# 7

# Learning models for psychophysics

ROBERT R. BUSH, R. DUNCAN LUCE, and RICHARD M. ROSE,
*University of Pennsylvania*

Psychophysics has been concerned almost exclusively with the properties of steady-state behavior, whereas learning theory has concentrated on the processes by which such behavior evolves. To be sure, other distinctions between the two fields can be made, but none are quite as striking. Psychophysicists typically ignore (and seldom record) the learning portion of their data. Learning theorists often make bold assumptions about asymptotic behavior but rarely test them. It appears to us that one should consider psychophysical behavior to be the end-product of a learning process. Thus, stochastic learning models should predict at least certain kinds of psychophysical relations. This paper provides one example of how this can be done. Comparable approaches have been taken by Atkinson (1963), Atkinson, Carterette, and Kinchla (1962), and Luce (1963a).

All psychophysical experiments involve two or more stimulus presentations to a subject. In one of the simplest designs, the two-alternative recognition experiment, one of two stimuli is presented on each trial, and the subject is asked to report which stimulus it is. Usually, the stimuli are "confusable" in the sense that the subject makes some errors of recognition. His proportions of correct recognitions for the two presentations are usually considered the basic data. Variables such as the similarity of the two stimuli, the relative frequency of the two presentations, and the payoff conditions are basic parameters of the experiment as it is usually conceived.

Analogous experimental designs in learning are commonly called "discrimination" experiments. The simplest example of such an experiment is one in which an animal is presented with one of two stimuli on a random half of the trials and is continuously reinforced for a particular response when it

This work was supported by grants NSF G-14839 and G-17637 from the National Science Foundation to the University of Pennsylvania. During part of the time this work was being done, R. M. Rose held a National Science Foundation pre-doctoral fellowship.





occurs on those trials. The two stimulus presentations are seldom considered to be "confusable" in an absolute sense. They may be "similar," but it is usually assumed that an animal will asymptotically "learn the discrimination" perfectly. His task is to learn the appropriate response for each stimulus. In human recognition studies, on the other hand, the subject is taught the appropriate identification between stimuli and responses through instructions and pretraining. His task is to recognize the stimuli when he sees or hears them, and his asymptotic judgments are presumed to be imperfect.

Formal theories of recognition behavior and of discrimination learning have been rather different. The psychophysical models have been built on the concept of imperfect discrimination; they assume overlapping probability distributions (Tanner and Swets, 1954) or choice axioms that lead to "confusion matrices" (Luce, 1959). The discrimination-learning models all involved two processes—a conditioning mechanism plus a process in which either a "similarity index" decreases (Bush and Mosteller, 1951), "irrelevant cues" are "adapted out" (Restle, 1955), or "observing responses" become modified (Wyckoff, 1952). Asymptotic perfection was always postulated or predicted. Thus, in the psychophysical sense, discrimination was assumed always to be perfect.

The models presented below assume imperfect asymptotic discrimination, like the psychophysical models, but they also assume a mechanism of behavioral change, like the learning models. The goal is to describe both the pre-asymptotic and asymptotic behavior of subjects in recognition and discrimination experiments. After investigating several models for the two-alternative recognition design, we generalize the simplest one to a design involving $m$ stimulus presentations and $m$ responses for which there is a one-to-one correspondence between the two sets. This model is further generalized to include possible effects of partial reinforcement. Finally, the model is extended to partial identification designs, usually called psychophysical discrimination experiments, for which there are $m$ stimulus presentations and $k(<m)$ responses.

## 1. An experimenter-controlled-event model

Suppose that there are two stimulus presentations, $s_1$ and $s_2$, and two responses, $r_1$ and $r_2$. Let $r_1$ be "correct" (i.e., rewarded) when $s_1$ is presented, and let $r_2$ be correct when $s_2$ is presented. Let $Z_1, Z_2, \cdots$ be a sequence of random variables such that, for all $n$, $Z_n$ equals 1 if $s_1$ is presented on trial $n$ and equals 0 if $s_2$ is presented on trial $n$. Assume that on every trial $n$, we have conditional probabilities,

$$p_n = \Pr\{r_1 \mid s_1\}, \qquad q_n = \Pr\{r_1 \mid s_2\}. \tag{1}$$

Both of these probabilities are modified on each trial regardless of whether $s_1$ or $s_2$ is presented. On an $s_1$ trial, $p_n$ is assumed to increase by direct "con-

ditioning" and $q_n$ is assumed to increase by a process of "stimulus generalization." On an $s_2$ trial, the reverse is true.

Experimenter-controlled events are assumed in this section, by which we mean that the changes in $p_n$ and $q_n$ are independent of which response, $r_1$ or $r_2$, is made and depend only upon which stimulus is presented by the experimenter. This suggests that the experimental procedure should be "noncontingent" in the sense that information given to the subject after his response should depend only on the stimulus presented on that trial.

