

## A Generalization of a Theorem of Dimensional Analysis

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Any function relating several variables for which admissible scale transformations are specified is assumed to satisfy a functional equation that requires admissible transformations of the independent variables to effect only admissible transformations of the dependent variable. When there are not any dimensional constants that, in effect, cancel out the admissible transformations of the independent variables, this equation severely limits the possible functions relating the variables. The equation is solved for any finite number of variables that are either ratio or interval scales.

Consider a functional relation  $u$  holding among variables whose measurement theories establish that they can each be represented numerically and that each representation is unique up to some set (often, group) of transformations. An argument about the invariance of  $u$  when admissible transformations are effected on the representations of the independent variables leads to a restrictive functional equation for  $u$ , provided that no dimensional constants enter into the relation in such a way as to cancel out the effects of the transformations. Within physics, where the variables are all assumed to be measured on ratio scales, the study and use of such functional equations is called dimensional analysis (Bridgman, 1922, 1931, 1932; Sedov, 1959). Within psychology the problem has been generalized to wider classes of measurement theories (Luce, 1959, 1962; Rozeboom, 1962). Although a general discussion of the considerations leading to the functional equation will not be repeated here, it is necessary to be quite explicit about the equation itself.

Let  $x_i$ ,  $i = 1, \dots, n + 1$ , denote typical numerical values of the  $n$  independent variables and the one dependent variable; let

$$u : \prod_{i=1}^n R_i \rightarrow R_{n+1},$$

where the  $R_i$  are (appropriate) subsets of the reals, be the functional relation; and let  $\mathbf{T}_i$  denote the set of admissible transformations under the measurement theory for variable  $i$ . We suppose that  $\mathbf{T}_i$  and  $R_i$  satisfy the requirement that if  $x \in R_i$  and  $T \in \mathbf{T}_i$ ,

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then  $Tx \in R_i$ . It will be convenient to use the following notation for a function  $f$  of  $n$  variables:

$$f_n(x_i) \equiv f(x_1, x_2, \dots, x_n),$$

where  $x_i \in R_i$ . Using this notation, the basic restriction on  $u$  is: for each set of  $T_i \in \mathbf{T}_i$ ,  $i = 1, \dots, n$ , there is a  $D_n(T_i) \in \mathbf{T}_{n+1}$  such that for all  $x_i \in R_i$ ,  $i = 1, \dots, n$ ,

$$u_n(T_i x_i) = D_n(T_i) u_n(x_i). \quad (1)$$

Note that juxtaposition, e.g.,  $T_i x_i$ , here means the image of  $x_i$  under the transformation  $T_i$ . Later, when we confine our attention to linear transformations, much the same notation is used to mean numerical multiplication.

It should be noted that Eq. 1 involves the assumption that the independent variables are indeed independent of each other in two strong senses. First, it is assumed that we may choose any combination of their values, i.e., no constraint holds among their values. Second, it is assumed that we may choose any combination of their admissible transformations, i.e. no constraint holds among their representations. In physics, the second postulate is described by saying that they are a set of fundamental variables, and the first by saying that they are not related by a physical law. These two notions of independence are distinct. For example, suppose  $m$  and  $v$  denote mass and velocity, then if momentum is conserved,  $mv = \text{constant}$ , a constraint holds between the values of  $m$  and  $v$ , but there is no constraint on the admissible measurement transformations since mass, length, and time have distinct dimensions. Conversely, in a system in which  $m$  and kinetic energy,  $\frac{1}{2}mv^2$ , are both treated as independent variables, clearly their values may be selected independently, but an admissible transformation on mass automatically forces one on the energy scale.

Within dimensional analysis, the nature of the constraints on physical laws when the independent variables are all ratio scales but they do not constitute a fundamental set of variables is covered by the so-called  $\Pi$ -theorem. So far as I am aware, its analogue for scales weaker than ratio ones has never been developed.

These remarks make it clear that the results to be given are of significance only when we are dealing with a set of fundamental independent variables that are in no way constrained by an empirical law. At present, this appears to mean that they are of significance to psychology only in psychophysical problems in which the independent variables are physical ones whose independence is determined by physical theory. Later, when we understand better what psychological variables are fundamental, our results should play a role in limiting the nature of psychological theories that involve nonratio scale variables, just as dimensional analysis limits physical and other theories when the variables are on a ratio scale.

For the case  $n = 1$  with  $\mathbf{T}_1$  and  $\mathbf{T}_2$  the affine and positive linear groups (corresponding to ratio and interval scales) and with  $u$  continuous, the solutions to Eq. 1 have

been given (Luce, 1959). Apparently they have not been worked out for general  $n$  involving one or more interval scales; this is done here.

We shall say that a function  $f$  of  $n$  variables *depends upon* each of its arguments if for every choice of  $n - 1$  of the arguments there are at least two values of the  $n$ th variable for which the function has different values.

