

## A Model for Detection in Temporally Unstructured Experiments with a Poisson Distribution of Signal Presentations<sup>1,2</sup>

R. DUNCAN LUCE

*University of Pennsylvania, Philadelphia, Pennsylvania*

Suppose that a subject responds, after a random delay, to some random fraction  $b$  of internal detection states and that during the response period he ignores any new detection states. These states are assumed to occur both with probability  $q$  whenever a signal is presented and spontaneously according to a Poisson process with intensity  $\nu$ . Expressions are developed for two inter-response densities and one signal-response density when the signal presentations are Poisson distributed, and equations are derived to estimate  $bq$  and  $b\nu$  and to test the model. Two methods whereby  $b$  might be estimated independently of  $\nu$  and  $q$  are discussed.

### 1. INTRODUCTION

The most widely used experimental abstractions of the familiar task of detecting difficult-to-detect signals currently employ well-defined, short time intervals during which the subject knows that a signal may be presented.<sup>3</sup> Following each such interval he is required to respond concerning his belief about what was just presented. This temporal structuring does not correspond very closely to most detection situations confronted in real life, since short time periods during which signals may occur are not usually identified. Aside from the literature on vigilance tasks (see Broadbent and Gregory, 1963, and the references given there), which is focussed to some degree

<sup>1</sup> This research was supported in part by National Science Foundation grant NSF GB 1462 to The University of Pennsylvania; much of the work was carried out while I was a participant in two Senior Seminars in Mathematical Psychology held under the sponsorship of the Social Science Research Council at Stanford University during the summers of 1963 and 1964. I am indebted to D. M. Green, A. R. Jonckheere, W. D. Larkin, Richard Rose, Saul Sternberg, and Jack Yellott for discussions while the work was in progress, and to the referees for their helpful criticisms and corrections.

<sup>2</sup> A summary of these results was given at the Symposium über psychologische Probleme kybernetischer Forschung, Berlin, September 3-6, 1964. That paper is to be published in translation (Luce, 1965).

<sup>3</sup> I shall follow the convention of the field of acoustics where the experimentally presented stimulus, which is buried in a statistically stationary (noise or constant) background, is called a "signal." For some experimental realizations this term may seem strange.

on the degradation of performance over time when the signals are very infrequent, the only temporally unstructured detection experiment with which I am familiar is that of Egan, Greenberg, and Schulman (1961). The somewhat arbitrary analyses used in all of the published experiments, along with some unpublished theoretical work of Mr. Jack Yellott of Stanford University on a signal detection model for temporally unstructured detection experiments, have, in part, prompted the present analysis in terms of a simple threshold theory.

Egan *et al.* spoke of "the method of free response," Yellott used the phrase "continuous presentation conditions," and Broadbent and others refer to "vigilance experiments." Each expression is suggestive of a central feature of the design, but I feel that the essence of the matter is better summarized from the subject's point of view as a lack of temporal structuring except, perhaps, for vigilance experiments in which attention may be the dominant factor. The other central fact about any such experiment is the temporal schedule of signal occurrences used by the experimenter. Since, within the limits of the technology available to him, it can be anything he wants, he should select a schedule that is in some sense optimal for him. There is little doubt that a Poisson distribution (i.e., the expected number of signals occurring during any period of time is proportional to the length of that period or, equivalently, the distribution of inter-signal times is exponential) is both a good abstraction of many real detection situations and, for the model presented here, one that makes the mathematical analysis especially tractable.

Before entering into the details of the present model, it may be helpful if I indicate what I consider unsatisfactory about the existing analyses. The difficulties derive from two general propositions which everyone seems to accept. The first is that some responses are not initiated by a signal presented by the experimenter; these are the so-called noise-induced or spontaneous responses. The second is that when a response is initiated by a signal, the time that elapses between the presentation of the signal and the resulting response is not always the same. Together, these propositions<sup>4</sup> imply that (1) we can never be certain whether a particular response was due to a signal or whether it was spontaneously generated, and (2) even if we were assured that a response was due to some signal and if the signals come at all close together, we can not be certain to which signal it was due. This ambiguity is inherent in the situation, and no cute tricks of analysis can dispel it.

<sup>4</sup> It should be noted that neither of the basic propositions can really be considered a fact. The first is factually true for experiments with defined trials and this is accepted as presumptive evidence that it is also true in temporally unstructured experiments, but of that we cannot really be certain unless we assume that reaction time is truly a constant. The second—the nonconstancy of reaction time—could, of course, be false even though the observed reaction time to a signal is variable, since if the first proposition is true the signal may not have initiated the response.

The published analyses do not, in my opinion, sufficiently acknowledge this ambiguity. Egan *et al.* plotted the frequency of response times following a signal, and they found a function which appears to be a reaction time distribution “added” to a uniform base rate of random responding. To get an estimate of the distribution of reaction times, they simply subtracted the estimated base rate from the observed frequency function. Although this may very well turn out to be a good approximation to the distribution of reaction times, it is not obvious that in fact it is. Until a detailed theoretical analysis of the underlying processes is provided, the matter is bound to remain one of opinion. The procedure used by Broadbent and Gregory (1963)—in which time is arbitrarily rendered discrete, the resulting intervals are treated as trials of a Yes-No (actually, in their study, a rating scale) experiment, and the data are summarized as the resulting “estimates” of hit and false alarm probabilities—is, at best, a stopgap analysis. Not only does this method ignore the ambiguity mentioned above, but the resulting numbers are not invariant under changes in the arbitrary choice of the length of the time intervals.