Assuming linear operators and recalling that both $p_n$ and $q_n$ increase on $s_1$ trials and decrease on $s_2$ trials, we have

$$(2)\qquad p_{n+1} = \begin{cases}(1-\theta_1)p_n + \theta_1 & \text{if } s_1 \text{ on trial } n,\\ (1-\phi_2)p_n & \text{if } s_2 \text{ on trial } n,\end{cases}$$

$$q_{n+1} = \begin{cases}(1-\phi_1)q_n + \phi_1 & \text{if } s_1 \text{ on trial } n,\\ (1-\theta_2)q_n & \text{if } s_2 \text{ on trial } n.\end{cases}$$

More compactly, we can write

$$(3)\qquad \begin{aligned} p_{n+1} &= Z_n[(1-\theta_1)p_n + \theta_1] + (1-Z_n)(1-\phi_2)p_n,\\ q_{n+1} &= Z_n[(1-\phi_1)q_n + \phi_1] + (1-Z_n)(1-\theta_2)q_n.\end{aligned}$$

We consider $\theta_1$ and $\theta_2$ to be conditioning-rate parameters—sampling ratios in Estes' sense—and $\phi_1$ and $\phi_2$ to be generalized conditioning-rate parameters. To facilitate an interpretation we wish to impose, define

$$(4)\qquad \eta_1 = \frac{\phi_1}{\theta_1},\qquad \eta_2 = \frac{\phi_2}{\theta_2}.$$

We consider these quantities to be similarity or confusability indices, both of which are assumed to be less than unity. We can think of $\eta_1$ as the similarity of $s_1$ to $s_2$, and of $\eta_2$ as the similarity of $s_2$ to $s_1$; we need not assume that similarity is symmetric at this point.

Once the sequence $\{Z_n\}$ is specified, our model allows us to compute all of the values of $p_n$ and $q_n$ in terms of the rate parameters, $\theta_1$ and $\theta_2$, the similarity indices, $\eta_1$ and $\eta_2$, and the initial probabilities, $p_1$ and $q_1$. A case of special interest is that in which the $Z_n$ form a Bernoulli chain. We let

$$(5)\qquad P = \Pr\{Z_n = 1\},\qquad n = 1, 2, \cdots.$$

THEOREM 1. *If Eq.* (2) *holds and if* $\{Z_n\}$ *forms a Bernoulli chain, then in the limit as* $n \to \infty$, *the expectation of* $p_n$ *approaches*

$$(6)\qquad p = \frac{1}{1 + b\eta_2},$$



*and the expectation of* $q_n$ *approaches*

$$(7)\qquad q = \frac{\eta_1}{b + \eta_1},$$

*where*

$$(8)\qquad b = \left(\frac{1-P}{P}\right)\frac{\theta_2}{\theta_1}.$$

PROOF. If one takes the expectations of both sides of Eq. (3) over all possible binomial sequences, sets the expectation on trial $n+1$ equal to that on trial $n$ as $n \to \infty$, and uses Eqs. (4), the assertion is obtained immediately.‖

This theorem is interesting, in part, because the asymptotic formulas are identical to those obtained from Luce's choice model (Luce, 1959, 1963b). In his theory, $p$ and $q$ are probabilities rather than expected probabilities, and his interpretations of the parameters are somewhat different from the interpretations suggested by the present model. The two similarity indices, $\eta_1$ and $\eta_2$, could be taken as equal, but, in any case, they represent properties of the stimuli. Luce's signal parameter can be interpreted in this same way. The parameter $b$ in the equations for $p$ and $q$ was called a "bias" parameter by Luce. His choice model does not specify the dependence of $b$ on the presentation probability $P$, but a simple extension of that model which uses an expected utility principle for describing the effects of the payoffs makes $b$ proportional to $(1-P)/P$ as in Eq. (8). In the present model, one would expect $\theta_i$ to be a function of the two payoffs that are possible when $s_i$ is presented and to be independent of the payoffs associated with the other stimulus. More specifically, one might expect $\theta_1$ to be an increasing function of the difference between the payoffs for the events $s_1r_1$ and $s_1r_2$.

The ROC curve or "iso-sensitivity" curve is obtained by eliminating $b$ from Eqs. (6) and (7), which yields

$$(9)\qquad \left(\frac{1-p}{p}\right)\left(\frac{q}{1-q}\right) = \eta_1\eta_2.$$

For fixed stimulus presentations but variable presentation probability or variable payoffs, the observed asymptotes should lie on this curve. The curve is symmetric about the line $p + q = 1$, and it lies above the line $p = q$ because $\eta_1\eta_2 < 1$.

Similarly, an "iso-bias" curve is obtained if we let $\eta = \eta_1 = \eta_2$ and eliminate it from Eqs. (6) and (7), which yields

$$(10)\qquad \left(\frac{1-p}{p}\right)\left(\frac{1-q}{q}\right) = b^2 = \left(\frac{1-P}{P}\right)^2\left(\frac{\theta_2}{\theta_1}\right)^2.$$

By holding $P$ and the payoffs fixed and varying the similarity of the stimuli, we might hope to observe points on this curve, but we have no evidence that $\theta_1$ and $\theta_2$ do not change as the stimuli are changed.

## 2. Single-sequence analyses, parameter estimation, and evaluation of the model

Theorem 1 was concerned with expected response probabilities, the expectations being over all possible binomial sequences of stimulus presentations. In an experiment, then, one should estimate the expected response probabilities by running a large number of subjects and by using the data on only a single trial after learning is essentially complete. But, for good reason, this is not how psychophysics is done. Data from different subjects are seldom combined because of known or presumed individual differences in the psychophysical parameters. Instead, many observations are made on a single subject.

It has been shown (Rose, this volume, p. 405) that for the model of Eq. (2) the distributions of $p$ and $q$ values generated by almost all single sequences of $Z_n$ are identical to the asymptotic distributions of $p$ and $q$ values, respectively. This result simplifies the estimation of parameters when only one subject is run and allows the experimenter to compare several statistics of his psychophysical data with predictions generated by the model.

