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Two Extensions of Conjoint Measurement[1,2]

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Both extensions modify the axioms of Luce and Tukey for additive conjoint measurement. The first yields a theory for more than two coordinates. The main problem is to find a weak generalization of the cancellation property; the one suggested is weaker than Krantz's generalization. The second extension weakens the solution-of-equations axiom, which has been justly criticized as too strong for most potential applications. A much more plausible version is suggested. This weakening is compensated for by adding the (necessary) independence property as an axiom and by postulating the existence of a part of a dual standard sequence. The usual representation and uniqueness theorems are proved.

1. INTRODUCTION

Theories of (additive) conjoint measurement state conditions—either necessary or sufficient or both—on an ordering of multicoordinate stimuli for which a numerical representation exists that is additive over the coordinates. The earliest discussions of such additive representations are probably those in the economic literature of the nineteenth century, but for various reasons the problem was not pursued very deeply at that time. Most of what we know about conjoint measurement has been uncovered within the past few years.

Briefly, the results are these. Adams and Fagot (1959) derived a number of necessary algebraic conditions for an additive representation to exist, and Scott (1964) gave necessary and sufficient (cancellation) conditions when the two-coordinate stimuli are finite in number. Tversky (1964) independently proved the same result, and later (1967) he gave, among other generalizations, necessary and sufficient conditions for the

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[2] I am deeply indebted to Dr. Michael Levine for discussing with me the ideas presented in Secs. 7-9, which involve a weakening of the solutions-of-equations Axiom of Luce and Tukey (1964). In particular, he suggested how the original system can be imbedded in a system (Definition 9A) that satisfies the axioms of Luce and Tukey. He has a different proof of the representation Theorem 9F which is based on the same construction but which uses the concatenation operation introduced by Krantz (1964).

infinite case. From an empirical point of view, Scott's and Tversky's results are not especially satisfactory because the number of cancellation conditions increases rapidly with the number of stimuli, and so the problem remains to find sufficient conditions that are empirically more usable.

Debreu (1959, 1960) provided the first set of general sufficient conditions, both for the 2- and k-coordinate cases, but they involve topological assumptions about the structure of the coordinates which do not seem inherent to the problem. For the 2-coordinate case, Luce and Tukey (1964)[3] presented a set of four sufficient algebraic conditions that assure the existence of additive measurement, and their proof provides a constructive device (dual standard sequences) that may be useful in estimating the additive scales. Krantz (1964) used the theory of ordered groups to give an alternative proof of the L-T theorem, and he generalized the result so that the coordinates of the stimuli need not be known in advance provided that there are $n + 1$ relations that are interlocked in a certain way. For $n = 2$, these results are very closely related to ones in the algebraic theory of webs (Aczél, Pickert, and Radó, 1960); there the basic cancellation axiom is known as the Thompsen condition. Roskies (1965) modified the L-T axioms so as to provide a multiplicative representation which cannot be transformed into an additive one. Luce (1966) stated interconnections between conjoint measures on the stimuli and extensive measures on each of the coordinates that are sufficient to lead to the usual multiplicative power relations of many of the laws of classical physics.

Various extensions of these results are needed, both to satisfy one's mathematical curiosity and to make the theorems more useful for applications. This paper is concerned with two extensions. First, the L-T theorem is extended to k coordinates. The result presented here seems somewhat more satisfactory than Krantz's, because the cancellation axiom is rather weaker than his, and than Debreu's, because it does not involve topological assumptions about the coordinates. Second, the L-T theorem is modified by weakening the very restrictive solution-of-equations axiom; a few words of comment may be appropriate.

Luce and Tukey proved that there exists an additive representation of a relation $\geqslant$ on $\mathscr{A}_1 \times \mathscr{A}_2$ provided that (1) $\geqslant$ is a weak ordering, (2) $\geqslant$ satisfies the simplest possible cancellation property, (3) $\geqslant$ is Archimedean in an appropriate sense, and (4) $=$ fulfills the following solution-of-equations condition: if A, B are in $\mathscr{A}_1$ and P is in $\mathscr{A}_2$, then there exists X in $\mathscr{A}_2$ such that $(A, P) = (B, X)$, and a similar property holds when the second coordinate is given and the first is to be solved for. One consequence of these assumptions, which at least for psychological applications is often unacceptable, is that the numerical scales are unbounded, and it is not difficult to see that this stems entirely from Conditions (3) and (4). Moreover, in most potential applications it is clear that Condition (4) is not generally satisfied. For example, sup-

[3] For brevity, I shall refer to this paper as L-T.

pose that the stimuli are pure tones and they are ordered by loudness, then given a tone and a intensity, it is well known that it is not always possible to select a frequency so that the resulting tone has a loudness equal to that of the given one. So it seems clear that we must weaken Condition (4) to the point where it is empirically realizable, without, however, completely losing the possibility of constructing an additive representation. In particular, we do not want to weaken it so much that very many new axioms, even though they are necessary conditions, must be added in order to prove the result. However, unless the L-T axiomatization is very inefficient, it is clear that we will be unable to weaken Condition (4) much without having to add more conditions, but for empirical reasons we do not want to add many more.

The organization of the paper is this. Sections 2-5 present a generalization to k coordinates. Section 6 gives a result about the relationship among standard sequences when the Archimedean condition does not hold. This is of some inherent interest as well as being used in the proof of the representation when the solution-of-equations axiom is weakened. The latter problem is examined in Secs. 7-9.

2. ORDERING, SOLUTION, AND CANCELLATION AXIOMS FOR k COORDINATES

Suppose $\mathscr{A}_i$, $i = 1, \cdots, k$, are sets, and let $\geqslant$ be a binary relation over the Cartesian product $\prod_{i=1}^{k} \mathscr{A}_i$.

(2A) Ordering Axiom 1. *$\geqslant$ is a weak ordering of $\prod_{i=1}^{k} \mathscr{A}_i$.*

(2B) Solution Axiom 2. *For any integer j, $1 \leqslant j \leqslant k$, and any A^i in $\mathscr{A}_i$, $1 \leqslant i \leqslant k$, and any B^i in $\mathscr{A}_i$, $1 \leqslant i \leqslant k$, $i \neq j$, there exists an X^j in $\mathscr{A}_j$ such that*

$$(A^1, \cdots, A^{j-1}, A^j, A^{j+1}, \cdots, A^k) = (B^1, \cdots, B^{j-1}, X^j, B^{j+1}, \cdots, B^k).$$

(2C) Cancellation Axiom 3. *Consider any two inequalities*

$$(A_i^1, A_i^2, \cdots, A_i^k) \geqslant (B_i^1, B_i^2, \cdots, B_i^k), \qquad i = 1, 2,$$

with the property that for each $j = 1, 2, \cdots, k$ there exists a corresponding permutation J of 1 and 2 such that

$$A_i^j = B_{J(i)}^j$$

except for at most one i, which if it exists we denote by i^*. Then,

$$(A^1, A^2, \cdots, A^k) \geqslant (B^1, B^2, \cdots, B^k),$$

where if i^* exists, $A^j = A_{i^*}^j$ and $B^j = B_{J(i^*)}^j$, and if i^* does not exist, $A^j = B^j =$ any element of $\mathscr{A}_j$.

REMARK. When $k = 2$ it is easy to see that Axioms 1 and 2 reduce to those of L-T. For $k = 2$, the only significant case of Axiom 3 is of the form: if $(A_1{}^1, A_1{}^2) \geqslant (B_1{}^1, B_1{}^2)$ and $(B_1{}^1, A_2{}^2) \geqslant (B_2{}^1, A_1{}^2)$, then $(A_1{}^1, A_2{}^2) \geqslant (B_2{}^1, B_1{}^2)$, which is the cancellation axiom assumed in L-T. Also note that Axiom 3 implies the transitivity of $\geqslant$.