Whatever its failings may be, the model described here at least seriously acknowledges the two propositions mentioned above and the resulting ambiguity about which signal, if any, actually initiated a given response. Specifically, the only data to which we shall suppose that we have access are the signal and response time series. From these, certain inter-response and signal-response distributions can be estimated and so derived theoretical relations among these distributions can be tested.

## 2. THE TWO-STATE THRESHOLD MODEL

Several studies (Békésy, 1930; Larkin and Norman, 1964; Miller and Garner, 1944; Neisser, 1955; Norman, 1962a, b, 1963, 1964; and Stevens, Morgan and Volkman, 1941) of detection in temporally structured experiments have lent some support to the neural quantum theory (first stated by Békésy, 1930; for a summary, see Luce, 1963b) which argues that at least some simple signals which vary in only one physical dimension produce a (significantly) discrete representation within the subject. Moreover, according to the theory, this discreteness is reflected in his response behavior when certain functions are plotted. Evidence against the theory has also been presented (Blackwell, 1953a, b; Corso, 1956; and a number of signal detection studies that can be found in Swets, 1964), which in turn has been reanalyzed and re-evaluated in terms of a discrete representation (Luce, 1963a, b; Norman, 1962b, 1964). The issue is far from settled, but neural quantum theory appears to have sufficient viability to warrant investigating its consequences for temporally unstructured experiments.

Assuming that such a theory is correct, the experimental data from two-response (Yes-No and two-alternative forced-choice) situations suggest that, when signal intensity is held constant, it is reasonable to treat the subject as having a stable criterion

that divides the number of neural quanta activated into two classes,  $D$  and  $\bar{D}$ , which can be referred to as detect and nondetect states, respectively. Thus, were we able to observe these internal states directly, which of course we cannot, we would find that at any instant the subject is in one or the other state and that his response pattern differs considerably depending upon which he is in. For the temporally unstructured situation, I shall suppose, as a matter of fact, that most of the time he is in the  $\bar{D}$ -state and that only occasionally—usually when a signal is presented but also sometimes when there is none—does he pass into the  $\bar{D}$ -state, and then only for a brief period. When he enters a  $\bar{D}$ -state, there is some tendency for this to initiate a process which ultimately leads the subject to respond to the effect that he believes that a signal has just been presented (at the time the response process was initiated). As a mathematical idealization, I shall suppose that the occurrence of a  $D$ -state can be represented as a mathematical point—i.e., its duration is nil—and so the over-all process can be treated as a renewal counting process in the technical sense (Parzen, 1962, p. 160). Because of the nature of the response process that is later postulated, the results are little affected if  $D$ -states have finite duration provided only that it is short relative to the latency of the response mechanism and to the mean time between successive  $D$ -states.

Clearly, we anticipate that the occurrence of  $D$ -states is considerably more likely when a signal is presented than when there is none. We shall suppose that each of these two different rates of occurrence can be summarized by a number. First, we postulate a time-independent conditional probability,

$$q = \Pr(D \mid \text{signal at time } \tau),$$

that a  $D$ -state occurs when a signal (of fixed intensity and duration) is presented at time  $\tau$ . Should the signals be varied in some manner from presentation to presentation, then  $q$  would depend on which signal was presented. We assume, however, that the same signal is used for each presentation. Second, in the absence of a signal, fluctuations either within the subject, in the background stimulation (e.g., noise), or in both cause random occurrences of  $D$ -states. Furthermore, we shall suppose that these spontaneous occurrences are uniformly distributed in the sense that during any time period  $t$  the expected number is  $t\nu$ , where  $\nu$  is known as the intensity (or mean rate) of the process (see Parzen, 1962, p. 140). This is a well known necessary and sufficient condition for the occurrence of  $D$ -states to form a Poisson process, and so the distribution of inter-occurrence times is exponential with mean  $1/\nu$  (Parzen, 1962, pp. 135, 174, 177). Such an assumption seems highly appropriate for large  $t$  provided that we believe spontaneous detections to be statistically stationary in the large; it may, however, break down in the small if there are any subtle sequential effects, such as neural refractory periods, local facilitation, and the like. Too little seems to be known about these local effects, especially for structures as abstract and

complex as  $D$ -states, to make sensible alternative assumptions, so I shall examine the simplest assumption here.

As was mentioned earlier, we suppose that the experimenter uses a Poisson schedule of signals; let  $\lambda$  denote the intensity of this process. Since a random proportion  $q$  of the signals lead to  $D$ -states, the  $D$ -states due to the signals are Poisson distributed with intensity  $q\lambda$ . If, as we shall suppose, the spontaneous occurrence of  $D$ -states is independent of their occurrence due to signals, the over-all distribution of  $D$ -states is Poisson with intensity  $\eta = q\lambda + \nu$  (Parzen, 1962, p. 35).

The analysis of temporally structured detection experiments strongly suggests that, if the neural quantum theory is correct, responses are not generally in one-to-one correspondence with  $D$ -states. Rather, subjects appear to choose to respond to some proportion of the  $D$ -states and, under some circumstances, to some different proportion of the  $\bar{D}$ -states. In the temporally unstructured situation, where for the most part the subject is in the  $\bar{D}$ -state, it seems plausible to assume that he responds to some proportion of the  $D$ -states and to none of the  $\bar{D}$ -states. Specifically, we shall suppose that the conditional probability that the response mechanism  $R$  is activated when a  $D$ -state occurs at time  $\tau$  is  $b(\tau)$ , i.e.,

$$b(\tau) = \Pr(R \mid D \text{ at time } \tau).$$

For most of the analysis we assume that  $b(\tau)$  is a constant,  $b$ .