The model presented in Sec. 1 has six parameters: two initial probabilities, $p_1$ and $q_1$, two rate parameters, $\theta_1$ and $\theta_2$, and two similarity indices, $\eta_1$ and $\eta_2$. Because the branching process generated by this model is ergodic, the parameters $p_1$ and $q_1$ are of little interest in an analysis of one long sequence of responses. For this reason we restrict our attention to methods of estimating $\theta_1$, $\theta_2$, $\eta_1$, and $\eta_2$.

Let $X_1, X_2, \cdots$ be a sequence of random variables such that, for all $n$, $X_n$ equals 1 if the response is $r_1$ when stimulus $s_1$ is presented and equals 0 if the response is $r_2$ when $s_1$ is presented. Similarly, let $Y_1, Y_2, \cdots$ be a sequence of random variables such that, for all $n$, $Y_n$ equals 1 if the response is $r_1$ on an $s_2$ trial and equals 0 if the response is $r_2$ on an $s_2$ trial. (There is no need to define $X_n$ on $s_2$ trials or $Y_n$ on $s_1$ trials.) Suppose we run a single subject for $N$ trials using the same stimulus presentation set, the same payoff matrix, and a particular presentation sequence $\{Z_n\}$. The proportion of correct responses to $s_1$ is

$$\hat{p}_N = \frac{\sum_{n=1}^{N} Z_n X_n}{\sum_{n=1}^{N} Z_n}, \tag{11}$$

and the proportion of incorrect responses to $s_2$ is

$$\hat{q}_N = \frac{\sum_{n=1}^{N} (1 - Z_n) Y_n}{N - \sum_{n=1}^{N} Z_n}. \tag{12}$$



Using an arrow to indicate convergence in measure as $N$ becomes large, we can easily prove

THEOREM 2. *If Eq.* (2) *holds and* $\{Z_n\}$ *forms a Bernoulli chain, then*

$$\hat{p}_N \to p \text{ and } \hat{q}_N \to q$$

PROOF. By the Law of Large Numbers (Feller, 1957, p. 141),

$$\frac{1}{N}\sum_{n=1}^{N} Z_n \to P.$$

Rose (this volume, p. 411) has shown that

$$\frac{1}{N}\sum_{n=1}^{N} Z_n X_n \to Pp$$

and that

$$\frac{1}{N}\sum_{n=1}^{N} (1 - Z_n) Y_n \to (1 - P)q. \quad \|$$

Because of this result, we can use Eqs. (11) and (12) to estimate $p$ and $q$. Moreover, because we can only "observe" the values of $p_n$ when $s_1$ is presented and the values of $q_n$ when $s_2$ is presented, it is clear that $\hat{p}_n$ and $\hat{q}_n$ are maximum likelihood estimators of $p$ and $q$, respectively. Equations (9) and (10) allow us to use these estimators to obtain maximum likelihood estimators of the product $\eta_1\eta_2$ and of the bias parameter $b$. Because $P$ is known, we then have a maximum likelihood estimator of the ratio $\theta_2/\theta_1$ from Eq. (8).

One must look beyond the asymptotic probabilities to obtain estimators for either $\eta_1$ or $\eta_2$ and either $\theta_1$ or $\theta_2$. We use sequential properties of the model for this purpose. It has been shown (Rose, this volume, p. 412) that

$$\begin{aligned}
&\frac{1}{N}\sum_{n=2}^{N} X_n Z_{n-1} Z_n \to P^2[(1 - \theta_1)p + \theta_1], \\
&\frac{1}{N}\sum_{n=2}^{N} X_n(1 - Z_{n-1})Z_n \to (1 - P)P[(1 - \eta_2\theta_2)p], \\
&\frac{1}{N}\sum_{n=2}^{N} Y_n Z_{n-1}(1 - Z_n) \to P(1 - P)[(1 - \eta_1\theta_1)q + \eta_1\theta_1], \\
&\frac{1}{N}\sum_{n=2}^{N} Y_n(1 - Z_{n-1})(1 - Z_n) \to (1 - P)^2[(1 - \theta_2)q].
\end{aligned} \tag{13}$$

Similarly, it has been shown that

$$\frac{1}{N}\sum_{n=3}^{N} X_n Z_{n-2} Z_{n-1} Z_n \to P^3[(1 - \theta_1)^2 p + 2\theta_1 - \theta_1^2], \tag{14}$$

and so on. Any two of the four sums in relations (13) can be used to estimate the remaining two parameters, and the other two relations can be used to test how well the model fits the data. In addition, other relations, such as relation (14), can be used in testing goodness of fit.

A common "randomizing" practice is to repeat the same sequence of stimuli every $K$ trials, where $K$ is something like 20 or 30. Supposedly, the subject cannot learn a sequence of this length, and so longer ones are deemed unnecessary. Often the proportions of events are fixed within the block of trials, and sometimes other restrictions on "randomness" are imposed. In a two-alternative recognition experiment, suppose we use a repeated sequence $\{Z_1, Z_2, \cdots Z_K\}$ with $K_1$ presentations of $s_1$ and $K_2 = K - K_1$ presentations of $s_2$. If $X_n$ is the response random variable introduced above, the proportion of correct $s_1$ recognitions in a block is

$$\bar{X}_i = \frac{1}{K_1}\sum Z_n X_n, \tag{15}$$

where the sum is over the $i$th block of $K$ trials. Now, if the process has reached equilibrium—if the subject is "at asymptote"—the model in Sec. 1 predicts that the sequence of response probabilities $\{p_1, p_2, \cdots, p_K\}$ is repeated in each block of $K$ trials. Thus, the random variables $\bar{X}_i$ have a common distribution and they are independent. The mean is

$$E(\bar{X}_i) = \frac{1}{K_1}\sum Z_n p_n = \bar{p}, \tag{16}$$

and the variance is

$$\begin{aligned}\operatorname{var}(\bar{X}_i) &= \frac{1}{K_1^2}\sum Z_n p_n(1-p_n) \\ &= \frac{1}{K_1}\left\{\bar{p}(1-\bar{p}) - \frac{1}{K_1}\sum Z_n(p_n-\bar{p})^2\right\}.\end{aligned} \tag{17}$$

For a given repeated sequence $\{Z_n\}$ and known model parameters, one can compute these two moments numerically and compare them with sample moments. It is worth noting that for a fixed mean, the variance is maximized when the $p_n$ all equal that mean; the effect of changing $p_n$'s is to *decrease* the variance of the block proportion.