(2D) THEOREM. *Suppose that Axioms 1-3 hold. If for A^i in $\mathscr{A}_i$, $i \neq j$, and X, Y in $\mathscr{A}_j$,*

$$(A^1, \cdots, A^{j-1}, X, A^{j+1}, \cdots, A^k) \geqslant (A^1, \cdots, A^{j-1}, Y, A^{j+1}, \cdots, A^i)$$

then for any B^i in $\mathscr{A}_i$, $i \neq j$,

$$(B^1, \cdots B^{j-1}, X, B^{j+1}, \cdots, B^k) \geqslant (B^1, \cdots, B^{j-1}, Y, B^{j+1}, \cdots, B^k).$$

PROOF. By Axiom 2, there exists an F in $\mathscr{A}_j$ such that

$$(A^1, \cdots, A^{j-1}, X, A^{j+1}, \cdots, A^k) = (B^1, \cdots, B^{j-1}, F, B^{j+1}, \cdots, B^k).$$

By the hypothesis and transitivity (Axiom 1)

$$(B^1, \cdots, B^{j-1}, F, B^{j+1}, \cdots, B^k) \geqslant (A^1, \cdots, A^{j-1}, Y, A^{j+1}, \cdots, A^k).$$

Axiom 3 applied to these two equations yields the result.

This result, which is known as the condition of *independence*, ensures that the following relation is well defined and is a weak ordering of $\mathscr{A}_j$.

(2E) DEFINITION. *Suppose that Axioms 1-3 hold and that X, Y are in $\mathscr{A}_j$. Define $X \geqslant Y$ if for some A^i in $\mathscr{A}_i$, and therefore for all A^i in $\mathscr{A}_i$, $i \neq j$,*

$$(A^i, \cdots, A^{j-1}, X, A^{j+1}, \cdots, A^k) \geqslant (A^1, \cdots, A^{j-1}, Y, A^{j+1}, \cdots, A^k).$$

REMARK. The use of this definition is referred to as "by independence". The term "transfer" was used in L-T, but other authors have called it "independence" and that seems the better term.

3. STANDARD SEQUENCES

If we fix an element from each $\mathscr{A}_j$, $j \neq r$, s, then by Theorem 2D we may suppress the notation for these coordinates, since the ordering of ordered k-tuples that vary only on the r and s coordinates is unaffected by the arbitrary, but fixed, choice for the other coordinates. In terms of just these two coordinates, we repeat the important notion defined in IXA of L-T.

(3A) DEFINITION. *A double infinite sequence of pairs* $\{A_i^r, A_i^s\}$, $i = 0, \pm 1, \pm 2, \cdots$, *where for each* i, A_i^r *is in* $\mathscr{A}_r$ *and* A_i^s *is in* $\mathscr{A}_s$, *is a* dual standard sequence (*abbreviated* dss) *if for each* i,

(i) $(A_i^r, A_{i+1}^s) = (A_{i+1}^r, A_i^s)$,

(ii) $(A_{i+1}^r, A_{i-1}^s) = (A_i^r, A_i^s)$.

The dss $\{A_i^r, A_i^s\}$ *is said to be* on (A_0^r, A_0^s) through A_1^r, *or* through A_1^s, *or* through (A_1^r, A_1^s) *as may be appropriate. It is said to be* increasing *if, for all* i, $A_i^r < A_{i+1}^r$ *and* $A_i^s < A_{i+1}^s$.

(3B) THEOREM. *If Axioms 1 to 3 hold, if* $\{A_i^r, A_i^s\}$ *is a* dss, *and if* m, n, p, *and* q *are integers, then* $(A_m^r, A_n^s) = (A_p^r, A_q^s)$ *whenever* $m + n = p + q$.

PROOF. Theorem IXD of L-T.

REMARK. The use of 3B is referred to "by dss."

(3C) THEOREM. *If Axioms 1 to 3 hold, there is a* dss *on* (A_0^r, A_0^s) *through* A_1^r, *or through* A_1^s, *or through* (A_1^r, A_1^s) *provided that* $(A_0^r, A_1^s) = (A_1^r, A_0^s)$.

PROOF. Theorem IXJ of L-T.

(3D) THEOREM. *If Axioms 1-3 hold and if* $\{A_i^r, A_i^s\}$ *and* $\{A_i^s, A_i^t\}$ *are* dss's, *then* $\{A_i^r, A_i^t\}$ *is a* dss.

PROOF. Suppress the notation for all coordinates except r, s, and t. Let m, n, p, and q be integers such that $m + n = p + q$. By dss and Theorem 2D,

$$(A_m^r, A_n^s, A_0^t) = (A_p^r, A_q^s, A_0^t)$$

and

$$(A_0^r, A_q^s, A_n^t) = (A_0^r, A_n^s, A_q^t).$$

An application of Axiom 3 to these two equations yields

$$(A_m^r, A_0^s, A_n^t) = (A_p^r, A_0^s, A_q^t),$$

thus showing that $\{A_i^r, A_i^t\}$ is a dss.

(3E) THEOREM. *If Axioms 1-3 hold and if* $\{A_i^r, A_i^s\}$ *is a* dss, *then for every* $t \neq r$, s *there exist* A_i^t *in* $\mathscr{A}_t$ *such that* $\{A_i^s, A_i^t\}$ *is a* dss.

PROOF. Suppress the notation for all coordinates except r, s, and t. Select any element of $\mathscr{A}_t$ as A_0^t. Define A_i^t to be the solution to

$$(A_0^r, A_0^s, A_i^t) = (A_0^r, A_i^s, A_0^t).$$

Suppose m, n, p, and q are integers such that $m + n = p + q$. Then

$$\begin{aligned}
(A_0^r, A_0^s, A_n^t) &= (A_0^r, A_n^s, A_0^t) && \text{(def. of } A_n^t\text{)}\\
&= (A_n^r, A_0^s, A_0^t) && \text{(by dss);}\\
(A_q^r, A_0^s, A_0^t) &= (A_0^r, A_q^s, A_0^t) && \text{(by dss)}\\
&= (A_0^r, A_0^s, A_q^t) && \text{(def. of } A_q^t\text{);}\\
(A_q^r, A_0^s, A_n^t) &= (A_n^r, A_0^s, A_q^t) && \text{(by Axiom 3);}\\
(A_n^r, A_m^s, A_0^t) &= (A_q^r, A_p^s, A_0^t) && \text{(by dss);}\\
(A_0^r, A_m^s, A_n^t) &= (A_0^r, A_p^s, A_q^t) && \text{(by Axiom 3);}
\end{aligned}$$

thus proving $\{A_i^s, A_i^t\}$ is a dss.

Because of this result, we introduce the following notion.

(3F) DEFINITION. *A sequence* $\{A_i^r\}$, $i = 0, \pm 1, \cdots$, *is a* standard sequence (ss) *of* $\mathscr{A}_r$ *if it is a member of some* dss. *It is* increasing *if* $A_i^r < A_{i+1}^r$ *for all* i.

Note that if A_0^r, $r = 1, \cdots, k$, are given, we can construct ss's $\{A_i^r\}$ such that for all $r, s = 1, 2, \cdots, k$, $r \neq s$, $\{A_i^r, A_i^s\}$ are dss's. This we do by constructing $\{A_i^1, A_i^2\}$ by 3C. Then for any $r \neq 1, 2$, $\{A_i^r\}$ is constructed as in 3E, and according to 3D all of the pairwise combinations are dss's.

A referee has pointed out that Theorems 3D and 3E can be readily proved using, instead of Axiom 3, the following two-coordinate generalization of independence: if

$$\begin{aligned}
&(A^1, \cdots, A^{i-1}, W, A^{i+1}, \cdots, A^{j-1}, X, A^{j+1}, \cdots, A^k)\\
&\geqslant (A^1, \cdots, A^{i-1}, Y, A^{i+1}, \cdots, A^{j-1}, Z, A^{j+1}, \cdots, A^k)
\end{aligned}$$

then

$$\begin{aligned}
&(B^1, \cdots, B^{i-1}, W, B^{i+1}, \cdots, B^{j-1}X, B^{j+1}, \cdots, B^k)\\
&\geqslant (B^1, \cdots, B^{i-1}, Y, B^{i+1}, \cdots, B^{j-1}, Z, B^{j+1}, \cdots, B^k).
\end{aligned}$$

This property cannot, however, simply be substituted for Axiom 3 because it does not lead to the L-T axioms for two coordinates, which are needed to prove Theorems 3B and 3C.

4. THE EXISTENCE OF REAL-VALUED FUNCTIONS FOR k COORDINATES

(4A) DEFINITION. *If* $\{A_i^r\}$ *is a set of elements of* $\mathscr{A}_r$, *its* convex cover $\mathscr{C}\{A_i^r\}$ *consists of all* A^r *in* $\mathscr{A}_r$ *such that* $A_m^r \leqslant A^r \leqslant A_n^r$ *for some* m *and* n.

