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A CHOICE THEORY ANALYSIS OF SIMILARITY JUDGMENTS*

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The selection of one of several stimuli as most similar to a reference stimulus is assumed to satisfy a choice axiom that permits assigning ratio scale values to each variable-reference stimuli pair. The logarithm of this scale is treated as a distance measure, leading to the following testable conclusions about the pairwise choice probabilities as the reference stimulus is varied. First, the plot is a symmetrically truncated ogive with horizontal tails. Second, if two pairs of choice stimuli have the same midpoint, the ogive of one pair is part of the ogive of the other. In terms of this model, the hysteresis and midpoint displacement effects in the method of bisection are discussed, and relations with Coombs' unfolding techniques are explored.

The experimental technique to be considered is the trivial generalization of the complete method of triads [10] in which a subject is confronted by a finite set T of stimuli from which he must select one as "most similar" to a reference stimulus a. In the method of triads, T consists of only two stimuli.

For each subject in a given experiment and for every T and a, suppose a probability distribution $P_T(\cdot;\, a)$ governs his responses. Thus, $P_T(x;\, a)$ is the probability that, out of T, he selects x as most similar to a. With a held fixed and T treated as a variable, these are simply choice probabilities—not unlike those postulated in many models for ordinary discrimination experiments. Of the various theories that have been proposed to relate such choice probabilities one to another, the following choice axiom, which has been investigated in [5], is assumed.

If the probabilities are all different from 0 and 1, then for $x \in S \subset T$,

$$P_T(x;\, a) = P_S(x;\, a)P_T(S;\, a),$$

where

$$P_T(S;\, a) = \sum_{x \varepsilon S} P_T(x;\, a).$$

(The choice axiom can be stated to cover the case where some of the probabilities are 0 or 1, but we will confine ourselves to the more restricted case where they are different from 0 and 1.) An important, though simple, con-

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sequence of this assumption is that a positive ratio scale exists over the alternatives which, via a simple formula, reproduces all of the probabilities. Because the stimulus a is a parameter in our problem, we must assume that the scale also depends upon a; hence, we write the scale value for stimulus x as $v(x, a)$. The theorem asserts that for $S \subset T$,

$$P_S(x; a) = \frac{v(x, a)}{\sum_{y \epsilon S} v(y, a)}. \tag{1}$$

In the remainder of this paper this assumption will be accepted as correct for similarity judgments, and several assumptions about $v(x,a)$ will be investigated. Actually, of course, our interest is in relations among the probabilities when a is varied. One hopes that such relations may be found because a stimulus can serve both as a reference and, in other presentations, as one of the comparison stimuli; however, it is not at all easy to guess the relations directly. Apparently it is simpler to assign a reasonable interpretation to v, then to impose assumptions upon v that seem plausible in the light of the interpretation, and finally to determine the restrictions thus implied on the probabilities themselves. Although this technique is familiar and has been used to advantage in the past, it is not at all evident to the writer exactly why it works.

Because the subject is asked to render a similarity judgment, it seems possible that $v(x, a)$ is some sort of measure of the similarity, or dissimilarity, of x and a. Of the two, it must be the first because, with a and $T-\{x\}$ fixed and x variable, $v(x, a)$ varies in the same direction as $P_T(x; a)$. Although there is no clear evidence or necessary reason, it is widely held that a measure of similarity must be in some sense symmetric. In this case, the immediate formalization that comes to mind is

$$v(x, y) = v(y, x);^* \tag{2}$$

however, another possibility should also be considered. In contrast to most scales that have been studied in psychology, ours is a ratio scale, which for some purposes means that multiplicative inverses are appropriate symmetric pairs. It is not clear that this is wrong here, so one should also consider the assumption

$$v(x, y) = 1/v(y, x). \tag{3}$$

We shall investigate both assumptions, first the latter and, after rejecting it, then the former.

