varepsilon\, \mathbf{a}, b' \,\varepsilon\, \mathbf{b} \text{ such that } a' \cap b' = \varnothing\}.$$

B is nonempty since we have assumed that an a exists for which $a > \varnothing$ and $\bar{a} > \varnothing$. If $(\mathbf{a}, \mathbf{b}) \,\varepsilon\, B$ and, with no loss of generality, $a \cap b = \varnothing$, then we define the binary operation $\circ$ by: $\mathbf{a} \circ \mathbf{b} = \mathbf{a} \cup \mathbf{b}$. Two applications of Corollary 1 of Lemma 3 show that $\circ$ is well defined.

We now prove that $\langle A^*, B, R, \circ\rangle$, where $A^* = (A/\sim) - \boldsymbol{\phi}$ and R is the restriction of $\gtrsim/\sim$ to A^*, is an extensive system without a maximal element (Definition 3). Note that A^* excludes events of qualitative probability zero; they are reinstated later.

(1) R is obviously a strict ordering of A^* since, by Axiom 1, $\gtrsim$ is a weak ordering of A.

(2) Suppose that $(\mathbf{a}, \mathbf{b}) \,\varepsilon\, B$, $a \cap b = \varnothing$, and $(\mathbf{a} \circ \mathbf{b}, \mathbf{c}) \,\varepsilon\, B$. By definition of B, there exist $d \,\varepsilon\, \mathbf{a} \cup \mathbf{b}$ and $c' \,\varepsilon\, \mathbf{c}$ such that $c' \cap d = \varnothing$. Since $a \cup b \supseteq b$ and $a > \varnothing$, Lemma 5 implies that $d \sim a \cup b > b$. This, together with $c' \sim c$ and $c' \cap d = \varnothing$, implies that (Axiom 4) there exist $b', c'' \,\varepsilon\, A$ for which $b' \,\varepsilon\, \mathbf{b}$, $c'' \,\varepsilon\, \mathbf{c}$, and $b' \cap c'' = \varnothing$. Thus, $(\mathbf{b}, \mathbf{c}) \,\varepsilon\, B$.

Next, we establish that $(\mathbf{a}, \mathbf{b} \circ \mathbf{c}) \,\varepsilon\, B$. Since $a > \varnothing$, $b \sim b'$, and $a \cap b = \varnothing$, Corollary 2 of Lemma 3 yields $d \sim a \cup b > b'$. But $c' \sim c''$ and $c' \cap d = \varnothing$, so by Axiom 4 there exist $b'' \,\varepsilon\, \mathbf{b}$, $c''' \,\varepsilon\, \mathbf{c}$, and $e \,\varepsilon\, A$ such that $e \sim d \cup c'$, $e \supseteq b'' \cup c'''$, and $b'' \cap c''' = \varnothing$. Suppose that $d > (e - c''')$. Since $c' \sim c'''$ and $c' \cap d = \varnothing$, Corollary 2 of Lemma 3 yields $d \cup c' > (e - c''') \cup c''' = e \sim (d \cup c')$, which is impossible by Axiom 1. The supposition $(e - c''') > d$ and $c''' \sim c'$ leads to a similar contradiction. So $d \sim (e - c''')$. Now, suppose $a' > a$, where $a' = e - (b'' \cup c''')$. Since $b'' \sim b$ and $a' \cap b'' = \varnothing$, Corollary 2 of Lemma 3 implies that $d \sim (e - c''') = (a' \cup b'') > (a \cup b) \sim d$, which is impossible. And if $a > a'$, then $d \sim (a \cup b) > (a' \cup b'') \sim d$, which is also impossible. So, $a \sim a'$. Since $a' \,\varepsilon\, \mathbf{a}$, $b'' \cup c''' \,\varepsilon\, \mathbf{b} \circ \mathbf{c}$, and $a' \cap (b'' \cup c''') = \varnothing$, the assertion is proved. Moreover,

$$\begin{aligned} \mathbf{a} \circ (\mathbf{b} \circ \mathbf{c}) &= \mathbf{a}' \cup \mathbf{b}'' \cup \mathbf{c}''' \\ &= \mathbf{e} \\ &= \mathbf{d} \cup \mathbf{c}' \\ &= (\mathbf{a} \circ \mathbf{b}) \circ \mathbf{c}. \end{aligned}$$

(3) Suppose that $(\mathbf{a}, \mathbf{c}) \,\varepsilon\, B$ and $\mathbf{a}R\mathbf{b}$, where with no loss of generality, $a \cap c = \varnothing$. If $\mathbf{a} = \mathbf{b}$, it follows immediately that $\mathbf{a} \circ \mathbf{c} = \mathbf{c} \circ \mathbf{b}$. So we assume

$a > b$. Since $c \sim c$, Axiom 4 asserts the existence of $b' \varepsilon \mathbf{b}$ and $c' \varepsilon \mathbf{c}$ such that $b' \cap c' = \varnothing$. So $(\mathbf{c}, \mathbf{b}) \varepsilon B$, and by Corollary 2 of Lemma 3, $a \cup c > b' \cup c'$, whence, by definition, not $\mathbf{c} \circ \mathbf{b} R \mathbf{a} \circ \mathbf{c}$.