LEMMA. *Let  $R_+$  be the positive reals, and let  $F_n : R_+^n \rightarrow R_+$  be continuous in and dependent upon each of its arguments. If for every  $x_i, y_i \in R_+, i = 1, \dots, n$ ,*

$$F_n(x_i y_i) = F_n(x_i) F_n(y_i), \quad (2)$$

*then there exist  $\beta_i \neq 0$  such that*

$$F_n(x_i) = \prod_{i=1}^n x_i^{\beta_i}. \quad (3)$$

PROOF. Define

$$\begin{aligned} F_{n-1}(x_i) &= F_n(x_1, \dots, x_{n-1}, 1), \\ F(x) &= F_n(x, \dots, x). \end{aligned}$$

Using Eq. 2 three times and introducing these definitions,

$$\begin{aligned} F_n(x y_i) &= F_{n-1}(y_i/y_n) F(x y_n) \\ &= F_{n-1}(y_i/y_n) F(y_n) F(x). \end{aligned}$$

Because the range of  $F_n$  is  $R_+$ , we may divide out  $F_{n-1}(y_i/y_n)$  to get

$$F(x y) = F(x) F(y). \quad (4)$$

Note that  $F$  is continuous because  $F_n$  is, so the solution to Eq. 4 is

$$F(x) = x^{\beta_n}.$$

Thus,

$$F_n(x_i) = F_{n-1}(x_i/x_n) x_n^{\beta_n}.$$

To complete the proof, it is sufficient to show that  $F_{n-1}$  satisfies Eq. 2. By Eq. 2,

$$\begin{aligned} F_{n-1}(x_i y_i) &= F_n(x_1 y_1, \dots, x_{n-1} y_{n-1}, 1) \\ &= F_n(x_1, \dots, x_{n-1}, 1) F_n(y_1, \dots, y_{n-1}, 1) \\ &= F_{n-1}(x_i) F_{n-1}(y_i). \end{aligned}$$

Equation 3 follows immediately by a simple induction. Since  $F_n$  depends upon each of its arguments,  $\beta_i \neq 0$ . Q.E.D.

In the following three theorems, we let

$$u : \prod_{i=1}^n R_i \rightarrow R_{n+1} .$$

where

$$R_i = \begin{cases} R_+ & \text{positive reals when } \mathbf{T}_i \text{ is the affine group (ratio scale)} \\ R & \text{reals when } \mathbf{T}_i \text{ is the positive linear group (interval scale).} \end{cases}$$

Furthermore, we assume that  $u$  is continuous in and dependent upon each of its arguments and that Eq. 1 holds.

Our first theorem is substantially the same as one in dimensional analysis (Bridgman, 1922, p. 22; Sedov, 1959, p. 9). It is included not only for completeness, but also because the proof is different and the assumptions are slightly weaker (continuity rather than differentiability).

**THEOREM 1.** *If both the independent and dependent variables are ratio scales, then there exist  $\alpha > 0$ ,  $\beta_i \neq 0$  such that*

$$u_n(x_i) = \alpha \prod_{i=1}^n x_i^{\beta_i} . \tag{5}$$

**PROOF.** In this case, Eq. 1 asserts that for  $k_i, x_i \in R_+$ , there exists a  $D_n(k_i) \in R_+$  such that

$$u_n(k_i x_i) = D_n(k_i) u_n(x_i) . \tag{6}$$

Set  $x_i = 1$  in Eq. 6,

$$u_n(k_i) = \alpha D_n(k_i) , \tag{7}$$

where  $\alpha = u(1, \dots, 1) > 0$ . Clearly,  $D_n$  is continuous in and dependent upon each of its arguments. Moreover,

$$\begin{aligned} D_n(k_i x_i) &= u_n(k_i x_i) / \alpha \\ &= D_n(k_i) u_n(x_i) / \alpha \\ &= D_n(k_i) D_n(x_i) . \end{aligned}$$

By the lemma and Eq. 7, Eq. 5 follows immediately.

Q.E.D.

**THEOREM 2.** *If the independent variables are ratio scales and the dependent variable is an interval scale, then either there exist  $\alpha \neq 0$ ,  $\beta_i \neq 0$ , and  $\gamma$  such that*

$$u_n(x_i) = \alpha \prod_{i=1}^n x_i^{\beta_i} + \gamma \tag{8}$$

or there exist  $\beta_i \neq 0$  and  $\gamma$  such that

$$u_n(x_i) = \sum_{i=1}^n \beta_i \log x_i + \gamma. \quad (9)$$

PROOF. In this case, Eq. 1 asserts that for  $k_i, x_i \in \mathcal{R}_+$  there exist  $D_n(k_i) \in \mathcal{R}_+$  and  $C_n(k_i) \in \mathcal{R}$  such that

$$u_n(k_i x_i) = D_n(k_i) u_n(x_i) + C_n(k_i). \quad (10)$$

Set  $x_i = 1$  in Eq. 10,

$$u_n(k_i) = D_n(k_i) a + C_n(k_i),$$

where  $a = u(1, \dots, 1)$ . Using this and writing Eq. 10 in two ways,

$$\begin{aligned} u_n(k_i x_i) &= D_n(k_i) u_n(x_i) + C_n(k_i) \\ &= D_n(k_i) [D_n(x_i) a + C_n(x_i)] + C_n(k_i) \\ &= D_n(x_i) u_n(k_i) + C_n(x_i) \\ &= D_n(x_i) [D_n(k_i) a + C_n(k_i)] + C_n(x_i). \end{aligned}$$