If the stimulus-presentation schedule is generated by a nonrepeating Bernoulli chain $\{Z_n\}$, the preceding analysis is not appropriate. The sequence $\{p_n\}$ does not contain repeating subsequences and so the random variables $\bar{X}_i$ do not have a common distribution.

### 3. An experimenter–subject-controlled event model

The model presented and analyzed in the preceding sections postulated that changes in the response probabilities depend only on the stimulus presented and not on the response made. That model is easily generalized to include respones contingencies, but the analysis is not so easily generalized. The axioms are



$$p_{n+1} = \begin{cases}(1-\theta_{11})p_n + \theta_{11} & \text{if } s_1 \text{ and } r_1, \\ (1-\theta_{12})p_n + \theta_{12} & \text{if } s_1 \text{ and } r_2, \\ (1-\phi_{21})p_n & \text{if } s_2 \text{ and } r_1, \\ (1-\phi_{22})p_n & \text{if } s_2 \text{ and } r_2,\end{cases}$$

$$q_{n+1} = \begin{cases}(1-\phi_{11})q_n + \phi_{11} & \text{if } s_1 \text{ and } r_1, \\ (1-\phi_{12})q_n + \phi_{12} & \text{if } s_1 \text{ and } r_2, \\ (1-\theta_{21})q_n & \text{if } s_2 \text{ and } r_1, \\ (1-\theta_{22})q_n & \text{if } s_2 \text{ and } r_2.\end{cases} \tag{18}$$

For given $p_n$ and $q_n$ and for fixed $P = \Pr(s_1)$, the expected values of $p_{n+1}$ and $q_{n+1}$ are

$$\begin{aligned}E(p_{n+1}) = p_n &+ P(1-p_n)\{\theta_{11}p_n + \theta_{12}(1-p_n)\} \\ &- (1-P)p_n\{\phi_{21}q_n + \phi_{22}(1-q_n)\},\end{aligned} \tag{19}$$

and

$$\begin{aligned}E(q_{n+1}) = q_n &+ P(1-q_n)\{\phi_{11}p_n + \phi_{12}(1-p_n)\} \\ &- (1-P)q_n\{\theta_{21}q_n + \theta_{22}(1-q_n)\}.\end{aligned} \tag{20}$$

We can obtain approximations, $p$ and $q$, to the asymptotic mean response probabilities by assuming that

$$\begin{aligned}p &= p_n = E(p_{n+1}), \\ q &= q_n = E(q_{n+1}).\end{aligned} \tag{21}$$

This is the expected operator approximation discussed elsewhere (Bush and Mosteller, 1955, pp. 138–48). Equations (19) and (20) then give

$$\begin{aligned}\frac{1-p}{p} &= \left(\frac{1-P}{P}\right)\left(\frac{\phi_{21}q + \phi_{22}(1-q)}{\theta_{11}p + \theta_{12}(1-p)}\right), \\ \frac{1-q}{q} &= \left(\frac{1-P}{P}\right)\left(\frac{\theta_{21}q + \theta_{22}(1-q)}{\phi_{11}p + \phi_{12}(1-p)}\right).\end{aligned} \tag{22}$$

In the experimenter-controlled-event model, we found that $(1-p)q/p(1-q)$ was constant. For the more general model, that ratio is also constant provided that

$$\begin{aligned}\eta_1 &= \frac{\phi_{11}}{\theta_{11}} = \frac{\phi_{12}}{\theta_{12}}, \\ \eta_2 &= \frac{\phi_{21}}{\theta_{21}} = \frac{\phi_{22}}{\theta_{22}}.\end{aligned} \tag{23}$$

This assumption says that the similarity indices are independent of the response made, which clearly seems demanded by our interpretation of the model. The equation for the iso-sensitivity curve is then

$$\left(\frac{1-p}{p}\right)\left(\frac{q}{1-q}\right) = \eta_1\eta_2, \tag{24}$$

which is in agreement with Eq. (9) except that $p$ and $q$ in that equation are exact asymptotic means, whereas $p$ and $q$ in Eq. (24) are approximate values. Of course, if we take $\theta_{11} = \theta_{12} = \theta_1$ and $\theta_{21} = \theta_{22} = \theta_2$, the model reduces to the earlier one and so the equation is exact.