*The following notational convention is employed. When a stimulus is to be considered fixed, letters such as $a, b, \cdots$ are used; when it is variable, $x, y, \cdots$ are used. Thus, (2) is not written $v(x, a) = v(a, x)$, as one might have expected, because we no longer want to consider a single, fixed reference stimulus.

The Assumption $v(x, y) = 1/v(y, x)$

Rewriting assumption (3) in the form $v(x, y)v(y, x) = 1$, then for four stimuli, w, x, y, z,

$$\frac{v(x, z)v(z, x)}{v(y, z)v(z, y)} = \frac{1}{1} = \frac{v(x, w)v(w, x)}{v(y, w)v(w, y)}.$$

Cross-multiplying,

$$\frac{v(x, z)v(z, x)}{v(y, z)v(w, x)} = \frac{v(x, w)v(z, y)}{v(y, w)v(w, y)}.$$

Let $P(x, y; z) = P_{\{x,y\}}(x; z)$, etc., then from (1) and the last equation,

$$\frac{P(x, y; z)\, P(z, w; x)}{P(y, x; z)\, P(w, z; x)} = \frac{P(x, y; w)\, P(z, w; y)}{P(y, x; w)\, P(w, z; y)}. \tag{4}$$

The following "thought" experiment should convince the reader that (4) cannot be correct. Consider a unidimensional continuum such as sound intensity, and let x and y be stimuli chosen several jnds apart, with y more intense than x. Then choose z to be their midpoint in the sense that z is more intense than x but not as intense as y and $P(x, y; z) = \frac{1}{2}$. Finally choose w more intense than y and such that y is the midpoint of z and w, i.e., $P(z, w; y) = \frac{1}{2}$. Schematically, this situation is shown below.

Substituting these two values in (4),

$$\frac{P(z, w; x)}{P(w, z; x)} = \frac{P(x, y; w)}{P(y, x; w)}.$$

Because $P(w, z; x) = 1 - P(z, w; x)$ and $P(y, x; w) = 1 - P(x, y; w)$, it follows immediately that $P(z, w; x) = P(x, y; w)$. However, on the intensity continuum it was assumed that $x < z < w$, i.e., that z is closer than w to x, so one anticipates that $P(z, w; x) > \frac{1}{2}$. Equally well, $x < y < w$, so $P(x, y; w) < \frac{1}{2}$. But this contradicts what has just been shown to follow from (3). Although the experiment has not been done, the results seem so certain that one can safely reject the original assumption (3).

The Assumption $v(x, y) = v(y, x)$

A necessary and sufficient condition that (2) *hold over* x, y, z *is that*

$$P(x, y; z)P(y, z; x)P(z, x; y) = P(x, z; y)P(z, y; x)P(y, x; z). \tag{5}$$

PROOF. Using (2),

$$1 = \frac{v(x, y)v(y, z)v(x, z)}{v(y, z)v(x, z)v(x, y)} = \frac{v(x, y)v(y, z)v(z, x)}{v(z, y)v(x, z)v(y, x)}.$$

Substituting from (1) and cross-multiplying yields the result.

Conversely, $v(x, z)$ and $v(y, z)$ are determined by the choice probabilities up to an arbitrary multiplicative constant. Similarly, $v(x, y)$ and $v(z, y)$ are determined up to a multiplicative constant, which may be chosen so that $v(z, y) = v(y, z)$. Finally, the constant for $v(y, x)$ and $v(z, x)$ can be chosen so that $v(x, y) = v(y, x)$, leaving only the relation between $v(x, z)$ and $v(z, x)$ unspecified. But (5) ensures that they must be equal.

Although (5) is similar in form to an important condition implied by the choice axiom, derived in ([5], p. 16), they are actually logically independent.

No "thought" experiment seems to reject (5). This is not to say that the condition is correct, but only that if (2) is wrong, it is more subtly wrong than (3).

The notion of symmetry embodied in (2) will be assumed in the remainder of the paper.

Betweeness

The ideas in this section are closely related to nonprobabilistic notions in [3] and [6]. The continued use of the word "between" is justified because the present definition reduces to the usual one when the probabilities are 0 and 1. It should be noted that in terms of some definitions of distance, this definition does not preclude three stimuli forming certain types of triangles.