Since it is well known, again from temporally structured experiments, that responses to signals involve a delay—a reaction time—we shall suppose that once a response mechanism is initiated there is a random delay  $t$  before the response actually occurs. Let  $r(t)$  denote the (unknown) probability density function of the reaction time  $t$ .

A crucial assumption—crucial in the sense that it makes the mathematics especially simple and that it may well be wrong—is that whenever the response mechanism is activated, the process is carried through to completion—to a response—with no further interference from or recording of events in the sensory parts of the system. Put another way, once the response mechanism begins, the person in effect becomes insensitive to signals or to apparent signals (i.e., spontaneous  $D$ -states) until the response finally occurs, at which point he again becomes sensitive with, however, no memory of the occurrence of  $D$ -states prior to the response. This corresponds to the assumption that characterizes nonparalyzable nuclear particle counters (Parzen, 1962, p. 164).

Several alternative assumptions are possible, and they should be investigated in detail, but in all cases that I have considered the mathematics is vastly more complicated than that discussed here. One possible assumption is that once a response mechanism is initiated, it terminates either when the response occurs or when a second response mechanism is initiated, whichever occurs first. Another possibility is that each initiated response process terminates in a response and that when two or more processes are initiated they are carried out in parallel. Clearly, the time order of the

responses need not be the same as the order of the  $D$ -states that initiated them. Numerous other possibilities are easy to devise, although these two are perhaps the simplest and therefore the most reasonable to investigate first.

In summary, then, the model postulated involves the following assumptions:

1. At each instant the subject is in exactly one of two possible states,  $D$  or  $\bar{D}$ .
2. The  $D$ -states occur only at discrete points in time as the result of two statistically independent processes, namely:
  - (a) a spontaneous (internal) process that is independent of signal presentations and that is Poisson distributed with intensity  $\nu$ ; and
  - (b) a signal process characterized by the property that with probability  $q$  a  $D$ -state occurs whenever a signal is presented.
3. If a  $D$ -state occurs at time  $\tau$  and if no response process is already under way, then with probability  $b(\tau)$  a response process is initiated; if a response process is already under way when a  $D$ -state occurs, no new response process is initiated.
4. When a response process is activated at time  $\tau$ , the probability density that it terminates in a response at time  $\tau + t$  is independent both of  $\tau$  and of the occurrence of any  $D$ -states in the interval  $(\tau, \tau + t)$ ; this density, which of course depends upon  $t$ , is denoted  $r(t)$ , and it is assumed to satisfy:

$$(a) \int_0^{\infty} r(t)dt = 1;$$

$$(b) r(0) = 0;$$

$$(c) r \text{ is continuous.}$$

These assumptions are all quite general ones about people, except possible 4b and c which place empirically weak restrictions on the reaction time distribution. The next assumption selects a specific schedule of signal presentations, and the final one further restricts the reaction time distribution.

5. The distribution of signals is Poisson with intensity  $\lambda$ , and so by Assumption 2 the distribution of  $D$ -states is Poisson with intensity  $\eta = q\lambda + \nu$ .

6. The tail of the reaction time distribution approaches 0 sufficiently rapidly so that

$$\int_0^{\infty} e^{\eta t} r(t) dt < \infty,$$

where  $\eta$  is defined in Assumption 5.

## 3. THE STANDARD YES-NO EXPERIMENT

Although our primary concern is with temporally unstructured experiments, the theory would not be especially attractive if we were unable to specialize it to temporally structured ones. Obviously, in doing so we will have to modify Assumption 3 to apply only to  $D$ -states that occur during specified intervals, to ignore Assumptions 4 and 6, and to abandon 5. In the Yes-No design, well-defined time intervals of duration  $\epsilon$  are marked off for the subject (usually by easily detected signals in some modality other than the one being studied) and he is to report whether or not a signal has been presented during that interval. For the occurrence of  $D$ -states, two conditional probabilities are relevant (see Luce, 1963b):

$$q_{\epsilon}(s) = \Pr(D \mid \text{signal } s \text{ is presented during a specified interval of duration } \epsilon)$$

$$q_{\epsilon}(n) = \Pr(D \mid \text{no signal is presented during a specified interval of duration } \epsilon).$$

It is known that when the inter-occurrence times are exponentially distributed, as they are in a Poisson process, then the excess life from an arbitrary time  $\tau$  to the next occurrence of an event is also exponentially distributed, and with the same intensity as that of the original process (Parzen, 1962, p. 173). Thus, if the subject attends for an interval of  $\delta$  seconds that includes the potential signal interval of  $\epsilon$  seconds and if no signal is presented, we have

$$q_{\epsilon}(n) = \int_0^{\delta} \nu e^{-\nu t} dt = 1 - e^{-\nu \delta}, \quad (1)$$

