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COMMENTS ON ROZEBOOM'S CRITICISMS OF "ON THE POSSIBLE PSYCHOPHYSICAL LAWS"[1]

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The publication of Rozeboom's (1962) criticisms of my paper concerning limitations on scientific laws of certain types (Luce, 1959), together with what we both feel are misinterpretations and misleading uses of those results, prompt me to attempt to clarify some of the issues. At the outset, let me admit that much of the confusion is my fault, due in part to an inadequate understanding of the problem and in part to a not wholly satisfactory presentation of the ideas and results. These failures have been aggravated by some readers who seem to have ignored the second half of that paper and, thereby, have oversimplified its content.

Rozeboom takes me to task for an ambiguous statement of the two criteria that were suggested as a possible "principle of theory construction"—despite these ambiguities, which he has discussed in detail, we both agree about the mathematical "nugget" to be extracted from them, namely his Equation 2. We both emphasize that there are physical laws that either fail to satisfy this equation or satisfy it vacuously, and he thus concludes, correctly I now believe, that it should never have been termed a "principle." We also both point out that whether Equation 2 has any bite in limiting the form of a law has something to do with the existence of dimensional parameters in the law. We seem to differ mainly on the emphasis to place upon this distinction. Because I believe that it is extremely important, I should like to make a few remarks about it.

Among the various lawful statements that can be found in any science, Rozeboom and I are restricting our attention to what may be called function laws—the assertion of a functional relation among several variables, each of which can be represented as a simple numerical scale with certain invariance properties dictated by its measurement theory. As a specific example, from which we will generalize, consider the familiar law $E = mc^2$, where E denotes energy, m denotes mass, and c the velocity of light in a vacuum, all measured in some consistent set of units. It will be convenient to think of this law in the form $E - mc^2 = 0$. The first thing involved in this physical law is the assertion that there exists a particular function of three variables, call it ψ, such that all physically realizable values of E, m, and c satisfy

$$\psi(E, m, c) = 0$$

But there is a good deal more intended than just this simple mathematical equation. For example, not all of the arguments in the equation are of the same type: they are constrained in different ways. We usually speak of E as the dependent variable, m as the independent variable, and c as a dimensional parameter. By these words we mean such things as this: that certain of the symbols can assume arbitrary (positive) values and that the law then constrains others to have particular values, that other symbols represent constants that cannot be experimentally manipulated, and that certain measurement transformations of some of the symbols can be carried out provided that the transformations are admissible according to the measurement theories for these variables and that certain variables and parameters are transformed in prescribed ways. In other words, the statement of a law involves not only the usually written equation, but also certain specifications about which values can be arbitrarily selected, about which variables can be

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freely transformed, and about which values and transformations are prescribed when these choices are made.

To generalize from the example, we distinguish three kinds of symbols that enter into our functional relation (the distinctions among them are rendered more precise later): There are $m + 1$ variables, one of which is designated as

1. the dependent variable, y, and it assumes values in a set Y of real numbers, and the other $m \geqslant 1$ are designated as

2. independent variables, x_i, and they assume values in, respectively, sets X_i of real numbers; in addition, there are

3. $n \geqslant 0$ dimensional parameters, a_j.

Associated with each of these quantities is a group of admissible transformations. In the case of a variable, the group is determined by its measurement theory whereas, in the case of a parameter, it is simply included as part of the statement of the law. We denote these groups by $\mathbf{Y}$, $\mathbf{X}_i$, and $\mathbf{A}_i$. The more common transformation groups found in science include the affine group (multiplication by a positive constant), the full linear group, and the group of strictly monotonic increasing transformations—corresponding, respectively, to what are called ratio, interval, and ordinal scales.

Once the variables, parameters, and transformation groups are listed, the statement of a law then involves two distinct assertions. First, a particular function ψ of $1 + m + n$ variables is given which has the property that for any physically realizable choice of $x_i \epsilon X_i$, $i = 1, 2, \cdots, m$, any physically realizable value $y(x_1, \cdots, x_m) \epsilon Y$ of the dependent variable satisfies

$$\psi[y(x_1, \cdots, x_m), x_1, \cdots, x_m, a_1 \cdots, a_n] = 0$$

Second, for any set of (admissible) transformations $T_i \epsilon \mathbf{X}_i$, $i = 1, 2, \cdots, m$, of the independent variables, there exists an (admissible) transformation $U(T_1, \cdots, T_m) \epsilon \mathbf{Y}$ of the dependent variable and a set of (admissible) transformations $S_j(T_1, \cdots, T_m) \epsilon \mathbf{A}_j$, $j = 1, 2, \cdots, n$, of the dimensional parameters such that

$$\psi(y, x_1, \cdots, x_m, a_1, \cdots, a_n) = 0$$

implies

$$\psi[U(T_1, \cdots, T_m)y, T_1x_1, \cdots, T_mx_m, S_1(T_1, \cdots, T_m)a_1, \cdots, S_n(T_1, \cdots, T_m)a_n] = 0.$$

The first part of such a law, the function relating the variables, is always stated explicitly, but the second part is often implicit. It is simply a convention that scientists know and take for granted. For example, if a decay law for a particular radioactive material is

$$q = q_0 e^{-0.14t}$$

when the time t since a certain event is measured in seconds, then everyone knows that if we measure time in hours we must change the time constant from 0.14 to (0.14)(360) = 50.4.