In the following definitions and results, probabilities of $\frac{1}{2}$ are excluded because such symmetric cases are difficult to handle neatly in stating results.

DEFINITION 1. Let x, y, and z be stimuli, then y is *between* x and z if and only if $P(y, x; z) > \frac{1}{2}$ and $P(y, z; x) > \frac{1}{2}$. Denote this as $x \mid y \mid z$.

DEFINITION 2. Three stimuli form a *similarity intransitivity*, or briefly, an *intransitivity*, if and only if for some labeling x, y, and z, $P(x, y; z) > \frac{1}{2}$, $P(y, z; x) > \frac{1}{2}$, and $P(z, x; y) > \frac{1}{2}$.

Given three stimuli, at most one is between the other two; and, if they do not form an intransitivity, and none of the pairwise probabilities is $\frac{1}{2}$, *then exactly one is between the other two.*

PROOF. Without loss of generality, suppose both that x is between y and z and that y is between x and z; then, by definition, $P(x, y; z) > \frac{1}{2}$ and $P(y, x; z) > \frac{1}{2}$. Adding, $1 < P(x, y; z) + P(y, x; z) = 1$, a contradiction.

Now, suppose that none of the probabilities is $\frac{1}{2}$ and that no stimulus is between the other two. With no loss of generality suppose $P(x, y; z) > \frac{1}{2}$. Because x is not between y and z, it follows that $P(z, x; y) > \frac{1}{2}$. And because z is not between x and y, $P(y, z; x) > \frac{1}{2}$. Thus, the three elements form an intransitivity, contrary to assumption, so one must be between the other two.

If three stimuli satisfy (5), *then they do not form an intransitivity.*

PROOF. Suppose, on the contrary, x, y, and z do form an intransitivity as in definition 2, then

$$\frac{P(x, z; y)}{P(z, x; y)} < 1, \quad \frac{P(y, x; z)}{P(x, y; z)} < 1, \quad \frac{P(z, y; x)}{P(y, z; x)} < 1,$$

so,

$$\frac{P(x, z; y)P(y, x; z)P(z, y; x)}{P(z, x; y)P(x, y; z)P(y, z; x)} < 1,$$

contrary to (5).

This last result establishes that the choice axiom plus the symmetry condition $v(x, y) = v(y, x)$ implies some degree of unidimensionality in the responses, at least in the sense that intransitivities of three stimuli are impossible. Actually, this observation is really little more than a precursor to stating the usual, much stronger notion of unidimensionality: given a distance measure, then for y between x and z the distance from x to z is the sum of the distances from x to y and from y to z. The crucial question, of course, is what is meant by distance. Again, two possibilities come to mind. First, because $v(x, y)$ becomes larger as x and y become more similar, v itself cannot be a measure of distance, but $1/v(x, y)$ could be. Second, because there is evidence from other sources that the logarithm of the v-scale acts much like the interval scales that arise in Fechnerian and Thurstonian scaling, and because these scales have, in one way or another, been treated as measures of distance, $-\log v$ is a possibility. It will be shown that the former interpretation is untenable; then the consequences of the latter for the unidimensional case will be examined.

*The Assumption That $1/v$ Is a Distance Measure**

If $1/v$ is a distance measure, in the usual sense, then $1/v(x, x) = 0$, so

$$P(x, y; x) = \frac{1}{1 + [v(y, x)/v(x, x)]} = 1,$$

for any y, however similar to x. Although it is probably unnecessary to cite data to convince the reader that this is wrong, they do exist in [7].

The Assumption That $-\log v$ Is a Unidimensional Distance Measure

In order that $d(x, y) = -\log v(x, y)$ act like a measure of distance of a unidimensional continuum, it is necessary that

(i) $d(x, y) = d(y, x)$,
(ii) $d(x, y) \geq 0$ and $d(x, x) = 0$,
(iii) if $x \mid y \mid z$, then $d(x, z) = d(x, y) + d(y, z)$.