(4B) Definition of ϕ_r. *Suppose that Axioms 1-3 hold. Let* $\{A_i^r\}$, $r = 1, 2, \cdots, k$, *be fixed increasing* ss's *such that every pair forms a dss. For any* X^r *in* $\mathscr{A}_r$, *let* $\{X_i^r\}$ *be a* ss *through* A_0^r *and* X. *Define*

$$\phi_r(X^r) = \inf \left\{ \frac{i}{m} \,\middle|\, A_i^r \geqslant X_m^r, m > 0 \right\} = \sup \left\{ \frac{i}{n} \,\middle|\, A_i^r \leqslant X_n^r, n > 0 \right\}.$$

(4C) Existence Theorem. *Suppose that Axioms 1 to 3 hold, that* $p_r > 0$, $r = 1, \cdots k$, *and* q *are given real numbers, and that* $\{A_i^r\}$, $r = 1, \cdots, k$, *are increasing* ss's *such that any pair forms a* dss. *There exist real-valued functions* θ *and* ϕ_r *defined on* $\prod_{r=1}^k \mathscr{C}\{A_i^r\}$ *and on* $\mathscr{C}\{A_i^r\}$, $r = 1, \cdots k$, *respectively, such that*

(i) $\theta(A^1, A^2, \cdots, A^k) = \sum_{r=1}^k p_r \phi_r(A^r) + q$;

(ii) $(A^1, A^2, \cdots, A^k) \geqslant (B^1, B^2, \cdots, B^k)$ *implies*

$$\theta(A^1, A^2, \cdots, A^k) \geqslant \theta(B^1, B^2, \cdots, B^k);$$

(iii) $A^r \geqslant B^r$ *implies* $\phi_r(A^r) \geqslant \phi_r(B^r)$, $r = 1, \cdots, k$;

(iv) $\phi_r(A_1^r) > \phi_r(A_0^r)$, $r = 1, \cdots, k$.

(4D) Corollary. *Suppose that the assumptions of* 4C *hold and that* $\{G_j^r\}$ *is a* ss *with* G_j^r *in* $\mathscr{C}\{A_i^r\}$ *for all* j, *then for any function* ϕ_r *that satisfies part* (iii) *of* 4C *over* $\mathscr{C}\{A_i^r\}$,

$$\phi_r(G_n^r) - \phi_r(G_0^r) = n[\phi_r(G_1^r) - \phi_r(G_0^r)].$$

(4E) Uniqueness Theorem. *Suppose that the assumptions of* 4C *hold. If* θ, ϕ_r *and* θ', ϕ_r' *are two sets of functions satisfying* 4C *with the same* p_r, q *and* $\{A_i^r\}$, *then there are real constants* $a > 0$ *and* b_r *such that*

$$\theta' = a\theta + \sum_{r=1}^k b_r$$

and

$$\phi_r' = a\phi_r + b_r,$$

i.e., *the* $k + 1$ *scales are interval scales with consistent units.*

The proofs of the above results are only slight rewordings of those given in Sec. XI of L-T. The functions defined in 4B are used in the proofs in the obvious way.

5. THE ARCHIMEDEAN AXIOM AND ITS CONSEQUENCES FOR k COORDINATES

We next add the following axiom:

(5A) Archimedean Axiom 4. *For any nontrivial* ss $\{A_i^r\}$, $1 \leqslant r \leqslant k$, *and any* B *in* $\mathscr{A}_r$, *there exist integers* m *and* n *such that* $A_m^r \leqslant B \leqslant A_n^r$.

(5B) LEMMA. *If Axioms 1 to 3 hold, Axiom 4 is equivalent to: if* $\{B_i^r\}$ *is a nontrivial* ss, *then* $\mathscr{C}\{B_i^r\} = \mathscr{A}_r$.

PROOF. A slight rewording of XIIA in L-T.

It is clear that with Axiom 4 appended, the θ and ϕ_r of 4C are defined on $\prod_{r=1}^{k} \mathscr{A}_r$ and $\mathscr{A}_r$, respectively. Moreover, parts (ii) and (iii) become equivalences rather than implications. The proof is the same as that of XIIB in L-T. The uniqueness theorem is unchanged.

6. RELATIONS AMONG STANDARD SEQUENCES

(6A) THEOREM. *Suppose that Axioms 1-3 hold. Let* $\{A_i\}$ *and* $\{B_i\}$ *be two increasing* ss's *on one coordinate, and let* ϕ_A *and* ϕ_B *be the real-valued functions defined by* 4B *relative to* $\{A_i\}$ *and* $\{B_i\}$, *respectively. Then either*

(i) $\mathscr{C}(A_i) \cap \mathscr{C}(B_i) = \phi$;

(ii) $\mathscr{C}(A_i) = \mathscr{C}(B_i)$, *in which case there exist constants* $a > 0$ *and* b *such that* $\phi_A = a\phi_B + b$;

(iii) $\mathscr{C}(A_i) \subset \mathscr{C}(B_i)$, *in which case* ϕ_B *is constant over* $\mathscr{C}(A_i)$; *or*

(iv) $\mathscr{C}(B_i) \subset \mathscr{C}(A_i)$, *in which case* ϕ_A *is constant over* $\mathscr{C}(B_i)$.

PROOF. The proof is broken down into three lemmas in each of which the hypotheses of Theorem 6A are implicitly assumed.

(6B) LEMMA. *Exactly one of the following is true:*

(i) $\mathscr{C}(A_i) \cap \mathscr{C}(B_i) = \phi$;

(ii) $\mathscr{C}(A_i) = \mathscr{C}(B_i)$;

(iii) *there exists an integer* k *such that for all integers* i, $B_k \geqslant A_i > B_{k-2}$; *or*

(iv) *there exists an integer* k *such that for all integers* i, $A_k \geqslant B_i > A_{k-2}$.

PROOF. Suppose that neither (i) nor (ii) holds, then we first show that there exists an integer k such that for all integers i at least one of the following four possibilities is true:

$$(1)\ B_k \geqslant A_i, \qquad (2)\ B_k \leqslant A_i, \qquad (3)\ A_k \geqslant B_i, \qquad \text{or} \qquad (4)\ A_k \leqslant B_i.$$

For suppose none were true, then by the negation of (1) for any k there exists i such that $A_i > B_k$ and by the negation of (2) there exists j such that $B_k > A_j$, hence $\mathscr{C}(B_i) \subseteq \mathscr{C}(A_i)$. In a similar way, by using the negation of (3) and (4), $\mathscr{C}(A_i) \subseteq \mathscr{C}(B_i)$, hence $\mathscr{C}(A_i) = \mathscr{C}(B_i)$, contrary to assumption.

We proceed on the assumption that (1) holds; the argument in the other three cases is similar. Let k be the smallest integer such that for all i, $B_k \geqslant A_i$. It exists since otherwise the ss $\{B_i\}$ would not be increasing or $\mathscr{C}(B_i) \cap \mathscr{C}(A_i) = \phi$, both of which are contrary to hypothesis. So, for at least one j, $A_j > B_{k-1}$. Let $\{A_i^*\}$, $\{B_i^*\}$ be ss's on a second coordinate such that $A_0^* = B_0^*$ and $\{A_i, A_i^*\}$ and $\{B_i, B_i^*\}$ are dss's. Since $B_k \geqslant A_i$, for all integers i,

$$(B_k, A_0^*) \geqslant (A_i, A_0^*) = (A_i, B_0^*),$$

and so by IXF of L-T,

$$(B_{k+m}, A_n^*) \geqslant (A_{i+n}, B_m^*),$$

for any integers m and n. Now for any integer i, consider

$$\begin{aligned}
(A_{2j-i}, A_0^*) &= (A_j, A_{j-i}^*) && \text{(by dss)} \\
&> (B_{k-1}, A_{j-i}^*) && \text{(by choice of } A_j) \\
&\geqslant (A_j, B_{-1}^*) && (m = -1, n = j - i \text{ in above equation)} \\
&> (B_{k-1}, B_{-1}^*) && \text{(by choice of } A_j) \\
&= (B_{k-2}, B_0^*) && \text{(by dss)} \\
&= (B_{k-2}, A_0^*) && \text{(by } A_0^* = B_0^*),
\end{aligned}$$

so by independence, $A_{2j-i} > B_{k-2}$. Since j is fixed and i is any integer, this proves (iii).