or rewriting,

$$\nu = -\frac{\ln[1 - q_{\epsilon}(n)]}{\delta}. \quad (2)$$

Exactly what value to take for  $\delta$  is uncertain because, in part, it is under the control of the subject. Since, however, in many experiments the interval  $\epsilon$  indicated to the subjects equals the signal duration, a reasonable lower bound for  $\delta$  in these cases is  $\epsilon$ . In several such experiments  $\epsilon$  has been of the order of 150 ms and  $q_{\epsilon}(n)$  has been estimated between 0.05 and 0.10 (Larkin and Norman, 1964; Norman, 1962a, b, 1963). Substituting in Eq. 2, the mean time between  $D$ -states not due to signals,  $1/\nu$ , is estimated to be in the range from about 1.5 to 3 sec. If the effective period of observation,  $\delta$ , were actually longer, as many believe it may be, these estimates must be increased by the factor  $\delta/\epsilon$ . In any event, they appear to be consistent with the idea, postulated in neural quantum theory, that a quantal grid is fluctuating slowly and that whenever the internal representation of the signal or background crosses a sufficient number of quantal boundaries a  $D$ -state arises. Had  $1/\nu$  turned out to be much less than a second, it would have been difficult to view the occurrence of  $D$ -states as reflecting a slowly fluctuating process within the subject.

## 4. DISTRIBUTIONS OF TWO INTER-RESPONSE TIMES

Returning to the temporally unstructured experiment, the only observables are the succession of times at which signals and responses occur, and so it is from these, and these alone, that we must estimate the unknown parameters ( $q$  and  $\nu$ ) and the unknown functions ( $b$  and  $r$ ) and then attempt to test the adequacy of the model. For present purposes, I shall suppose that the process has been under way a sufficiently long time that the response bias  $b(\tau)$  has settled down to an asymptotic value  $b$  which is independent of time. Later, in Section 7, we study a possible mechanism for changes in  $b$  and determine an explicit expression for the asymptotic expected value of the bias.

Under this assumption that  $b$  is constant, I propose to determine in this section expressions for two inter-response distributions, namely, the density  $f(t)$  that the time is  $t$  between one response and the next, and the joint density  $g^*(t)$  that the time is  $t$  between one response and the next and that no signal occurs between the two responses.

To calculate  $f$ , let us suppose that a response has occurred at time  $\tau$  and that the next one occurs at time  $t + \tau$ . By our assumptions, this is possible if and only if a response mechanism was activated at some time  $\tau + x$ , where  $0 \leq x \leq t$ , and that the reaction time was  $t - x$ . Since the activation of the response mechanism is Poisson with intensity

$$b\eta = bq\lambda + b\nu, \quad (3)$$

we see that  $f$  is given by the following convolution:

$$\begin{aligned} f(t) &= \int_0^t b\eta e^{-b\eta x} r(t-x) dx \\ &= b\eta e^{-b\eta t} \int_0^t e^{b\eta y} r(y) dy, \end{aligned} \quad (4)$$

where we have set  $y = t - x$ . Note that by setting  $t = 0$  in Eq. 4,  $f(0) = 0$ , and that by Assumption 6  $f$  is asymptotically exponential with parameter  $-b\eta$ .

If  $\phi$  is any continuous non-negative function that is asymptotically exponential with parameter  $-\alpha$ , then its moment generating function (*m.g.f.*) is defined in the usual way for all  $\theta < \alpha$ :

$$M_\phi(\theta) = \int_0^\infty e^{\theta t} \phi(t) dt. \quad (5)$$

THEOREM 1. For all  $\theta < b\eta$ ,

$$M_r(\theta) = \left(1 - \frac{\theta}{b\eta}\right) M_f(\theta). \quad (6)$$

PROOF. By Eq. 4,  $f$  is the convolution of  $r$  and an exponential, and so by a well known theorem

$$M_f(\theta) = \frac{b\eta}{b\eta - \theta} M_r(\theta),$$

from which Eq. 6 follows immediately.

COROLLARY. For all  $t \geq 0$ ,

$$r(t) = f(t) + f'(t)/b\eta. \quad (7)$$

PROOF. Because  $f(0) = 0$ , it follows readily from Eq. 5 that

$$M_{f'}(\theta) = -\theta M_f(\theta).$$

Substituting this in Eq. 6 and taking the inverse transform yields Eq. 7.

The joint density  $g^*$  can be derived as follows. Consider two responses with no signal between them. Let the time of the first response be denoted by  $\tau$ , the time of the second be  $\tau + t$ , and the time of the first signal after  $\tau$  be  $\tau + x$ . Since  $x \geq t$ , the response must have been activated by the Poisson process with intensity  $b\nu$  and that is independent of the Poisson process governing the signals, so

$$\begin{aligned} g^*(t) &= \int_t^\infty \left[ \int_0^t b\nu e^{-b\nu y} r(t-y) dy \right] \lambda e^{-\lambda x} dx \\ &= e^{-\lambda t} g(t), \end{aligned}$$

where

$$\begin{aligned} g(t) &= \int_0^t b\nu e^{-b\nu y} r(t-y) dy \\ &= b\nu e^{-b\nu t} \int_0^t e^{b\nu y} r(y) dy. \end{aligned} \quad (8)$$

We will work with  $g$  rather than  $g^*$ . We can either estimate  $g^*$  and use  $g(t) = e^{\lambda t} g^*(t)$  to estimate  $g$  or we can estimate  $g$  directly by simply turning off the signals for portions of the experiment.