We can also obtain a curve analogous to the iso-bias curve of Eq. (10) by defining a quantity $b$ such that

$$p = \frac{1}{1+\eta_2 b}. \tag{25}$$

Thus,

$$\frac{1-p}{p} = \eta_2 b, \tag{26}$$

and Eq. (24) gives

$$\frac{1-q}{q} = \frac{b}{\eta_1}. \tag{27}$$

The curve analogous to the iso-bias curve is then

$$\left(\frac{1-p}{p}\right)\left(\frac{1-q}{q}\right) = \frac{\eta_2}{\eta_1}b^2. \tag{28}$$

The quantity $b$ is a complicated function of the parameters as can be seen by substituting back into Eqs. (22). In fact, $b$ is a solution of

$$b = \pm\left(\frac{1-P}{P}\right)\left(\frac{1+\eta_2 b}{\eta_1+b}\right)\left(\frac{\theta_{21}\eta_1+\theta_{22}b}{\theta_{11}+\theta_{12}\eta_2 b}\right). \tag{29}$$

Clearly, $b$ depends not only on the $\theta_{ij}$, but also on the similarity indices $\eta_1$ and $\eta_2$. Thus, it depends on the stimuli as well as on the payoffs and presentation probability $P$. Therefore $b$ should not be considered a "bias" parameter. Only in the specialization to the experimenter-controlled-event model might we expect $b$ to be independent of the stimulus properties.

## 4. A model with partial stimulus generalization

A possible objection to the models discussed so far is that the conditioning effects of one stimulus presentation are completely generalized to the other. Although the generalized-rate parameters were assumed to be smaller than the corresponding conditioning-rate parameters, the limit points of the operators were 1 or 0. Thus, if $s_1$ were presented for many trials, both $p_n$ and $q_n$ would approach 1. This assumption can be weakened, which we do for



experimenter-controlled events only. In place of Eq. (2), we write

$$p_{n+1} = \begin{cases}(1-\theta_1)p_n + \theta_1 & \text{if } s_1,\\ (1-\phi_2)p_n + \phi_2(1-\lambda_2) & \text{if } s_2,\end{cases}$$
$$q_{n+1} = \begin{cases}(1-\phi_1)q_n + \phi_1\lambda_1 & \text{if } s_1,\\ (1-\theta_2)q_n & \text{if } s_2.\end{cases} \tag{30}$$

The asymptotic mean probabilities are obtained as before, which yields

THEOREM 3. *If Eq.* (30) *holds and if* $\{Z_n\}$ *forms a Bernoulli chain, then in the limit as* $n \to \infty$ *the expectations of* $p_n$ *and* $q_n$ *approach, respectively,*

$$p = \frac{1+b\eta_2(1-\lambda_2)}{1+b\eta_2}$$
$$q = \frac{\eta_1\lambda_1}{b+\eta_1}, \tag{31}$$

*where, as before,*

$$\eta_1 = \frac{\phi_1}{\theta_1}, \qquad \eta_2 = \frac{\phi_2}{\theta_2}, \qquad b = \left(\frac{1-P}{P}\right)\frac{\theta_2}{\theta_1}. \tag{32}$$

When $\lambda_1 = \lambda_2 = 1$, the results reduce to the earlier ones, Eqs. (6) and (7).

An interesting special case arises when we let $\eta_1 = \eta_2 = 1$. Eliminating $b$ from Eq. (31), we obtain

$$p = (1-\lambda_2) + \frac{\lambda_2}{\lambda_1}q. \tag{33}$$

This is a straight line between the points $[(1-\lambda_2),0]$ and $(1,\lambda_1)$ in the $(p,q)$ plane. If $\lambda_1$ and $\lambda_2$ are considered to be stimulus parameters, this last equation is the iso-sensitivity curve. It does not include the points (0,0) and (1,1). Indeed, even when $\eta_1 \neq \eta_2 \neq 1$, the curve moves from $[(1-\lambda_2),0]$ when $P = 0$ to $(1,\lambda_1)$ when $P = 1$, but it is not a straight line. Data are needed to determine whether one can move arbitrarily close to the corners (0,0) and (1,1) in the $(p,q)$ plane. If so, then we must set $\lambda_1 = \lambda_2 = 1$, as in the first model presented in this paper.

## 5. An $m$-alternative experimenter-controlled recognition model

The model for two stimulus presentations and two responses, given in Sec. 1, is readily generalized to $m$ presentations and $m$ responses. Let the stimulus presentation set be

$$\{s_1, s_2, \cdots, s_m\},$$

let the response set be

$$\{r_1, r_2, \cdots, r_m\},$$

and require that $r_i$ be correct if and only if $s_i$ is presented. We abbreviate

$\Pr(r_j \mid s_1)$ on trial $n$ by $p_n(j \mid i)$. The basic axiom, which is a generalization of Eq. (2), is:

*If $s_k$ is presented on trial n, then for $i,j = 1,2,\cdots,m$,*

$$p_{n+1}(j \mid i) = p_n(j \mid i) + \eta(i,k)\theta(k)[\delta_{jk} - p_n(j \mid i)], \tag{34}$$

*where $\delta_{jk}$ is the Kronecker delta (equal to 1 when $j = k$, and 0 otherwise), where the similarity indices, $\eta(i,k)$, and rate parameters, $\theta(k)$, are in the unit interval, and where $\eta(i,i) = 1$.*

For $m = 2$, this reduces to Eqs. (2) provided that we introduce the definitions given by Eqs. (3) and let $\eta_1 = \eta(2,1)$, $\eta_2 = \eta(1,2)$, $\theta_1 = \theta(1)$, $\theta_2 = \theta(2)$, $p_n = p_n(1 \mid 1)$, and $q_n = p_n(1 \mid 2)$.

If we sum both sides of Eq. (34) over $j$, we obtain $\Sigma p_{n+1}(j \mid i) = \Sigma p_n(j \mid i) = 1$ because the expression in square brackets in Eq. (34) sums to 0.

Let $\{Z_n\}$ be a sequence of random variables such that $Z_n = k$ if stimulus $s_k$ is presented on trial $n$. We say that $\{Z_n\}$ forms a *generalized Bernoulli chain* when the $Z_n$ are independent and $P(k) = \Pr(Z_n = k)$ are constant probabilities and $\Sigma_{k=1}^m P(k) = 1$.