(6C) LEMMA. *If $\mathscr{C}(A_i) = \mathscr{C}(B_i)$ and if ϕ_A and ϕ_B are defined relative to the given ss's by* 4B, *then $\phi_A = a\phi_B + b$, where*

$$a = \frac{1}{\phi_B(A_1) - \phi_B(A_0)} > 0 \qquad \textit{and} \qquad b = \frac{-\phi_B(A_0)}{\phi_B(A_1) - \phi_B(A_0)}.$$

PROOF. With no loss of generality, suppose that for some X,

$$\phi_A(X) > \frac{\phi_B(X) - \phi_B(A_0)}{\phi_B(A_1) - \phi_B(A_0)}.$$

There exist integers n, m such that the rational number n/m lies between these two quantities, and does not equal either. Since

$$\phi_A(X) = \inf \left\{ \frac{i}{m} \,\middle|\, A_i \geqslant X_m{}^A, m > 0 \right\},$$

where $\{X_m{}^A\}$ is the ss with $X_0{}^A = A_0$, $X_1{}^A = X$, it follows that $A_n < X_m{}^A$. Then

$$\begin{aligned} n[\phi_B(A_1) - \phi_B(A_0)] + \phi_B(A_0) &= \phi_B(A_n) && \text{(Corollary 4D)} \\ &\leqslant \phi_B(X_m{}^A) && \text{(4C)} \\ &= m[\phi_B(X) - \phi_B(A_0)] + \phi_B(A_0), && \text{(Corollary 4D)} \end{aligned}$$

and so

$$\frac{n}{m} \leqslant \frac{\phi_B(X) - \phi_B(A_0)}{\phi_B(A_1) - \phi_B(A_0)},$$

contrary to choice. Therefore, $\phi_A = a\phi_B + b$, where a and b are defined as above.

(6D) LEMMA. *Suppose that an integer k exists such that for all integers i, $B_k \geqslant A_i > B_{k-2}$. If ϕ_B is the real-valued function of* 4B *relative to $\{B_i\}$, then for all integers i and j, $\phi_B(A_i) = \phi_B(A_j)$.*

PROOF. By Corollary 4D and by 4C, for all integers n,

$$\begin{aligned} n \mid \phi_B(A_1) - \phi_B(A_0) \mid &= \mid n[\phi_B(A_1) - \phi_B(A_0)] \mid \\ &= \mid \phi_B(A_n) - \phi_B(A_0) \mid \\ &\leqslant \mid \phi_B(B_k) - \phi_B(B_{k-2}) \mid \\ &= 2 \mid \phi_B(B_1) - \phi_B(B_0) \mid \\ &= \text{constant} < \infty. \end{aligned}$$

hence

$$\phi_B(A_n) - \phi_B(A_0) = 0.$$

7. WEAKENING OF THE SOLUTION-OF-EQUATIONS AXIOM

In this and the following two sections our attention will be focused on a set of conditions on an ordering of two-coordinate stimuli that are sufficient to imply the additive representation without, however, imposing the unrestricted solutions-of-equations axiom of L-T. We continue to work with a full Cartesian product $\mathscr{A}_1 \times \mathscr{A}_2$, but as is pointed out at the end of Sec. 9 this can be weakened considerably.

Many of the axioms, definitions, and results to follow require similar statements for both coordinates; we state one explicitly and merely remark that the other case is similar.

Let $\geqslant$ be a binary relation on $\mathscr{A}_1 \times \mathscr{A}_2$, where $\mathscr{A}_i$ are nonempty sets. Throughout, $A, \cdots, L$ are elements of $\mathscr{A}_1$ and $P, \cdots, Y$ are elements of $\mathscr{A}_2$.

(7A) ORDERING AXIOM 1. *$\geqslant$ is a weak ordering of $\mathscr{A}_1 \times \mathscr{A}_2$.*

(7B) CANCELLATION AXIOM 2. *If $(A, X) \geqslant (F, Q)$* and *$(F, P) \geqslant (B, X)$, then $(A, P) \geqslant (B, Q)$.*

(7C) RESTRICTED SOLUTIONS-OF-EQUATIONS AXIOM 3.

(i) *If there exist $\underline{F}, \bar{F}$ in $\mathscr{A}_1$ such that $(\bar{F}, Q) \geqslant (A, P) \geqslant (\underline{F}, Q)$, then there exists F in $\mathscr{A}_1$ such that $(F, Q) = (A, P)$.*

(ii) *Similar.*

(7D) INDEPENDENCE AXIOM 4.

(i) *$(A, P) \geqslant (B, P)$ if and only if $(A, Q) \geqslant (B, Q)$ for all Q in $\mathscr{A}_2$.*

(ii) *Similar.*

REMARK. In L-T, Axiom 4 was derived as Theorem VH from the first three axioms, but the proof depends upon having unrestricted solutions of equations. It is easy to show by examples that Axiom 4 does not follow from the present Axioms 1-3.[4] The following definition is justified by Axiom 4.

(7E) DEFINITION.

(i) *$A \geqslant B$ if and only if $(A, P) \geqslant (B, P)$* for some, and therefore all, P in $\mathscr{A}_2$.

(ii) *Similar.*

(7F) THEOREM. *If Axioms 1-4 hold, $\geqslant$ on $\mathscr{A}_i$ is a weak ordering.*

PROOF. Obvious.

(7G) DEFINITION.

(i) *A set $\{B_i \mid i$ in N, B_i in $\mathscr{A}_1\}$ is an* increasing standard sequence, ss, relative to *Q_0, Q_1 in $\mathscr{A}_2$ if*

(1) *N is an interval of at least two integers, conventionally* 0 *and* 1,

(2) *if $i, i+1$ are in N, then $B_{i+1} > B_i$,*

(3) *if $i, i+1$ are in N, $(B_{i+1}, Q_0) = (B_i, Q_1)$.*

(ii) *Similar.*

[4] A referee has pointed out that Axiom 3 above is somewhat similar in spirit to Debreu's (1960) Assumption 2.3 for choice probabilities.

(7H) Nontrivialness Axiom 5. *There exist (maximal) increasing* ss's $\{A_i \mid i \text{ in } N_1\}$ *and* $\{P_i \mid i \text{ in } N_2\}$ *such that*

(i) $\{A_i \mid i \text{ in } N_1\}$ *is a* ss *relative to* P_0 *and* P_1 *of* $\{P_i \mid i \text{ in } N_2\}$,

(ii) $\{P_i \mid i \text{ in } N_2\}$ *is a* ss *relative to* A_0 *and* A_1 *of* $\{A_i \mid i \text{ in } N_1\}$,

(iii) *neither* ss *is a proper subset of a* ss *for which (i) and (ii) hold,*

(iv) $\{-1, 0, 1, 2\} \subseteq N_1, N_2$.

Remark. In what follows, A_i and P_i will always refer to a fixed choice of *ss*'s fulfilling Axiom 5.

(7I) Archimedean Axiom 6.

(i) *If* $\{B_i \mid i \text{ in } N\}$ *is an increasing* ss *and there exist* C, C', *in* $\mathscr{A}_1$ *such that* $C \geqslant B_i \geqslant C'$ *for all* i *in* N, *then* N *is* finite.

(ii) *Similar.*

8. PRELIMINARY RESULTS

For brevity in this section, we suppress the blanket assumption that Axioms 1-6 hold.

(8A) Definition.

(i) *If* A *is in* $\mathscr{A}_1$, $A + 1$ *and* $A - 1$ *are those elements of* $\mathscr{A}_1$, *if they exist, that solve, respectively,*

$$(A + 1, P_0) = (A, P_1) \qquad \textit{and} \qquad (A - 1, P_1) = (A, P_0).$$

(ii) *Similar.*

(8B) Lemma.

(i) (1) *If* $A + 1$ *exists, then* $(A + 1) - 1$ *exists and* $= A$.

(2) *If* $A - 1$ *exists, then* $(A - 1) + 1$ *exists and* $= A$.

(3) *If* $A \leqslant B$ *and* $B + 1$ *exists, then* $A + 1$ *exists and* $A + 1 \leqslant B + 1$.

(4) *If* $A \leqslant B$ *and* $A - 1$ *exists, then* $B - 1$ *exists and* $A - 1 \leqslant B - 1$.