(4) Suppose that not $\mathbf{a} R \mathbf{b}$. Thus, $b > a$ and so by Lemma 4 there exist $a' \varepsilon \mathbf{a}$, $b' \varepsilon \mathbf{b}$ such thas $b' \supseteq a'$. By Lemma 5, $(b' - a') > \varnothing$, $a > \varnothing$ since $a \varepsilon A^*$, and $a' \cap (b' - a') = \varnothing$, so $\mathbf{b} = \mathbf{a} \circ (\mathbf{b}' - \mathbf{a}')$.

(5) Suppose that $(\mathbf{a}, \mathbf{b}) \varepsilon B$, where $a \cap b = \varnothing$. Since $b > \varnothing$, it follow from Lemma 5 that $a \cup b > a$, and so not $\mathbf{a} R \mathbf{a} \circ \mathbf{b}$.

(6) Finally, we show that $\{n \mid \mathbf{b} R n\mathbf{a}\}$ is finite, where $n\mathbf{a} = (n - 1)\mathbf{a} \circ \mathbf{a}$ and $1\mathbf{a} = \mathbf{a}$. We do this by showing that the existence of $n\mathbf{a}$ implies the existence of a standard series relative to a that has n elements. Because $\mathbf{a} \varepsilon A^*$, $a > \varnothing$, and so by Axiom 5 any such series must be finite; therefore there exists some integer m such that $n\mathbf{a}$ is not defined for $n > m$. Suppose that $2\mathbf{a}$ exists, then by definition of $\circ$ there exist $a_1 = b_1 \varepsilon \mathbf{a}$ and $c_1 \varepsilon \mathbf{a}$ such that $b_1 \cap c_1 = \varnothing$ and $a_2 = b_1 \cup c_1 \varepsilon 2\mathbf{a}$. Suppose that $n\mathbf{a}$ exists and that, for $i \leqq n - 1$, we have constructed a standard series a_i relative to a, with auxiliary b_{i-1} and c_{i-1}, and that $a_i \varepsilon i\mathbf{a}$. We extend it to $i = n$. By definition of $\circ$, there exist $b_{n-1} \varepsilon (n - 1)\mathbf{a}$ and $c_{n-1} \varepsilon \mathbf{a}$ such that $b_{n-1} \cap c_{n-1} = \varnothing$ and $b_{n-1} \cup c_{n-1} \varepsilon n\mathbf{a}$. Thus, if we set $a_n = b_{n-1} \cup c_{n-1}$, the series is extended.

By Theorem 2, there is a positive real-valued function φ on A^* such that

$$\mathbf{a} R \mathbf{b} \quad \text{if and only if} \quad \varphi(\mathbf{a}) \geqq \varphi(\mathbf{b}),$$

and

$$\text{if} \quad (\mathbf{a}, \mathbf{b}) \varepsilon B, \quad \text{then} \quad \varphi(\mathbf{a} \circ \mathbf{b}) = \varphi(\mathbf{a}) + \varphi(\mathbf{b}).$$

Select that φ for which $\varphi(\mathbf{X}) = 1$ and, for $a \varepsilon A$, define

$$\begin{aligned} p(a) &= \varphi(\mathbf{a}) \quad \text{if} \quad a > \varnothing \\ &= 0 \qquad \text{if} \quad a \sim \varnothing. \end{aligned}$$

Using the properties of φ and Lemma 6, it is easy to see that p fulfills the assertions of the theorem. Moreover p is unique since if another such function existed there would be an additive measure in the extensive system not related to φ by a multiplicative constant, thus violating part (iii) of Theorem 2. Q.E.D.