Note that  $g$  satisfies the same equation as does  $f$  (Eq. 4) except that  $\eta$  is replaced by  $\nu$ . Therefore, the following results are immediate consequences of Theorem 1 since  $b\nu \leq b\eta$ :

THEOREM 2. For all  $\theta < b\nu$ ,

$$M_r(\theta) = \left(1 - \frac{\theta}{b\nu}\right) M_s(\theta). \quad (9)$$

COROLLARY. For all  $t \geq 0$ ,

$$r(t) = g(t) + g'(t)/bv. \quad (10)$$

Let  $M_\phi^{(n)}$  denote the  $n$ th derivative of the m.g.f. of  $\phi$ .

THEOREM 3. For all  $\theta$  such that  $\theta < bv$ ,

$$\left(1 - \frac{\theta}{b\eta}\right) M_f^{(n)}(\theta) - \frac{n}{b\eta} M_f^{(n-1)}(\theta) = \left(1 - \frac{\theta}{bv}\right) M_g^{(n)}(\theta) - \frac{n}{bv} M_g^{(n-1)}(\theta). \quad (11)$$

PROOF. Equate Eqs. 6 and 9, and prove Eq. 11 by induction.

COROLLARY. If  $\mu_\phi^{(n)}$  denotes the  $n$ th raw moment of  $\phi$ , then

$$\mu_f^{(n)} - \frac{n}{b\eta} \mu_f^{(n-1)} = \mu_g^{(n)} - \frac{n}{bv} \mu_g^{(n-1)}. \quad (12)$$

PROOF. Set  $\theta = 0$  in Eq. 11.

Since the first two moments are of particular interest, we give these explicitly:

$$\mu_f^{(1)} - \mu_g^{(1)} = \frac{1}{b} \left( \frac{1}{\eta} - \frac{1}{v} \right), \quad (13)$$

$$\begin{aligned} \mu_f^{(2)} - \mu_g^{(2)} &= \frac{2}{b} \left( \frac{\mu_f^{(1)}}{\eta} - \frac{\mu_g^{(1)}}{v} \right) \\ &= \frac{2}{b} \left( \frac{1}{\eta} - \frac{1}{v} \right) \left( \mu^{(1)} + \frac{1}{bv} \right). \end{aligned} \quad (14)$$

## 5. A SIGNAL-RESPONSE TIME DISTRIBUTION

The final density that we examine is another joint one,  $h$ , that at least one signal occurs between two successive responses and that the elapsed time is  $t$  from the first such signal to the second response. Let the first response occur at  $\tau$ , the first signal at  $\tau + x$ , where  $0 \leq x < \infty$ , and the second response at  $\tau + x + t$ , where  $t \geq 0$ . There are three distinct ways in which the response process that terminates in the second response can be initiated:

- i. a spontaneous  $D$ -state occurs prior to the first signal, say at  $\tau + y$ , where  $0 \leq y < x$ , this initiates a response process, and the reaction time is  $t + x - y$ ;
- ii. i does not happen, the signal produces a  $D$ -state at  $\tau + x$ , this activates a response process, and the reaction time is  $t$ ; or

iii. neither i nor ii happens, either a spontaneously or signal induced  $D$ -state occurs at  $\tau + x + z$ , where  $0 \leq z \leq t$ , this activates a response process, and the reaction time is  $t - z$ .

Thus,

$$h(t) = \int_0^\infty \lambda e^{-\lambda x} \left\{ \int_0^x b v e^{-b v y} r(t + x - y) dy + e^{-b v x} b q r(t) + e^{-b v x} (1 - b q) \int_0^t b \eta e^{-b \eta z} r(t - z) dz \right\} dx. \quad (15)$$

Observe that by setting  $t = 0$  and using Eq. 8, we obtain

$$\begin{aligned} h(0) &= \int_0^\infty \lambda e^{-\lambda x} \int_0^x b v e^{-b v y} r(x - y) dy \\ &= \lambda \int_0^\infty e^{-\lambda x} g(x) dx \\ &= \lambda \int_0^\infty g^*(x) dx. \end{aligned} \quad (16)$$

To simplify Eq. 15, consider each of the three terms separately:

$$\begin{aligned} \text{i. } b \lambda v \int_0^\infty e^{-\lambda x} \int_0^x e^{-b v y} r(t + x - y) dy dx & \\ &= b \lambda v e^{-b v t} \int_0^\infty e^{-(\lambda + b v) x} \int_t^{t+x} e^{b v z} r(z) dz dx \\ &= b \lambda v e^{-b v t} \left[ - \frac{e^{-(\lambda + b v) x}}{(\lambda + b v)} \int_t^{t+x} e^{b v z} r(z) dz \Big|_0^\infty \right. \\ &\quad \left. + \frac{1}{(\lambda + b v)} \int_0^\infty e^{-(\lambda + b v) x} e^{b v (x+t)} r(x + t) dx \right] \\ &= \frac{b \lambda v}{(\lambda + b v)} \int_0^\infty e^{-\lambda x} r(x + t) dx \\ &= \frac{b \lambda v}{(\lambda + b v)} e^{\lambda t} \int_t^\infty e^{-\lambda y} r(y) dy, \end{aligned}$$

where the first term of the integration by parts is 0 because, by Assumption 6 and the definition of  $\eta$ ,

$$\begin{aligned} \int_t^\infty e^{b v z} r(z) dz &\leq \int_0^\infty e^{b v z} r(z) dz \\ &\leq \int_0^\infty e^{\eta z} r(z) dz \\ &< \infty. \end{aligned}$$

$$\text{i. } \lambda bqr(t) \int_0^\infty e^{-\lambda x} e^{-bv x} dx = \frac{\lambda bqr(t)}{\lambda + bv}.$$

iii. By Eq. 4,

$$\begin{aligned} & \lambda(1 - bq) \int_0^\infty e^{-(\lambda+bv)x} \int_0^t b\eta e^{-b\eta z} r(t-z) dz dx \\ &= \lambda(1 - bq) \int_0^\infty e^{-(\lambda+bv)x} f(t) dx \\ &= \lambda \left( \frac{1 - bq}{\lambda + bv} \right) f(t). \end{aligned}$$