(5) *If* i, $i + 1$ *are in* N_1, *then* $A_{i+1} = A_i - 1$ *and* $A_i = A_{i+1} + 1$.

(ii) *Similar.*

Proof. (1) By Axiom 4 and 8A, $(A + 1, P_1) \geqslant (A + 1, P_0) = (A, P_1)$, so by Axiom 3, the solution $(A + 1) - 1$ exists. Moreover,

$$[(A + 1) - 1, P_1] = (A + 1, P_0) = (A, P_1),$$

so by 7E, $A = (A + 1) - 1$.

(2) Similar to (1).

(3) Since by 7E and 8A, $(B + 1, P_0) = (B, P_1) \geqslant (A, P_1) > (A, P_0)$, Axiom 3 implies that $A + 1$ exists, and $(B + 1, P_0) \geqslant (A, P_1) = (A + 1, P_0)$ implies that $B + 1 \geqslant A + 1$.

(4) Similar to (3).

(5) Immediate from 7H and 8A.

(8C) COROLLARY. *Suppose* $A_0 \leqslant B$, $C < A_1$ *and* $P_0 \leqslant Q$, $R < P_1$, *then* $(B - 1, Q) < (C + 1, R)$ *and* $(B, Q - 1) < (C, R + 1)$.

PROOF. By (4) and (5) of 8B and Axiom 5, $B - 1$ exists and is $< A_0$, and by (3) and (5) of 8B and Axiom 5, $C + 1$ exists and is $\geqslant A_1$, hence

$$\begin{aligned} (C + 1, R) &\geqslant (A_1, P_0) && \text{(7E)} \\ &= (A_0, P_1) && \text{(Axiom 5)} \\ &> (B - 1, Q). && \text{(7E).} \end{aligned}$$

The other case is similar.

(8D) LEMMA.

(i) *If* B, $B - 1$ *exist, then there exists an integer* i *such that* $B - 1 \leqslant A_i < B$.

(ii) *Similar.*

PROOF. If $B - 1 \leqslant A_0 < B$ we are done. Suppose $A_0 < B - 1$, and the N be the set of integers i for which $A_0 \leqslant A_i < B - 1$. By Axiom 6, N is finite; let k be the largest integer in N. By Lemma 8B, and the choice of k,

$$B - 1 \leqslant A_{k+1} = A_k + 1 < (B - 1) + 1 = B.$$

The proof is similar for $B \leqslant A_0$.

(8E) DEFINITION.

(i) *If* i *is a positive, zero, or negative integer and* B *is in* $\mathscr{A}_1$, *then* $B - i$ *is defined recursively as follows provided that the defining elements exist in* $\mathscr{A}_1$:

(1) $i = 0$, $B - 0 = B$,

(2) $i \geqslant 1$, $B - i = [B - (i - 1)] - 1$.

(3) $i \leqslant -1$, $B - i = [B - (i + 1)] + 1$.

(ii) *Similar.*

(8F) THEOREM.

(i) *If B is in $\mathscr{A}_1$, then there exists an integer i such that $B - i$ exists and*

$$A_0 \leqslant B - i < A_1 .$$

(ii) *Similar.*

PROOF. If $A_0 \leqslant B < A_1$, $i = 0$ fulfills the assertion. If $B \geqslant A_1$, then $B - 1$ exists and so by Lemma 8D there exists an i such that $B - 1 \leqslant A_i < B$. By Lemma 8B and a finite induction on i, $B - i$ satisfies the theorem. If $B < A_0$, the argument is similar.

The next three lemmas are triple cancellation laws that are analogous to the double one of Axiom 2.

(8G) LEMMA. *If $(F, P) \geqslant (G, X)$, $(G, Y) \geqslant (F, Q)$, $(A, X) \geqslant (B, Y)$, and if U solves $(F, U) = (A, X)$ and V solves $(F, V) = (B, Y)$, then $(A, P) \geqslant (B, Q)$.*

PROOF. By Axiom 2, $(A, X) = (F, U)$ and $(F, P) \geqslant (G, X)$ imply $(A, P) \geqslant (G, U)$. Since $(F, U) = (A, X) \geqslant (B, Y) = (F, V)$, Axiom 4 implies $(G, U) \geqslant (G, V)$. And by Axiom 2, $(F, V) = (B, Y)$ and $(G, Y) \geqslant (F, Q)$ imply $(G, V) \geqslant (B, Q)$. The assertion follows from Axiom 1.

(8H) LEMMA. *If $(A, X) \geqslant (F, Y)$, $(G, Y) \geqslant (B, X)$, $(F, P) \geqslant (G, Q)$, and if K solves $(K, X) = (F, P)$ and L solves $(L, X) = (G, Q)$, then $(A, P) \geqslant (B, Q)$.*

PROOF. Similar to 8G.

(8I) LEMMA. *If $(F, Y) \geqslant (G, X)$, $(G, P) \geqslant (F, Q)$, $(A, X) \geqslant (B, Y)$, and if K solves $(K, P) = (F, Y)$ and U solves $(B, U) = (G, X)$, then $(A, P) \geqslant (B, Q)$.*

PROOF. By Axiom 2, $(B, U) = (G, X)$ and $(A, X) \geqslant (B, Y)$ imply $(A, U) \geqslant (G, Y)$; and $(F, Y) = (K, P)$ and $(G, P) \geqslant (F, Q)$ imply $(G, Y) \geqslant (K, Q)$; so by Axiom 1, $(A, U) \geqslant (K, Q)$. This together with $(K, P) = (F, Y) \geqslant (G, X) = (B, U)$ implies the result by Axiom 2.

(8J) LEMMA. *If i, $i - 1$ are in N_1 and j, $j + 1$ are in N_2, then*

$$(A_i , P_j) = (A_{i-1} , P_{j+1}).$$

PROOF. For $i = 1$ or for $j = 0$ the assertion is true by Axiom 5. We prove it generally by induction. Suppose $i \geqslant 2$, $j \geqslant 1$ and that the result is true for all (i', j') for which $(i, j) > (i', j') \geqslant (1, 0)$. By using this induction hypothesis three times,

$$(A_{i-2} , P_j) = (A_{i-1} , P_{j-1})$$
$$(A_{i-1} , P_j) = (A_{i-2} , P_{j+1})$$
$$(A_i , P_{j-1}) = (A_{i-1} , P_j).$$

Together these equations imply the result by Lemma 8I provided that there is a K such that $(K, P_j) = (A_{i-2}, P_j)$ and a U such that $(A_{i-1}, U) = (A_{i-1}, P_{j-1})$. Obviously, $K = A_{i-2}$ and $U = P_{j-1}$ suffice. The proofs are similar when $i \geqslant 2, j \leqslant -1$, etc.

(8K) LEMMA. (i) *If* B, $B-1$, P_j, P_{j-1} *exist, then* $(B-1, P_j) = (B, P_{j-1})$.

(ii) *Similar.*

PROOF. We prove the lemma by induction on j. For $j = 1$ it is true by Definition 8A. Depending on whether $j > 1$ or < 1 and $B > A_1$ or $\leqslant A_1$, there are four cases to consider. As the proofs are similar, we give only the one for $j > 1$, $B > A_1$. By Lemma 8D there exists an integer i such that $B - 1 \leqslant A_i < B$. Clearly, $A_i \geqslant A_1$, so A_{i-1} exists. Thus,

$$\begin{aligned} (A_i, P_{j-2}) &= (A_{i-1}, P_{j-1}) && \text{(8J)} \\ (A_{i-1}, P_j) &= (A_i, P_{j-1}) && \text{(8J)} \\ (B-1, P_{j-1}) &= (B, P_{j-2}). && \text{(induction hypothesis).} \end{aligned}$$

The result follows from Lemma 8I provided that there exist solutions K to $(K, P_j) = (A_i, P_{j-2})$ and U to $(B, U) = (A_{i-1}, P_{j-1})$. By 8B and Axiom 5, A_{i-2} exists since $i - 2 \geqslant 1 - 2 = -1$, so $K = A_{i-2}$ fulfills the first equation by two applications of Lemma 8J. If $j \geqslant 3$,

$$\begin{aligned} (B, P_{j-3}) &= (B-1, P_{j-2}) && \text{(induction hypothesis)} \\ &\leqslant (A_i, P_{j-2}) && \text{(7E)} \\ &= (A_{i-1}, P_{j-1}) && \text{(8J)} \\ &< (B, P_{j-2}), && \text{(7E)} \end{aligned}$$

so U exists by Axiom 3. If $j = 2$, the same argument holds provided that we can show $(B, P_{-1}) \leqslant (B - 1, P_0)$, which does not follow from the induction hypothesis. If the contrary were so, then

$$\begin{aligned} (A_{i-1}, P_0) &= (A_i, P_{-1}) && \text{(8J)} \\ (A_i, P_0) &= (A_{i-1}, P_1) && \text{(Axiom 5)} \\ (B, P_{-1}) &> (B-1, P_0) && \text{(hypothesis)} \end{aligned}$$

which, according to Lemma 8I, implies the contradiction $(B, P_0) > (B - 1, P_1)$ (see Definition 8A) provided that there exist solutions K to $(K, P_0) = (A_{i-1}, P_0)$ and U to $(B - 1, U) = (A_i, P_{-1})$. Clearly $K = A_{i-1}$ satisfies the first. Since $B > A_1$, then $A_i \geqslant B - 1 > A_{i-1}$, and so

$$(B-1, P_{-1}) \leqslant (A_i, P_{-1}) = (A_{i-1}, P_0) < (B-1, P_0),$$

which by Axiom 3 guarantees the existence of U.