THEOREM 4. *If Eq. (34) holds and if $\{Z_n\}$ forms a generalized Bernoulli chain, then*

$$\lim_{n\to\infty} E[p_n(j \mid i)] = \frac{\eta(i,j)b(j)}{\sum_{k=1}^m \eta(i,k)b(k)}, \tag{35}$$

*where*

$$b(k) = P(k)\theta(k).$$

PROOF.

$$\begin{aligned} E[p_{n+1}(j \mid i) \mid p_n(j \mid i)] &= \sum_{k=1}^m P(k)\{p_n(j \mid i) + \eta(i,k)\theta(k)[\delta_{jk} - p_n(j \mid i)]\} \\ &= p_n(j \mid i) + P(j)\eta(i,j)\theta(j) - p_n(j \mid i)\sum_{k=1}^m P(k)\eta(i,k)\theta(k). \end{aligned}$$

If we then take expectations over the possible values of $p_n(j \mid i)$ and go to the limit, we get Eq. (35).‖

Note that $b(k) = P(k)\theta(k)$ is a bias parameter analogous to Eq. (8). For $m = 2$, it was simpler to use the ratio

$$b = \frac{b(2)}{b(1)},$$

but the $m$-alternative formula is more symmetric if we do not.

Equation (35) is the general result obtained from the recognition choice model described in Luce (1963b), special cases of which have been discussed elsewhere (Luce, 1959; Shipley, 1960). Thus, the experimenter-controlled



linear learning model of Eq. (34) yields the same asymptotic means as the choice model when any number of alternatives are involved.

## 6. Partial-reinforcement recognition experiments

We can generalize the experimenter-controlled-event model of the preceding section to designs in which the subject is sometimes incorrectly told that stimulus $s_h$ was presented when in fact $s_k$ was. This sort of partial reinforcement is not common in psychophysical studies, but it is analogous to what is done in a number of learning experiments. Let $\Pi(h \mid k)$ denote the (constant) conditional probability that the subject is told that $s_h$ was presented, given that $s_k$ was, where

$$\sum_{h=1}^m \Pi(h \mid k) = 1 \qquad (k = 1,2,\cdots,m).$$

The learning axiom we propose is:

*If $s_k$ is presented and $s_h$ is reported to have been presented, then*

$$p_{n+1}(j \mid i) = p_n(j \mid i) + \eta(i,k)\theta(h)[\delta_{jh} - p_n(j \mid i)]. \tag{36}$$

Note that the similarity index, $\eta(i,k)$, depends on the stimulus, $s_k$, actually presented, whereas the learning rate parameter, $\theta(h)$, depends on the stimulus, $s_h$, said to have been presented. Our reason for these assumptions is that we interpret the $\eta(i,k)$ to be stimulus parameters (perceptual variables) and interpret the $\theta(h)$ to be outcome parameters (motivational variables).

THEOREM 5. *If Eq. (36) holds and if $\{Z_n\}$ forms a generalized Bernoulli chain, then*

$$\lim_{n\to\infty} E[p_n(j \mid i)] = \frac{\theta(j)\sum_{k=1}^m P(k)\Pi(j \mid k)\eta(i,k)}{\sum_{h=1}^m \theta(h)\sum_{k=1}^m P(k)\Pi(h \mid k)\eta(i,k)}. \tag{37}$$

PROOF. Taking expectations over the possible presentations and the possible experimenter reports, we obtain

$$\begin{aligned} E[p_{n+1}(j \mid i) \mid p_n(j \mid i)] &= \sum_{k=1}^m P(k)\sum_{h=1}^m \Pi(h \mid k)\{p_n(j \mid i) + \eta(i,k)\theta(h)[\delta_{jh} - p_n(j \mid i)]\} \\ &= p_n(j \mid i) + \theta(j)\sum_{k=1}^m P(k)\Pi(j \mid k)\eta(i,k) \\ &\quad - p_n(j \mid i)\sum_{h=1}^m \theta(h)\sum_{k=1}^m P(k)\Pi(h \mid k)\eta(i,k). \end{aligned}$$

Taking expectations over the values of $p_n(j \mid i)$ and going to the limit yields Eq. (37).

When $\Pi(h \mid k) = 1$ for $h = k$, Eq. (37) reduces to Eq. (35) for the continuous-reinforcement model. In general, however, the form of Eq. (37) is not the same as that obtained before; the bias parameters, $b(j) = P(j)\theta(j)$, and similarity parameters, $\eta(i,j)$, do not separate out as simple products.

Using the previous notation for $m = 2$ and letting $\Pi_1 = \Pi(1 \mid 1)$ and $\Pi_2 = \Pi(2 \mid 2)$, we obtain

$$p = \frac{\theta_1[P\Pi_1 + (1-P)(1-\Pi_2)\eta_2]}{\theta_1[P\Pi_1 + (1-P)(1-\Pi_2)\eta_2] + \theta_2[P(1-\Pi_1) + (1-P)\Pi_2\eta_2]},$$

$$q = \frac{\theta_1[P\Pi_1\eta_1 + (1-P)(1-\Pi_2)]}{\theta_1[P\Pi_1\eta_1 + (1-P)(1-\Pi_2)] + \theta_2[P(1-\Pi_1)\eta_1 + (1-P)\Pi_2]}. \tag{38}$$

Three special cases are of interest:

1. $\Pi_1 = \Pi_2 = 1$. Equation (38) reduces to those of Theorem 1 in Sec. 1.