Despite the fact that they are often left implicit, these transformation properties are important and must be made explicit in a full statement of a law. Indeed, one of the things that distinguishes theoretical science from a collection of simple numerical assertions are the relations among admissible transformations. In addition, the relations among transformations make the distinctions among the arguments of a law more than just linguistic conventions. Note that the independent variables are the arguments of ψ whose values in the first part and whose transformations in the second are selected arbitrarily; the value and transformation of the dependent variable are both determined via the law by those of the independent variable; and the value of a dimensional parameter is independent of the values of the variables and its transformation is determined by those applied to the independent variables. It is often possible to restate the law so that a different variable is treated as dependent; but this is not always possible, as for example, when the dependent variable is a periodic function of the independent ones.

Now consider a law (such as $F = ma$) that includes no dimensional parameters, and suppose that the solution y of

$$\psi(y, x_1, \cdots, x_m) = 0$$

is unique, i.e., there is a function ϕ of m variables such that

$$y = \phi(x_1, \cdots, x_m)$$

solves the above equation. Then the second part of the law can be rewritten as: for any $T_i \epsilon \mathbf{X}_i$, $i = 1, 2, \cdots, m$, then there exists $U(T_1, \cdots, T_m)\epsilon\mathbf{Y}$ such that if

$$y = \phi(x_1, \cdots, x_m)$$

then

$$U(T_1, \cdots, T_m)y = \phi(T_1x_1, \cdots, T_mx_m)$$

Substituting the first of these equations into the second, we obtain the following restriction on the function

$$U(T_1, \cdots, T_m)\phi(x_1, \cdots, x_m) = \phi(T_1x_1, \cdots, T_mx_m)$$

which is Rozeboom's Equation 2 for m independent variables; it includes the several special cases investigated in Luce (1959). For ratio and interval scales and $m = 1$, it was shown that, aside from numerical constants, there are very few possible solutions ϕ to this functional equation.

When dimensional parameters are admitted, the resulting equation is very much less restrictive. Specifically, with one independent variable and one dimensional parameter, the same argument leads to

$$U(T)\phi(x, a) = \phi[Tx, S(T)a]$$

which has many possible solutions. For example, suppose that both $y = \phi(x, a)$ and x are variables measured on ratio scales, and suppose that f is an arbitrary function of one variable. A solution to this equation is $\phi(x, a) = f(xa)$ provided that when x is transformed by $Tx = kx$, then U is the identity transformation and $S = T^{-1}$, i.e., $Uy = y$ and $Sa = a/k$.

Two conclusions seem justified. First, Rozeboom is correct in saying that the equation that results when there are no dimensional parameters is much too restrictive to be called a principle of theory construction. Second, and this I feel to be the main thrust of both my 1959 paper and this note, psychologists (as well as other scientists) either are restricted to a very few possible types of laws (in the case of one independent variable, to those given in my 1959 paper when the variables are measured on either ratio or interval scales) or they cannot avoid including dimensional parameters in the statement of their laws. Not only must these parameters be written explicitly, but a careful prescription should be included about how they transform as a function of admissible transformations of the independent variables, for otherwise the statement of the law is incomplete. Following the practice of physicists, the transformational aspects of laws are often not made explicit. This does not seem to create serious problems when the transformations are limited to the affine group (ratio scales), but matters are much less obvious when interval and weaker scales are involved. In these cases, which often seem to arise in psychology, it is essential that careful attention be given to the transformation properties of the variables and parameters.

Finally, the problem remains whether or not the number of dimensional parameters involved in the statement of a law is of any inherent importance. Some physicists seem to feel that such parameters are undesirable and that their total number in a science should be held to a minimum. My earlier results strongly suggest that the minimum cannot possibly be zero. Although psychologists have hardly begun to face these problems, inevitably they will. At that time, it would be useful to have a better understanding of the role of dimensional parameters than we have now. In physics they have meaning—the velocity of light, the density of a substance, the viscosity of a fluid, the gravitational constant, etc.—and they recur in various combinations in different laws. As yet, we have little of this in psychology.

SUMMARY

It is pointed out that certain scientific laws assert not only a functional relation among variables and dimensional parameters, but that they also state what transformations on the dependent variable and the parameters are necessary for the same function to be satisfied when admissible transformations are applied to the independent variables. It is shown when there are no dimensional parameters, these transformation conditions are very restrictive but that they are not when one or more dimensional parameters are present. The conclusion is that psychological laws will in general involve dimensional parameters and that it is necessary to state carefully the transformational aspects of the law.

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