Rearranging,

$$\frac{1 - D_n(k_i)}{C_n(k_i)} = \frac{1 - D_n(x_i)}{C_n(x_i)}.$$

Since  $k_i$  and  $x_i$  are arbitrary, either  $D_n(x_i) \equiv 1$  or there exists  $\gamma$  such that

$$C_n(x_i) = \gamma[1 - D_n(x_i)].$$

Observe that

$$a = u_n(1) = D_n(1) a + C_n(1),$$

so

$$\begin{aligned} C_n(1) &= a[1 - D_n(1)] \\ &= \gamma[1 - D_n(1)]. \end{aligned}$$

Thus, either  $a = \gamma$  or  $D_n(1) = 1$ . We now consider these several cases.

(i)  $a = \gamma$ . Then,

$$\begin{aligned} u_n(x_i) &= D_n(x_i) a + C_n(x_i) \\ &= D_n(x_i) a + a[1 - D_n(x_i)] \\ &= a, \end{aligned}$$

which contradicts the assumption that  $u$  depends upon each of its arguments.

(ii)  $a \neq \gamma$ ,  $D_n(1) = 1$ , but  $D_n(x_i) \neq 1$ . Then, as above,

$$u_n(k_i x_i) = D_n(k_i x_i) \alpha + \gamma,$$

where  $\alpha = a - \gamma \neq 0$ . Using Eq. 10,

$$\begin{aligned} u_n(k_i x_i) &= D_n(k_i) u_n(x_i) + C_n(k_i) \\ &= D_n(k_i) [D_n(x_i) a + C_n(x_i)] + C_n(k_i) \\ &= D_n(k_i) D_n(x_i) \alpha + \gamma. \end{aligned}$$

Equating and dividing  $\alpha \neq 0$ ,

$$D_n(k_i x_i) = D_n(k_i) D_n(x_i),$$

and so Eq. 8 follows from the lemma.

(iii)  $D_n(x_i) \equiv 1$ . Then,

$$u_n(x_i) = \gamma + C_n(x_i),$$

where now  $\gamma = u(1, \dots, 1)$ . Using Eq. 10,

$$\begin{aligned} u_n(k_i x_i) &= \gamma + C_n(k_i x_i) \\ &= u_n(x_i) + C_n(k_i) \\ &= \gamma + C_n(x_i) + C_n(k_i). \end{aligned}$$

Hence,

$$\exp C_n(k_i x_i) = \exp C_n(k_i) \exp C_n(x_i),$$

and so Eq. 9 follows from the lemma.

**THEOREM 3.** *If one or more of the independent variables are interval scales and the remainder are ratio scales, and if the dependent variable is either a ratio or interval scale, then except for one case it is impossible for Eq. 1 to hold. The exception occurs when  $n = 1$  and the dependent variable is an interval scale, in which case,*

$$u_1(x) = \alpha x + \gamma.$$

**PROOF.** In this case, Eq. 1 takes the form

$$u_n(k_i x_i + h_i) = D_n(k_i; h_i) u_n(x_i) + C_n(k_i; h_i), \quad (11)$$

where  $h_i \equiv 0$  for those independent variables that are ratio scales, and  $C_n \equiv 0$  if the dependent variable is a ratio scale. With no loss of generality, we may suppose that variable 1 is an interval scale, i.e.,  $h_1 \neq 0$ . When  $h_i = 0$  for all  $i$ , Eq. 11 reduces either to Eq. 6 or 10, so  $u$  is restricted at least to the extent of Theorem 2. Thus, there are only two possibilities to consider.

(i) Suppose  $u$  satisfies Eq. 8. Let  $k_i = 1$ ,  $h_i = 0$  for  $i = 2, \dots, n$ , then

$$\begin{aligned} u(k_1x_1 + h_1, x_2, \dots, x_n) &= \alpha(k_1x_1 + h_1)^{\beta_1} \prod_{i=2}^n x_i^{\beta_i} + \gamma \\ &= D(\alpha x_1^{\beta_1} \prod_{i=2}^n x_i^{\beta_i} + \gamma) + C, \end{aligned}$$

where the arguments of  $D$  and  $C$  are obvious. Rewriting,

$$\alpha \prod_{i=2}^n x_i^{\beta_i} [(k_1x_1 + h_1)^{\beta_1} - Dx_1^{\beta_1}] = C + (\alpha D - 1) \gamma.$$

But the  $x_i$  are arbitrary, hence for  $n \geq 2$  this equation can hold only if

$$(k_1x_1 + h_1)^{\beta_1} = Dx_1^{\beta_1},$$

which is possible only when  $\beta_1 = 0$ , and that contradicts the assumption that  $u$  depends upon each of its arguments.

(ii) A similar argument applies when  $u$  satisfies Eq. 9 provided that  $n \geq 2$ . The result for  $n = 1$  is established in Luce (1959).

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