(8L) THEOREM. *If* B, $B-1$, Q, $Q-1$ *exist, then* $(B-1, Q) = (B, Q-1)$.

PROOF. By Lemma 8D, there exist integers i, j such that $B-1 \leqslant A_i < B$ and $Q-1 \leqslant P_j < Q$. If we can choose m and n so that

$$\begin{aligned} (A_{m+1}, P_n) &= (A_m, P_{n+1}) && \text{(8J)} \\ (A_m, Q) &= (A_{m+1}, Q-1) && \text{(8K)} \\ (B-1, P_{n+1}) &= (B, P_n), && \text{(8K)} \end{aligned}$$

and so that there exist solutions K to $(K, Q) = (A_{m+1}, P_n)$ and U to $(B, U) = (A_m, P_{n+1})$, then the result follows by Lemma 8I.

There are several cases to consider.

(1) If $i+1$ is in N_1 and $j+1$ is in N_2, let $m = i$, $n = j$. Since

$$\begin{aligned} (A_{i+1}, Q) &> (A_{i+1}, P_j) && \text{(7E)} \\ &= (A_i, P_{j+1}) && \text{(8J)} \\ &\geqslant (A_i, Q), && \text{(7E and 8B)} \end{aligned}$$

K exists by Axiom 3. The proof of the existence of U is similar.

(2) If $i+1$ is not in N_1 and $j+1$ and $j-1$ are in N_2, let $m = i-1$, $n = j$. Since

$$\begin{aligned} (A_i, Q) &> (A_i, P_j) && \text{(7E)} \\ &= (A_{m+1}, P_n) && \text{(Definition of } m, n) \\ &= (A_{i-1}, P_{j+1}) && \text{(8J and Axiom 5)} \\ &\geqslant (A_{i-1}, Q), && \text{(7E and 8B)} \end{aligned}$$

K exists by Axiom 3. Since

$$\begin{aligned} (B, P_j) &> (A_i, P_j) && \text{(7E)} \\ &= (A_{i-1}, P_{j+1}) && \text{(8J and Axiom 5)} \\ &\geqslant (B-2, P_{j+1}) && \text{(7E, Axiom 5, and 8B)} \\ &= (B, P_{j-1}), && \text{(8K twice)} \end{aligned}$$

U exists by Axiom 3.

(3) If $i+1$, $i-1$ are in N_1 and $j+1$ is not in N_2, let $m = i$, $n = j-1$ and the proof is similar to (2).

(4) If $i+1$ is not in N_1 and $j+1$ is not in N_2, let $m = i-1$, $n = j-1$ and it is easy to show that

$$(A_{i-1}, Q) \geqslant (A_{m+1}, P_n) \geqslant (A_{i-2}, Q) \quad \text{and} \quad (B, P_{j-1}) \geqslant (A_m, P_{n+1}) \geqslant (B, P_{j-2}).$$

(5) If $i+1$ is not in N_1 and $j-1$ is not in N_2, then a somewhat different proof is needed. Since $i+1$ is not in N_1, we know from Axiom 5 that $i, i-1, i-2$ are in N_1, and since $j-1$ is not in N_2, $j, j+1, j+2$ are in N_2. Thus,

$$\begin{aligned} (B-1, P_{j+1}) &= (B-2, P_{j+2}) && \text{(8K)} \\ (B-2, Q) &= (B-1, Q-1) && \text{(8B and Part 1 above)} \\ (B-1, P_{j+2}) &= (B, P_{j+1}), && \text{(8K)} \end{aligned}$$

and the result follows by Lemma 8I provided that there exist solutions K to $(K, Q) = (B-1, P_{j+1})$ and U to $(B, U) = (B-2, P_{j+2})$. Since

$$\begin{aligned} (B, Q) &> (B, P_j) && \text{(7E)} \\ &= (B-1, P_{j+1}) && \text{(8K)} \\ &\geqslant (B-1, Q), && \text{(7E and 8B)} \end{aligned}$$

Axiom 3 insures the existence of K. And since

$$\begin{aligned} (B, P_{j+2}) &> (B-2, P_{j+2}) && \text{(7E and 8B)} \\ &= (B, P_j), && \text{(8K twice)} \end{aligned}$$

Axiom 3 also insures the existence of U.

(6) If $i-1$ is not in N_1 and $j+1$ is not in N_2, the proof is similar to (5).

(8M) Theorem. (i) *If A, $A-1$, B, $B-1$ exist, $(A, P) \geqslant (B, Q)$ is equivalent to $(A-1, P) \geqslant (B-1, Q)$.*

(ii) *Similar.*

Proof. Suppose $(A, P) \geqslant (B, Q)$. Either $P \leqslant Q$ or $P > Q$. Suppose, first, $P \leqslant Q$. By Axiom 5, either $Q-1$ or $Q+1$ exists. Suppose the former, then

$$\begin{aligned} (A-1, Q) &= (A, Q-1) && \text{(8L)} \\ (B, Q-1) &= (B-1, Q) && \text{(8L)} \\ (A, P) &\geqslant (B, Q), && \text{(hypothesis)} \end{aligned}$$

and Lemma 8H implies $(A-1, P) \geqslant (B-1, Q)$ provided that there exist solutions K to $(K, Q) = (A, P)$ and L to $(L, Q) = (B, Q)$. By $Q \geqslant P$ and the hypothesis, $(A, Q) \geqslant (A, P) \geqslant (B, Q)$, so Axiom 3 ensures that K exists, and $L = B$ satisfies the second equation. If $Q-1$ does not exist, then substitute $Q+1$ for Q in the first two equations above, and the result follows provided that there exist solutions K to $(K, Q+1) = (A, P)$ and $(L, Q+1) = (B, Q)$. Since,

$$\begin{aligned} (A-1, Q+1) &= (A, Q) && \text{(8L)} \\ &\geqslant (A, P) && \text{(7E)} \\ &\geqslant (B, Q) && \text{(hypothesis)} \\ &= (B-1, Q+1), && \text{(8L)} \end{aligned}$$

Axiom 3 insures the existence of K, and by Theorem 8L $L = B - 1$ solves the second equation.

Next, we suppose $P > Q$. If $A \geqslant B$, then by Lemma 8B, $A - 1 \geqslant B - 1$, and the result follows by Axiom 4. So we assume $A < B$, and let us suppose that $(A - 1, P) \geqslant (B - 1, Q)$ is false. If $Q - 1$ exists, then according to Lemma 8H,

$$\begin{aligned} (B, Q - 1) &= (B - 1, Q) && \text{(8L)} \\ (A - 1, Q) &= (A, Q - 1) && \text{(8L)} \\ (B - 1, Q) &> (A - 1, P), && \text{(hypothesis)} \end{aligned}$$

imply the contradiction $(B, Q) > (A, P)$ provided that there exists K such that $(K, Q - 1) = (B - 1, Q) = (B, Q - 1)$ and U such that $(L, Q - 1) = (A - 1, P)$. Clearly $K = B$ satisfies the first equation, and since

$$\begin{aligned} (B, Q - 1) &= (B - 1, Q) && \text{(8L)} \\ &> (A - 1, P) && \text{(hypothesis)} \\ &> (A - 1, Q) && \text{(7E and } P > Q) \\ &= (A, Q - 1), && \text{(8L)} \end{aligned}$$

Axiom 3 implies L exists. If $Q - 1$ does not exist, replace Q by $Q + 1$ before using Lemma 8H, and $K = B - 1$ satisfies $(K, Q) = (B - 1, Q)$ and

$$(B - 1, Q) > (A - 1, P) > (A - 1, Q)$$

shows that L exists for which $(L, Q) = (A - 1, P)$.