Thus, we have proved:

**THEOREM 4.** For  $t \geq 0$ ,

$$h(t) = \frac{\lambda}{(\lambda + bv)} \left[ bve^{\lambda t} \int_t^\infty e^{-\lambda y} r(y) dy + bqr(t) + (1 - bq)f(t) \right] \quad (17)$$

Setting  $t = 0$  in Eq. 17, recalling that  $r(0) = f(0) = 0$ , and integrating  $e^{-\lambda y}$  times Eq. 7, we obtain

$$\begin{aligned} h(0) &= \frac{\lambda bv}{(\lambda + bv)} \int_0^\infty e^{-\lambda y} r(y) dy \\ &= \lambda \left( \frac{\lambda + b\eta}{\lambda + bv} \right) \left( \frac{v}{\eta} \right) \int_0^\infty e^{-\lambda y} f(y) dy. \end{aligned} \quad (18)$$

As a preliminary to calculating the m.g.f. of  $h$ , we prove the following lemma.

**LEMMA 1.** For all  $\theta$  such that  $-\lambda < \theta \leq \eta$ ,

$$\lim_{t \rightarrow \infty} e^{(\theta + \lambda)t} \int_t^\infty e^{-\lambda x} r(x) dx = 0.$$

**PROOF.** By l'Hospital's rule,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_t^\infty e^{-\lambda x} r(x) dx}{e^{-(\theta + \lambda)t}} &= \lim_{t \rightarrow \infty} \frac{-e^{-\lambda t} r(t)}{-(\theta + \lambda)e^{-(\theta + \lambda)t}} \\ &= \lim_{t \rightarrow \infty} \frac{e^{\theta t} r(t)}{\theta + \lambda}. \end{aligned}$$

Since  $-\lambda < \theta \leq \eta$ , then for all  $t \geq 0$ ,

$$0 \leq \frac{e^{\theta t} r(t)}{\theta + \lambda} \leq \frac{e^{\eta t} r(t)}{\theta + \lambda},$$

so it is sufficient to prove that

$$\lim_{t \rightarrow \infty} e^{\eta t} r(t) = 0.$$

This follows from Assumption 6 and from the fact that  $r$  is continuous (Assumption 4c).

**THEOREM 5.** For all  $\theta$  such that  $-\lambda < \theta < b\eta$ ,

$$(\lambda + \theta)M_h(\theta) + h(0) = \lambda M_f(\theta)(1 - \theta^2 A), \quad (19)$$

where

$$A = \frac{q/\eta}{\lambda + b\nu}. \quad (20)$$

**PROOF.** By Eqs. 6 and 17,

$$\begin{aligned} M_h(\theta) &= \frac{\lambda}{(\lambda + b\nu)} \left[ b\nu \int_0^\infty e^{(\theta+\lambda)t} \int_t^\infty e^{-\lambda y} r(y) dy dt \right. \\ &\quad \left. + bq(1 - \theta/b\eta)M_f(\theta) + (1 - bq)M_f(\theta) \right]. \end{aligned}$$

The first term can be simplified as follows:

$$\begin{aligned} &\int_0^\infty e^{(\theta+\lambda)t} \int_t^\infty e^{-\lambda y} r(y) dy dt \\ &= \left\{ \frac{e^{(\theta+\lambda)t}}{(\theta + \lambda)} \int_t^\infty e^{-\lambda y} r(y) dy \right\}_0^\infty - \frac{1}{(\theta + \lambda)} \int_0^\infty e^{(\theta+\lambda)t} [-e^{-\lambda t} r(t)] dt \left\{ \right. \\ &= \frac{1}{(\theta + \lambda)} \left[ - \int_0^\infty e^{-\lambda y} r(y) dy + \int_0^\infty e^{\theta t} r(t) dt \right] \\ &= \frac{1}{(\theta + \lambda)} \left[ M_r(\theta) - \left( \frac{\lambda + b\nu}{\lambda b\nu} \right) h(0) \right] \\ &= \frac{1}{(\theta + \lambda)} \left[ \left( 1 - \frac{\theta}{b\eta} \right) M_f(\theta) - \left( \frac{\lambda + b\nu}{\lambda b\nu} \right) h(0) \right], \end{aligned}$$

where we have used Lemma 1 to show that the first term of the integration by parts is 0, Eq. 18 to introduce  $h(0)$ , and Eq. 7 to introduce  $M_f(\theta)$ . Substituting, rearranging, and simplifying yields Eq. 19.

(Note that it follows from  $\theta < b\eta$  that  $1 - \theta^2 A > 0$ .)