*The following argument is due to Clyde Coombs; it is simpler than that originally used.

The first condition is guaranteed by the symmetry of v, equation (2). The second is satisfied if $v(x, x) = v(y, y)$, for all x and y, and $v(x, y) \leq v(x, x)$, for one may choose the unit of v so that $v(x, x) = 1$. The condition $v(x, x) = v(y, y)$ is equivalent to

$$P(x, y; y) = P(y, x; x). \tag{6}$$

The third condition is equivalent to

$$\text{if} \quad x \mid y \mid z, \quad \text{then} \quad v(x, z) = v(x, y)v(y, z). \tag{7}$$

It should be noted that if $v(x, z) = v(x, y)v(y, z)$ and if all three v's are <1, then $x \mid y \mid z$.

To investigate the probability implications of (6) and (7), focus attention upon $P(a, b; x)$, treating a and b as fixed stimuli and the reference stimulus as a variable. Assuming that the order between a and b is fixed, there are two cases depending upon whether x is between a and b or not. First, look at the case where x is outside the interval defined by a and b.

If (6) *and* (7) *hold, then for* $a \mid b \mid x$,

$$P(a, b; x) = P(a, b; b),$$

and for $y \mid a \mid b$ *and* $a \mid b \mid x$,

$$P(a, b; x) = 1 - P(a, b; y).$$

PROOF.

$$\begin{aligned} P(a, b; x) &= \frac{1}{1 + [v(b, x)/v(a, x)]} \\ &= \frac{1}{1 + [1/v(a, b)]} \\ &= \frac{1}{1 + [v(b, b)/v(a, b)]} \\ &= P(a, b; b), \end{aligned}$$

which proves the first statement.

To prove the second,

$$\begin{aligned} P(a, b; x) &= \frac{1}{1 + [v(b, b)/v(a, b)]} \\ &= \frac{1}{1 + [v(a, a)/v(b, a)]} \\ &= P(b, a; a) \\ &= 1 - P(a, b; a) \\ &= 1 - P(a, b; y). \end{aligned}$$

Next, consider the case where x is between a and b. To do so, a notion used earlier will be formalized.

DEFINITION 3. The *midpoint* of stimuli a and b is that stimulus $\overline{ab}$ such that $a \mid \overline{ab} \mid b$ and $P(a, b; \overline{ab}) = \frac{1}{2}$.

It follows immediately that $v(a, \overline{ab}) = v(b, \overline{ab})$ and $P(a, \overline{ab}; b) = P(b, \overline{ab}; a)$.

Extending the betweeness notation in the obvious way, if $a \mid c \mid x \mid d \mid b$ *and* $\overline{ab} = \overline{cd}$, *then* $P(a, b; x) = P(c, d; x)$.

PROOF. Because $a \mid c \mid \overline{cd} \mid d \mid b$, (7) and $v(c, \overline{cd}) = v(d, \overline{cd})$ imply

$$\begin{aligned} v(c, b) &= v(c, \overline{cd})v(\overline{cd}, b) \\ &= v(d, \overline{cd})v(\overline{ab}, b) \\ &= v(d, \overline{ab})v(\overline{ab}, a) \\ &= v(d, a) \\ &= v(a, d). \end{aligned}$$

But, by (7),

$$v(c, b) = v(c, x)v(x, b)$$

and

$$v(a, d) = v(a, x)v(x, d).$$

Hence,

$$\frac{v(c, x)}{v(d, x)} = \frac{v(a, x)}{v(b, x)},$$

from which the result follows by (1).

The empirical import of these two results is most easily seen graphically. If one considers pairs of stimuli having the same midpoint, then, independent of these stimuli, there is some function—presumably ogival—which determines $P(a, b; x)$ for $a \mid x \mid b$. For x outside this region, P is either the constant $\alpha = P(a, b; a)$ or $P(a, b; b) = 1 - \alpha$. See Fig. 1. Both aspects of this prediction should be possible to test experimentally.