2. $\Pi_1 = \Pi_2 = 0$. This corresponds to a situation in which the subject always is told that the opposite stimulus was presented. The asymptotic response probabilities are

$$p = \frac{\eta_2}{\eta_2 + \frac{P}{1-P}\frac{\theta_2}{\theta_1}},$$

$$q = \frac{1}{1 + \frac{P}{1-P}\frac{\theta_2}{\theta_1}\eta_1}. \tag{39}$$

The equation for the iso-sensitivity curve is

$$\left(\frac{1-p}{p}\right)\left(\frac{q}{1-q}\right) = \frac{1}{\eta_1\eta_2}. \tag{40}$$

This is of the same form as Eq. (9) for the $\Pi_1 = \Pi_2 = 1$ case, except that the right-hand sides are reciprocals of one another. Thus, the iso-sensitivity curve for $\Pi_1 = \Pi_2 = 0$ is the reflection (across the line $p = q$) of the corresponding curve for $\Pi_1 = \Pi_2 = 1$. Another way of putting it is that the two iso-sensitivity curves are identical if $p$ is replaced by $1 - p$ and $q$ by $1 - q$.

3. $\Pi_1 = 1 - \Pi_2 = \Pi$. This corresponds to an experiment in which $s_1$ is reported with a fixed probability $\Pi$ independent of the presentation probability $P$, because the unconditional probability that the experimenter reports $s_1$ is

$$P\Pi_1 + (1-P)(1-\Pi_2) = \Pi.$$



The asymptotic response probabilities are

$$p = q = \frac{1}{1 + \left(\frac{1-\Pi}{\Pi}\right)\frac{\theta_2}{\theta_1}}. \tag{41}$$

Not only are $p$ and $q$ equal, but they are independent of the presentation probability $P$ and of the similarity indices. We conclude, therefore, that if $\Pi_1 = 1 - \Pi_2$, then $p = q$ for any experimental conditions—stimulus similarities, presentation probabilities, and payoff conditions. This is a very strong and directly testable prediction of the model. Furthermore, if $\theta_2/\theta_1$ depends only on the payoffs, as we hope, then the *value* of $p = q$ is independent of the stimulus similarities and presentation probabilities. This is an even stronger prediction; it is not implied by the model alone, but rather it follows from our interpretation of the model parameters.

## 7. An experimenter-controlled discrimination model

In psychophysical discrimination experiments, the set, $\mathscr{S}$, of stimuli is ordered by some physical relation, $>$, e.g., weight, size, intensity. Each of the $m$ presentations to the subject consists of a $k$-tuple of the form $s = \langle s_1, s_2, \cdots, s_k \rangle$, where each $s_i \in \mathscr{S}$. The subject is to report which of the presented stimuli he believes meets the discriminative criterion, such as which is the heaviest. This he does by indicating its location in the presentation. The number of alternative responses is then $k$, the number of stimuli in each presentation.

We denote the response set by $R$, its typical elements by $r$ and $r'$, the presentation set by $S$, and its typical elements by $s$ and $s'$. For each $r \in R$, let $S_r$ denote the subset of $S$ for which $r$ is the correct response, i.e., $s \in S_r$ if and only if $s_r > s_{r'}$ for all $r' \neq r$. Thus, the subsets $S_r$ form a partition of $S$ that is one-to-one with the set $R$.

The conditional probability on trial $n$ of response $r$ given stimulus presentation $s$ is denoted by $p_n(r \mid s)$. Our learning axiom is

*If $s' \in S_{r'}$ is presented on trial n, then for all $s \in S$ and $r \in R$,*

$$p_{n+1}(r \mid s) = p_n(r \mid s) + \eta(s,s')\theta(s')[\delta_{rr'} - p_n(r \mid s)], \tag{42}$$

*where, for all $s' \in S_{r'}$, $\theta(s')$ has the same value, say $\theta(r')$.*

We assume this about $\theta$ because, as in the partial reinforcement recognition model, we associate the rate parameters with responses and their outcomes.

THEOREM 6. *If Eq. (42) holds and if $\{Z_n\}$ forms a generalized Bernoulli chain, then*

$$\lim_{n\to\infty} E[p_n(r \mid s)] = \frac{\bar{\eta}(s,r)\theta(r)}{\sum_{r' \in R} \bar{\eta}(s,r')\theta(r')}, \tag{43}$$

*where*

$$\bar{\eta}(s,r) = \sum_{s' \epsilon S_r} P(s')\eta(s,s') \tag{44}$$

*and $P(s')$ is the probability that $s'$ is presented on a trial.*

PROOF.

$$\begin{aligned} E[p_{n+1}(r \mid s)] &= \sum_{s' \epsilon S} P(s')\Big(E[p_n(r \mid s)] + \eta(s,s')\theta(r')\{\delta_{rr'} - E[p_n(r \mid s)]\}\Big) \\ &= E[p_n(r \mid s)] + \theta(r) \sum_{s' \epsilon S_r} P(s')\eta(s,s') \\ &\quad - E[p_n(r \mid s)] \sum_{r' \epsilon R} \theta(r') \sum_{s' \epsilon S_{r'}} P(s')\eta(s,s') . \end{aligned}$$

The limit is then

$$\lim_{n \to \infty} E[p_n(r \mid s)] = \frac{\theta(r) \sum_{s' \epsilon S_r} P(s')\eta(s,s')}{\sum_{r' \epsilon R} \theta(r') \sum_{s' \epsilon S_{r'}} P(s')\eta(s,s')} .$$