**THEOREM 6.** For all  $\theta$  such that  $-\lambda < \theta < b\eta$  and  $n \geq 1$ ,

$$\begin{aligned} &(\lambda + \theta)M_h^{(n)}(\theta) + nM_h^{(n-1)}(\theta) \\ &= \lambda[(1 - A\theta^2)M_f^{(n)}(\theta) - 2nA\theta M_f^{(n-1)}(\theta) - n(n-1)AM_f^{(n-2)}(\theta)]. \quad (21) \end{aligned}$$

PROOF. Induction.

COROLLARY.

$$\int_0^{\infty} h(t) dt = M_h(0) = 1 - h(0)/\lambda, \quad (22)$$

$$\mu_h^{(1)} + \frac{1 - h(0)/\lambda}{\lambda} = \mu_f^{(1)}, \quad (23)$$

and for  $n \geq 2$ ,

$$\mu_h^{(n)} + \frac{n}{\lambda} \mu_h^{(n-1)} = \mu_f^{(n)} - n(n-1)A\mu_f^{(n-2)}. \quad (24)$$

PROOF. Set  $\theta = 0$  in Eqs. 19 and 21.

## 6. ESTIMATION AND TEST EQUATIONS

From Eqs. 13 and 14,

$$\frac{1}{b\nu} = \frac{\mu_f^{(2)} - \mu_g^{(2)}}{2(\mu_f^{(1)} - \mu_g^{(1)})} - \mu_f^{(1)}$$

so  $b\nu$  can be estimated. From Eq. 13 and this estimate of  $b\nu$ , we can estimate  $b\eta$  and, therefore,  $bq$  from the identity

$$bq = \frac{b\eta - b\nu}{\lambda}.$$

In the original formulation of the process, we had three unknowns:  $b$ ,  $\nu$ , and  $q$ . We are, however, able to estimate only two independent combinations of them since the parameter  $b$  never appears except as a multiplier of  $\nu$  and of  $q$ . Thus, an added equation must be obtained if all three are to be estimated. One possibility is to use Eq. 2 and data from the standard Yes-No experiment to estimate  $\nu$ . A second possibility arises from the learning considerations to be explored in Section 7.

Certain other equations provide tests of the model. Of special interest are Eqs. 16, 22, and 23 which are parameter free. Once the parameters are estimated, Eq. 12 for  $n \geq 3$  and Eq. 24 for  $n \geq 2$  provide added tests of the model. Their worth is limited by the accuracy with which we can estimate the higher moments of the three density functions.

## 7. INFORMATION FEEDBACK AND ASYMPTOTIC BIAS

In the above analysis we have supposed that the bias is a constant  $b$ , which presumably means that it is the asymptotic value of some function  $b(\tau)$ . We attempt now to derive an expression for the expectation of the bias function on the assumption that

the subject is informed as to when signals have been presented and that he engages in a linear learning process analogous to the one previously invoked in the analysis of Yes-No and forced-choice experiments (Luce, 1963a). To what extent such feedback will disrupt the process is unknown, but it is probably considerable and so if it is used at all then the functions  $f$ ,  $g$ , and  $h$  should be estimated from the feedback experiment.

Let  $b(\tau)$  denote the bias at time  $\tau$ . Suppose that information feedback is given a known time  $T$  after the actual presentation of each stimulus. We postulate that the subject adjusts his bias according to the following experimenter-controlled linear learning process:

$$b(\tau + \Delta\tau) = \begin{cases} \alpha b(\tau) + 1 - \alpha & \text{if signal feedback is given in the interval } (\tau, \tau + \Delta\tau) \\ & \text{and a } D\text{-state occurred during the interval} \\ & (\tau - T, \tau + \Delta\tau - T) \\ \alpha' b(\tau) & \text{if no signal feedback is given in the interval} \\ & (\tau, \tau + \Delta\tau) \text{ and a } D\text{-state occurred during the interval} \\ & (\tau - T, \tau + \Delta\tau - T) \\ b(\tau) & \text{otherwise.} \end{cases}$$

The probability of the first event is the probability that a signal occurs during an interval of length  $\Delta\tau$  times the probability that a  $D$ -state occurs during that interval conditional on the fact that a signal occurred during it, i.e.,

$$(1 - e^{-\lambda\Delta\tau})[q + (1 - q)(1 - e^{-\nu\Delta\tau})].$$

Similarly, the probability of the second event is

$$e^{-\lambda\Delta\tau}(1 - e^{-\nu\Delta\tau}).$$

The probability of the third event is simply 1 minus the sum of the other two. So the expectation of  $b(\tau + \Delta\tau)$  is given by

$$\begin{aligned} E[b(\tau + \Delta\tau)] &= b(\tau) - b(\tau)\{(1 - \alpha)(1 - e^{-\lambda\Delta\tau})[q + (1 - q)(1 - e^{-\nu\Delta\tau})] \\ &\quad + (1 - \alpha')e^{-\lambda\Delta\tau}(1 - e^{-\nu\Delta\tau})\} + (1 - \alpha)(1 - e^{-\lambda\Delta\tau})[q + (1 - q)(1 - e^{-\nu\Delta\tau})]. \end{aligned}$$

Taking expectations over  $b(\tau)$  and then the limit as  $\Delta\tau \rightarrow 0$ , we see that

$$\frac{dE[b(\tau)]}{d\tau} = (1 - \alpha)\lambda q\{1 - BE[b(\tau)]\}, \quad (25)$$

where

$$B = 1 + \left(\frac{1 - \alpha'}{1 - \alpha}\right) \frac{\nu}{\lambda q}. \quad (26)$$