The converse and part (ii) are proved similarly except that Lemma 8G rather than 8H is used in part (ii).

9. REPRESENTATION AND UNIQUENESS THEOREMS

To prove that an additive representation exists, we proceed roughly as follows. Let $I_1 = \{B \mid B \text{ in } \mathscr{A}_1 \text{ and } A_0 \leqslant B < A_1\}$ and $I_2 = \{Q \mid Q \text{ in } \mathscr{A}_2 \text{ and } P_0 \leqslant Q < P_1\}$. We construct a system that consists of countably many replicas of $\langle \geqslant, I_1 \times I_2 \rangle$, and we show that this system satisfies the L-T axioms and that the original system is isomorphically imbedded in the new system. The L-T representation restricted to this image of $\mathscr{A}_1 \times \mathscr{A}_2$ establishes the result.

(9A) Definition. *$\mathscr{A}_1^*$, $\mathscr{A}_2^*$, and $\geqslant$ on $\mathscr{A}_1^* \times \mathscr{A}_2^*$ are defined as follows:*

$$\mathscr{A}_1^* = \{(\mu, B) \mid \mu = 0, \pm 1, \pm 2, \cdots; B \text{ in } \mathscr{A}_1 \textit{ such that } A_0 \leqslant B < A_1\}$$

$$\mathscr{A}_2^* = \{(\nu, Q) \mid \nu = 0, \pm 1, \pm 2, \cdots; Q \text{ in } \mathscr{A}_2 \textit{ such that } P_0 \leqslant Q < P_1\}$$

$[(\mu, A), (\nu, P)] \geqslant [(\rho, Q), (\sigma, B)]$ *if and only if either*

(i) $\mu + \nu \geqslant \rho + \sigma + 2$,

(ii) $\mu + \nu = \rho + \sigma + 1$ *and* $(A, P) \geqslant (B - 1, Q)$,

(iii) $\mu + \nu = \rho + \sigma$ *and* $(A, P) \geqslant (B, Q)$,

(iv) $\mu + \nu = \rho + \sigma - 1$ *and* $(A - 1, P) \geqslant (B, Q)$.

(9B) THEOREM. *If Axioms 1-6 hold, then* $\langle \geqslant, \mathscr{A}_1^* \times \mathscr{A}_2^* \rangle$ *satisfies the Axioms of* L-T.

PROOF. Axiom 1 (weak ordering). Reflexiveness and connectedness are obvious. To show transitivity, suppose

$$[(\mu, A), (\nu, P)] \geqslant [(\rho, B), (\sigma, Q)]$$

and

$$[(\rho, B), (\sigma, Q)] \geqslant [(\kappa, C), (\lambda, R)].$$

There are 16 different cases, most of which involve only simple relations among the integers or these plus the use of transitivity and Lemma 8B in the original system. We omit these calculations and prove the remaining, somewhat different, cases.

(i) $\mu + \nu \geqslant \rho + \sigma + 2$, $\rho + \sigma = \kappa + \lambda - 1$, and $(B - 1, Q) \geqslant (C, R)$. If $\mu + \nu > \rho + \sigma + 2$, then $\mu + \nu \geqslant \kappa + \lambda + 2$ and transitivity follows. If $\mu + \nu = \rho + \sigma + 2$, then $\mu + \nu = \kappa + \lambda + 1$, and so we must show that $(A, P) \geqslant (C - 1, R)$. Suppose, on the contrary, $(C - 1, R) > (A, P)$, then by Lemma 8B $C = (C - 1) + 1$ exists and $A + 1$ exists by Axiom 5 because $A_0 \leqslant A < A_1$. So Theorem 8M implies $(C, R) > (A + 1, P)$ which together with $(B - 1, Q) \geqslant (C, R)$ implies that $(B - 1, Q) > (A + 1, P)$. This contradicts Corollary 8C.

(ii) $\mu + \nu = \rho + \sigma - 1$, $(A - 1, P) \geqslant (B, Q)$, $\rho + \sigma = \kappa + \lambda - 1$, and $(B - 1, Q) \geqslant (C, R)$. By Theorem 8M, $(B, Q) \geqslant (C + 1, R)$, and so by Axiom 1, $(A - 1, P) \geqslant (C + 1, R)$, which is impossible by Corollary 8C, hence this case is impossible.

Axiom 2 (Cancellation). Suppose $[(\mu, A), (\lambda, X)] \geqslant [(\kappa, F), (\sigma, Q)]$ and $[(\kappa, F), (\nu, P)] \geqslant [(\rho, B), (\lambda, X)]$, then we must show that

$$[(\mu, A), (\nu, P)] \geqslant [(\rho, B), (\sigma, Q)].$$

Again there are 16 cases, of which most involve only the simplest properties of the integers and a straightforward use of cancellation in the original system. The remaining ones are similar to the two that have been worked out for transitivity; in particular, they rest on Corollary 8C.

Axiom 3 (Solutions-of-Equations). Suppose (μ, A), (ρ, B), and (ν, P) are given, then the problem is to show that there exists (σ, X), $A_0 \leqslant X < A_1$, such that

$[(\rho, B), (\sigma, X)] = [(\mu, A), (\nu, P)]$. Let Y be a solution to $(B, Y) = (A, P)$, which by Axiom 3 exists because, according to Axiom 5 and Lemma 8B,

$$B - 1 < A_0 \leqslant A < A_1 \leqslant B + 1,$$

so

$$\begin{aligned} (B, P_2) &= (B + 1, P_1) && \text{(8K)} \\ &> (A, P) && \text{(7E)} \\ &> (B - 1, P_0) && \text{(7E)} \\ &= (B, P_{-1}). && \text{(8K)} \end{aligned}$$

Also, this argument shows that $P_{-1} \leqslant Y \leqslant P_2$. There are, therefore, three cases:

(i) if $P_{-1} \leqslant Y < P_0$, let $X = Y + 1$ and $\sigma = \mu + \nu - \rho - 1$,

(ii) if $P_0 \leqslant Y < P_1$, let $X = Y$ and $\sigma = \mu + \nu - \rho$,

(iii) if $P_1 \leqslant Y < P_2$, let $X = Y - 1$ and $\sigma = \mu + \nu - \rho + 1$.

By Definition 9A, (σ, X) solves the equation. The other half of the axiom is proved similarly.

Axiom 4 (Archimedean). Observe by Definition 9A $\{(\mu, A_0), (\mu, P_0)\}$, $\mu = 0, \pm 1, \cdots$, is a dss whose convex cover is $\mathscr{A}_1^* \times \mathscr{A}_2^*$. If the Archimedean axiom is false, then there is a dss whose convex cover is a proper subset of $\mathscr{A}_1^* \times \mathscr{A}_2^*$ and by Lemma 6B we know there is an integer k such that the infinity of first coordinate terms are bounded by the half open interval from (k, A_0) and $(k + 2, A_0)$. Thus, there must be an infinity of terms either between (k, A_0) and $(k + 1, A_0)$ or between $(k + 1, A_0)$ and $(k + 2, A_0)$. In either case, the terms must be of the form (σ, B_i), where $\sigma = k$ or $k + 1$ and $A_0 \leqslant B_i < A_1$. Moreover, $\{B_i\}$ is a ss in the original system; however, a bounded, infinite ss is impossible by Axiom 6, so the Archimedean axiom must hold in $\langle \geqslant, \mathscr{A}_1^* \times \mathscr{A}_2^* \rangle$.

(9C) Corollary. *There exist real-valued functions ϕ^* on $\mathscr{A}_1^*$ and ψ^* on $\mathscr{A}_2^*$ such that*

(i) $[(\mu, A), (\nu, P)] \geqslant [(\rho, B), (\sigma, Q)]$ *if and only if*

$$\phi^*(\mu, A) + \psi^*(\nu, P) \geqslant \phi^*(\rho, B) + \psi^*(\sigma, Q),$$

(ii) $(u, A) \geqslant (\rho, B)$ *if and only if* $\phi^*(u, A) \geqslant \phi^*(\rho, B)$,

(iii) $(\nu, P) \geqslant (\sigma, Q)$ *if and only if* $\psi^*(\nu, P) \geqslant \psi^*(\sigma, Q)$.