If $\bar{\eta}(s,r)$, Eq. (44), is substituted, Eq. (43) follows.‖

In general, the stimulus set $\mathcal{S}$ has more than $k$ elements, the number actually presented on any one trial. In an important special case, however, $\mathcal{S}$ has exactly $k$ elements and $S$ is a subset of the $k!$ possible orderings of $\mathcal{S}$. Each $s \epsilon S$ has the same set of stimuli and one of them, $\mathfrak{s}^*$, is the correct one, i.e., $\mathfrak{s}^* > \mathfrak{s}$ for all $\mathfrak{s} \epsilon \mathcal{S} - \{\mathfrak{s}^*\}$. Thus, $s \epsilon S_r$ if and only if $\mathfrak{s}_r = \mathfrak{s}^*$. In this case we assume that the index of similarity, $\eta(s,s')$, depends only on the correct stimulus, $\mathfrak{s}^*$, which is in position $r$ when $s' \epsilon S_r$ is presented, and on $\mathfrak{s}_r$, the stimulus in that same position in another presentation, $s$. That is, we assume that

$$\eta(s,s') = \rho(\mathfrak{s}_r, \mathfrak{s}^*), \qquad s' \epsilon S_r . \tag{45}$$

Therefore,

$$\bar{\eta}(s,r) = \sum_{s' \epsilon S_r} P(s')\eta(s,s') = \rho(\mathfrak{s}_r, \mathfrak{s}^*) \sum_{s' \epsilon S_r} P(s') . \tag{46}$$

The sum

$$P(r) = \sum_{s' \epsilon S_r} P(s')$$

is the probability that an element of $S_r$ is presented, i.e., the probability that $r$ is the correct response on trial $n$. Equation (43) now becomes

$$\lim_{n \to \infty} E[p_n(r \mid s)] = \frac{\rho(\mathfrak{s}_r, \mathfrak{s}^*)b(r)}{\sum_{r' \epsilon R} \rho(\mathfrak{s}_{r'}, \mathfrak{s}^*)b(r')} , \tag{47}$$

where

$$b(r) = P(r)\theta(r) .$$



If we note that $\mathfrak{s}^*$ is a parameter of the whole experiment, we can write

$$\rho(\mathfrak{s}_r, \mathfrak{s}^*) = \rho(r) ,$$

and so

$$\lim_{n \to \infty} E[p_n(r \mid s)] = \frac{\rho(r)b(r)}{\sum_{r' \epsilon R} \rho(r')b(r')} . \tag{48}$$

This is the same result obtained from the discrimination choice model described (for $m = 3$) in Luce (1959).

Another special case of interest is a design for which the $m$ presentation probabilities are all equal and for which the payoffs are such that the $\theta(r)$ are all equal. Then

$$\begin{aligned} \lim_{n \to \infty} E[p_n(r \mid s)] &= \frac{\sum_{s' \epsilon S_r} \eta(s,s')}{\sum_{r' \epsilon R} \sum_{s' \epsilon S_{r'}} \eta(s,s')} \\ &= \frac{\sum_{s' \epsilon S_r} \eta(s,s')}{\sum_{s' \epsilon S} \eta(s,s')} . \end{aligned} \tag{49}$$

In a classical discrimination experiment, often used to determine a psychometric function, each stimulus presentation consists of a standard stimulus followed by another stimulus; the subject is to report whether the second stimulus is greater or less than the first. Thus, $R = \{1,2\}$, and the assumptions that led to Eq. (49) should apply. From that result, we note that the asymptotic expected value of $p_n(r \mid s)$ is not "independent of irrelevant alternatives"; its value depends on the entire set of stimulus presentations used in the experiment. Thus, according to this theory, the psychometric function is not an invariant of the discrimination process, as is usually assumed by psychophysicists. It is an experimental question whether or not the psychometric function in fact depends on the presentation set $S$.

## REFERENCES


ATKINSON, R. C. A variable sensitivity theory of signal detection. *Psychol. Rev.*, 1963, **70**, 91–106.

ATKINSON, R. C., CARTERETTE, E. C., and KINCHLA, R. A. Sequential phenomena in psychophysical judgments: a theoretical analysis. *IRE Trans. on Information Theor.*, 1962, **IT-8**, 155–162.

BUSH, R. R., and MOSTELLER, F. A model for stimulus generalization and discrimination. *Psychol. Rev.*, 1951, **58**, 413–423.

BUSH, R. R., and MOSTELLER, F. *Stochastic models for learning*. New York: Wiley, 1955.

FELLER, W. *An introduction to probability theory and its applications.* (2nd ed., New York: Wiley, 1957.

LUCE, R. D. *Individual choice behavior: a theoretical analysis.* New York: Wiley, 1959.

LUCE, R. D. A threshold model for simple detection experiments. *Psychol. Rev.*, 1963, **70,** 61–79. (a)

LUCE, R. D. Detection and recognition. In R. D. Luce, R. R. Bush, and E. Galanter (Eds.), *Handbook of mathematical psychology.* Vol. 1. New York: Wiley, 1963. Pp. 103–189. (b)

RESTLE, F. A theory of discrimination learning. *Psychol. Rev.*, 1955, **62,** 11–19.

SHIPLEY, ELIZABETH F. A model for detection and recognition with signal uncertainty. *Psychometrika,* 1960, **25,** 273–289.

TANNER, W. P., Jr., and SWETS, J. A. A decision-making theory of visual detection. *Psychol. Rev.*, 1954, **61,** 401–409.

WYCKOFF, L. B. The role of observing responses in discrimination learning. Pt. I. *Psychol. Rev.*, 1952, **59,** 431–442.