In arriving at Eq. 25 we have used the fact that

$$\lim_{x \rightarrow 0} \frac{1 - e^{-\beta x}}{x} = \beta.$$

Solving Eq. 25, we obtain

$$E[b(\tau)] = \frac{1 - [1 - Bb(0)]e^{-(1-\alpha)\lambda q B\tau}}{B} \quad (27)$$

Thus,

$$b = \lim_{\tau \rightarrow \infty} E[b(\tau)] = \frac{1}{B} \quad (28)$$

If we are willing to assume that the learning rates are equal, i.e.,  $\alpha = \alpha'$ , then Eq. 28 reduces to

$$b = \frac{\lambda}{\lambda + \nu/q},$$

which provides the needed third equation to estimate the third parameter. Alternatively, if this assumption does not seem appropriate, we can use Eq. 2 to estimate  $\nu$  and Eq. 28 to estimate the ratio  $(1 - \alpha')/(1 - \alpha)$ . Since the experiment can be run with different values of  $\lambda$  and since the parameters  $q$ ,  $\nu$ , and  $(1 - \alpha')/(1 - \alpha)$  should be independent of the choice of  $\lambda$ , a variety of internal checks are possible. Note that if the learning rates are equal and if the effective signals are rare relative to the spontaneous rate of  $D$ -states, e.g.,  $\nu/\lambda q = 10$ , then the bias is very weak,  $b = 1/11$ , and so very few signals would be detected.

*Note added in proof.* The general problem of estimating an unknown distribution, such as  $r$ , from an observed distribution, such as  $f$ , when they are related as in Eq. 7 is discussed by Scheinok (1964).

#### REFERENCES

- BÉKÉSY, G. VON. Über das Fechnersche Gesetz und seine Bedeutung für die Theorie der akustischen Beobachtungsfehler und die Theorie des Hörens. *Annalen der Physik*, 1930, **7**, 329-359. (Translated in Békésy, G. von. *Experiments in hearing*. New York: McGraw-Hill, 1960.)
- BLACKWELL, H. R. *Psychological thresholds: Experimental studies of methods of measurement*. Engineering Research Bulletin 36. Ann Arbor: Univer. of Michigan, 1953. (a)
- BLACKWELL, H. R. Evaluation of the neural quantum theory in vision. *American Journal of Psychology*, 1953, **66**, 397-408. (b)
- BROADBENT, D. E., AND GREGORY, MARGARET. Vigilance considered as a statistical decision. *British Journal of Psychology*, 1963, **54**, 309-323.
- CORSO, J. F. The neural quantum theory of sensory discrimination. *Psychological Bulletin*, 1956, **53**, 371-393.
- EGAN, J. P., GREENBERG, G. Z., AND SCHULMAN, A. I. Operating characteristics, signal detectability, and the method of free response. *Journal of the Acoustical Society of America*, 1961, **33**, 993-1007.
- LARKIN, W. D., AND NORMAN, D. A. An extension and experimental analysis of the neural quantum theory. In R. C. Atkinson (Ed.), *Studies in mathematical psychology*. Stanford, Calif.: Stanford Univer. Press, 1964. Pp. 188-200.

- LUCE, R. D. A threshold theory for simple detection experiments. *Psychological Review*, 1963, **70**, 61-79. (a)
- LUCE, R. D. Detection and recognition. In R. D. Luce, R. R. Bush, and E. Galanter (Eds.), *Handbook of mathematical psychology*, Vol. I. New York: Wiley, 1963. Pp. 103-189. (b)
- LUCE, R. D. Eine theoretische Analyse der Detektion in zeitlich nichtstrukturierten Experimenten. *Zeitschrift für Psychologie*, 1965, **171**, in press.
- MILLER, G. A., AND GARNER, W. R. Effect of random presentation on the psychometric function: Implications for a quantal theory of discrimination. *American Journal of Psychology*, 1944, **57**, 451-467.
- NEISSER, U. A methodological study of the quantal hypothesis in auditory psychophysics. Unpublished Ph.D. dissertation, Harvard University, 1955.
- NORMAN, D. A. A stochastic learning and a quantal model of signal detection. *Proceedings of the Joint Automatic Control Conference, Symposium on discrete adaptive systems*. American Institute of Electrical Engineering, 1962. (a)
- NORMAN, D. A. Sensory thresholds and response biases in detection experiments: A theoretical and experimental analysis. Unpublished Ph.D. dissertation (available from University Microfilms, Ann Arbor, Michigan) Univer. of Penn., 1962. (b)
- NORMAN, D. A. Sensory thresholds and response bias. *Journal of the Acoustical Society of America*, 1963, **35**, 1432-1441.
- NORMAN, D. A. Sensory thresholds, response biases, and the neural quantum theory. *Journal of Mathematical Psychology*, 1964, **1**, 88-120.
- PARZEN, E. *Stochastic processes*. San Francisco: Holden-Day, 1962.
- SCHEINOK, P. Estimation of a component of a convolution, when the other component is of exponential type. *Technometrics*, 1964, **6**, 222-224.
- STEVENS, S. S., MORGAN, C. T., AND VOLKMANN, J. Theory of the neural quantum in the discrimination of loudness and pitch. *American Journal of Psychology*, 1941, **54**, 315-335.
- SWETS, J. A. (Ed.) *Signal detection and recognition by human observers*. New York: Wiley, 1964.

RECEIVED: March 4, 1965