6. Relation to Savage's system. In [11], Savage has shown that Axioms 1–3 plus the condition that $\gtrsim$ is fine and tight are sufficient to prove Theorem 1. Recall that $\gtrsim$ is *fine* if for every $a > \varnothing$, there exists a partition $\{b_1, \cdots, b_n\}$ of X such that for $i = 1, \cdots, n$, $a \gtrsim b_i$; that a and b are *almost equivalent*, denoted $a \sim^* b$, if for every c, d such that $c > \varnothing$, $d > \varnothing$, and $a \cap c = b \cap d = \varnothing$, then $a \cup c \gtrsim b$ and $b \cup d \gtrsim a$; and that $\gtrsim$ is *tight* if $a \sim^* b$ implies $a \sim b$.

THEOREM 3. *If $\gtrsim$ is a qualitative probability on 2^X that is fine and tight, then $\langle X, 2^X, \gtrsim \rangle$ is a regular system of qualitative probability.*

PROOF. Since Axioms 1—3 are assumed, it is sufficient to prove 4 and 5.

Suppose that $a \cap b = \varnothing$, $a > c$, and $b \gtrsim d$. Since $\gtrsim$ is fine, Theorem 3, p. 37

of [11] states that there exists a unique probability measure p that almost agrees with $\gtrsim$ (i.e., $a \gtrsim b$ implies $p(a) \geqq p(b)$), that $a \sim^* b$ if and only if $p(a) = p(b)$, that for any a and any number ρ, $0 \leqq \rho \leqq 1$, there exists $b \subseteq a$ such that $p(b) = \rho p(a)$, and that if $a > \varnothing$, $p(a) > 0$. Since $a > c$ and $b \gtrsim d$, $p(a) \geqq p(c)$ and $p(b) \geqq p(d)$. Let ρ, ρ' be such that $\rho p(a) = p(c)$ and $\rho' p(b) = p(d)$. So there exist $c' \subseteq a$ and $d' \subseteq b$ such that $p(c') = \rho p(a) = p(c)$ and $p(d') = \rho' p(b) = p(d)$. Therefore, $c' \sim^* c$ and $d' \sim^* d$, but since $\gtrsim$ is tight, $c' \sim c$ and $d' \sim d$, thereby proving Axiom 4 with $e = a \cup b$.

Axiom 5 is immediate. Q.E.D.

It is obvious that the converse of Theorem 3 is false since Savage's assumptions imply that X is infinite whereas the ordering induced by the uniform distribution on a finite set satisfies Definition 2.

REFERENCES

[1] BEHREND, F. A. (1956). A contribution to the theory of magnitudes and the foundation of analysis. *Math. Zeit.* **63** 345–362.

[2] CAMPBELL, N. R. (1920). *Physics: the elements.* Cambridge University Press, Cambridge. Reprinted (1957) as *Foundations of Science: the philosophy of theory and experiment.* Dover, New York.

[3] CAMPBELL, N. R. (1928). *An account of the principles of measurement and calculation.* Longmans, Green, London.

[4] DE FINETTI, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Ann. Inst. Poincaré.* **7** 1–68. English translation in H. E. Kyburg, Jr., & H. E. Smokler (Eds.) (1964) *Studies in subjective probability.* Wiley, New York. 93–158.

[5] KOOPMAN, B. O. (1940). The bases of probability. *Bull. Amer. Math Soc.* **46** 763–774.

[6] KOOPMAN, B. O. (1940). The axioms and algebra of intuitive probability. *Annals of Math.* **41** 269–292.

[7] KOOPMAN, B. O. (1941). Intuitive probabilities and sequences. *Annals of Math.* **42** 169–187.

[8] KRAFT, C. H., PRATT, J. W., and SEIDENBERG, A. (1959). Intuitive probability on finite sets. *Ann. Math. Stat.* **30** 408–419.

[9] LUCE, R. D., and MARLEY, A. A. J. (1967). Extensive measurement when concatenation is restricted and maximal elements may exist. In S. Morgenbesser, P. Suppes, & M. G. White (Eds.) *Essays in Honor of Ernest Nagel.* St. Martins Press, New York.

[10] LUCE, R. D., and SUPPES, P. (1965). Preference, utility and subjective probability. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Handbook of Mathematical Psychology.* **2** Wiley, New York. 249–410.

[11] SAVAGE, L. J. (1954). *The foundations of statistics.* Wiley, New York.

[12] SUPPES, P. (1951). A set of independent axioms for extensive quantities. *Portugaliae Mathematica.* **10** 163–172.

[13] SUPPES, P., and ZINNES, J. L. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Handbook of Mathematical Psychology.* **1** Wiley, New York. 1–76.

[14] VILLEGAS, C. (1964). On qualitative probability σ-algebras. *Ann. Math. Stat.* **35** 1787–1796.