Proof. Theorem VID of L-T.

(9D) Corollary. *If ϕ^* and ψ^* are such that $\phi^*(0, A_0) = \psi^*(0, P_0) = 0$ and $\phi^*(1, A_0) = \psi^*(1, P_0) = 1$, then*

$$\phi^*(\mu, A) = \mu + \phi^*(0, A) \qquad \text{and} \qquad \psi^*(\nu, P) = \nu + \psi^*(0, P).$$

PROOF. Observe by Definition 9A,

$$[(\mu, A), (0, P_0)] = [(\mu - 1, A), (1, P_0)],$$

so by Corollary 9C, and the hypothesis,

$$\begin{aligned}\phi^*(\mu, A) &= \phi^*(\mu, A) + \psi^*(0, P_0)\\ &= \phi^*(\mu - 1, A) + \psi^*(1, P_0)\\ &= \phi^*(\mu - 1, A) + 1.\end{aligned}$$

A finite induction completes the proof.

(9E) DEFINITION. (i) *If A is in $\mathscr{A}_1$ and i is the integer such that $A_0 \leqslant A - i < A_1$ (such an i exists by Theorem* 8F*), define*

$$\phi(A) = \phi^*(0, A - i) + i,$$

where ϕ^* is the function of Corollary 9C *for which $\phi^*(0, A_0) = 0$, $\phi^*(1, A_0) = 1$.*

(ii) *Similar.*

(9F) REPRESENTATION THEOREM. *If $\langle \geqslant, \mathscr{A}_1 \times \mathscr{A}_2 \rangle$ satisfies Axioms 1-6 and if ϕ, ψ are defined by* 9E, *then*

(i) $(A, P) \geqslant (B, Q)$ *if and only if* $\phi(A) + \psi(P) \geqslant \phi(B) + \psi(Q)$,

(ii) $A \geqslant B$ *if and only if* $\phi(A) \geqslant \phi(B)$,

(iii) $P \geqslant Q$ *if and only if* $\psi(P) \geqslant \psi(Q)$.

PROOF. By Corollary 9D and Definition 9E, the proof of (i) is equivalent to showing $(A, P) \geqslant (B, Q)$ if and only if $[(\mu, A - \mu), (\nu, P - \nu)] \geqslant [(\rho, B - \rho), (\sigma, Q - \sigma)]$, where μ, ν, ρ, and σ are chosen (by Theorem 8F) so that $A_0 \leqslant A - \mu$, $B - \rho < A_1$ and $P_0 \leqslant P - \nu$, $Q - \sigma < P_1$. First, suppose $(A, P) \geqslant (B, Q)$. There are 5 possible relations among the integers:

(1) $\mu + \nu \geqslant \rho + \sigma + 2$ in which case the conclusion holds by Definition 9A.

(2) $\mu + \nu = \rho + \sigma + 1$. A finite induction using Theorems 8L and 8M yields $(A - \mu, P - \nu) \geqslant [(B - \rho) - 1, Q - \sigma]$, which proves the conclusion by 9A.

(3) $\mu + \nu = \rho + \sigma$. Similar to (2).

(4) $\mu + \nu = \rho + \sigma - 1$. Similar to (2).

(5) $\mu + \nu \leqslant \rho + \sigma - 2$. Let $\kappa = \rho + \sigma - \mu - \nu \geqslant 2$, then a finite induction using Theorems 8L and 8M results in

$$[(A - \mu) - 1, P - \nu] \geqslant [(B - \rho) + \kappa - 1, Q - \sigma] \geqslant [(B - \rho) + 1, Q - \sigma],$$

which is impossible by Corollary 8C.

The converse is similar.

Parts (ii) and (iii) follow immediately from (i) and Definition 7E.

(9G) UNIQUENESS THEOREM. *If ϕ, ψ and ϕ', ψ' are two pairs of functions that each satisfy Theorem* 9F, *then there exist a positive constant $a > 0$ and constants b and c such that*

$$\phi = a\phi' + b \qquad \text{and} \qquad \psi = a\psi' + c.$$

PROOF. Under positive linear transformations there is no loss of generality in assuming that the given functions satisfy $\phi(A_0) = \phi'(A_0) = \psi(P_0) = \psi'(P_0) = 0$ and $\phi(A_1) = \phi'(A_1) = \psi(P_1) = \psi'(P_1) = 1$. Note that because of part (i) of 9F and the fact that $(A_0, P_1) = (A_1, P_0)$, the slopes of the transformations are the same within each pair. Moreover, $(A - 1, P_1) = (A, P_0)$, Corollary 8C, and a finite induction show that

$$\phi(A) = i + \phi(A - i) \qquad \phi'(A) = i + \phi'(A - i)$$

$$\psi(P) = j + \psi(P - j) \qquad \psi'(P) = j + \psi'(P - j),$$

where $A_0 \leqslant A - i < A_1$ and $P_0 \leqslant P - j < P_1$, so it is sufficient to show that ϕ and ψ are unique in these intervals. We deal explicitly only with ϕ.

For $A_0 \leqslant A < A_1$, Axiom 3 states that there exists A^*, $P_0 \leqslant A^* < P_1$ such that $(A_0, A^*) = (A, P_0)$. Following the construction used in Theorem IXJ of L-T, we can construct a maximal ss $\{A_i^1 \mid i \text{ in } N\}$ relative to P_0, A^* such that $A_0^1 = A_0$, $A_1^1 = A$, and for i in N, $A_0 \leqslant A_i^1 \leqslant A_1$. By Axiom 5, N is finite; let

$$n_1 = \max\{i \mid i \text{ in } N\}.$$

Following the proof of Corollary XIM of L-T, we see that

$$\phi(A^1_{n_1}) = n_1\phi(A_1^1) = n_1\phi(A).$$

The residue remaining between $A^1_{n_1}$ and A_1 can be reflected down to A_0 by defining A^{2*} to be the solution to $(A^1_{n_1}, A^{2*}) = (A_0, P_0)$ and A^2 to be the solution to $(A^2, P_0) = (A_0, A^{2*})$. It is easy to see that both solutions exist by Axiom 3 and that $A^2 < A$. By Theorem 9F,

$$n_1\phi(A) + \psi(A^{2*}) = 1 + 0, \qquad \phi(A^2) + 0 = 0 + \psi(A^{2*}),$$

so

$$n_1\phi(A) + \phi(A^2) = 1.$$

We repeat the process on A^2, letting n_2 be the integer that corresponds to n_1. We proceed inductively. If for any i, $(A^i)_{n_i} = A_1$, we set all $n_{i+j} = \infty$ for $j = 1, 2, \cdots$.

Moreover, since $A^i < A^{i-1}$, we see that $n_{i+1} \geqslant n_i + 1$. It follows immediately by induction that

$$\phi(A) = \sum_{i=1}^{\infty} (-1)^{i+1} \Big/ \prod_{j=1}^{i} n_j ,$$

and this series is convergent because $n_{i+1} \geqslant n_i + 1$ implies that it is absolutely convergent. Thus, because of the uniqueness of the n_i, ϕ is unique, thereby proving the theorem.

In closing, three remarks should be made. First, the construction given in the proof of Theorem 9G may be a practical way to estimate the function ϕ when we are dealing with a (bounded) situation that satisfies Axioms 1-6, but not the L-T axioms. Second, it seems plausible that this definition of ϕ, rather than the indirect one of Definition 9E, can be used to prove the Representation Theorem (9F), but I have been unable to see how to construct such a proof. Third, in the event that the ordering $\geqslant$ is over a proper subset S of $\mathscr{A}_1 \times \mathscr{A}_2$, the representation and existence theorems hold for all "rectangular" subsets $\mathscr{A}_1' \times \mathscr{A}_2'$ of S for which the axioms are true. If two such regions overlap nontrivially, the scales can be transformed linearly so that they coincide over the region of overlap, and in this way the representation can be extended throughout much, if not all, of any subset S that is likely to be of interest to psychologists provided that the coordinates are physically "continuous."

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