If one only requires that $-\log v(x, y)$ act like a distance measure, not a unidimensional one, in the sense that part (iii) of the axiom is replaced by the triangle inequality, namely, for $x \mid y \mid z$, $d(x, z) \leq d(x, y) + d(y, z)$, then using much the same methods it is easy to show that

(i) $P(a, b; b) = 1 - P(a, b; a)$,

(ii) for $a \mid b \mid x$, $\quad P(a, b; b) \geq P(a, b; x)$,

(iii) and for $y \mid a \mid b$, $\quad P(a, b; a) \leq P(a, b; y)$.

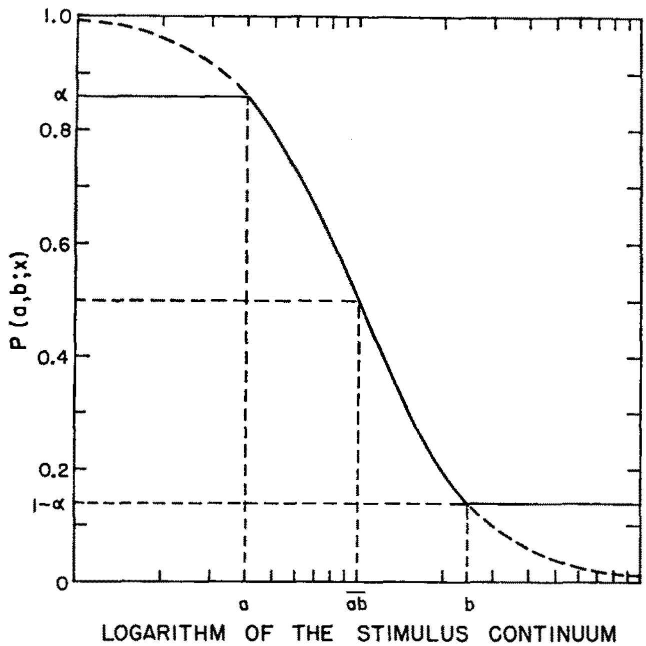

FIGURE 1

Theoretical Plot of $P(a, b; x)$ as a Function of x

Thus, the effect of this change in assumptions is to round the corners of the function in Fig. 1 as the reference stimulus passes by a or b.

More detail about the form of $P(a, b; x)$ when x is between a and b can be determined by the following argument. When the reference stimulus x is extremely far from a and b, either above both or below both, the subject really only has to discriminate between a and b. For example, if beyond a doubt x is larger than both a and b, then he will report as most similar to x the one he believes to be larger, i.e., if $a \mid b \mid x$,

$$P(a, b) = P(a, b; x).$$

But it has just been shown that

$$P(a, b; x) = P(a, b; b).$$

So,

$$P(a, b) = P(a, b; b).$$

Thus, if one assumes the discrimination probabilities also satisfy the choice axiom and if one denotes the corresponding scale values by $v(x)$, then for $v(a) \leq v(b)$ and $a \mid b \mid x$

$$\frac{1}{1 + [1/v(a, b)]} = \frac{1}{1 + [v(b)/v(a)]}.$$

The other possible cases yield the same result, namely

$$v(a, b) = \begin{cases} v(a)/v(b) & \text{if} \quad v(a) \leq v(b) \\ v(b)/v(a) & \text{if} \quad v(a) \geq v(b). \end{cases} \tag{8}$$

Equation (8), then, establishes a basic connection between the discrimination and the similarity data if the present theory is correct. Indeed, similarity distance, $-\log v(a, b)$, is simply the absolute value of the difference of the logarithms of the discriminative scale values—what have been called Fechnerian scale values [5]. Thus, the model is substantially like Coombs' unfolding technique, where $-\log v(a, b)$ is the folded scale and $\log v(a)$ the unfolded one.

The existing evidence [7] is against the assumption $P(a, b) = P(a, b; b)$, but rather would suggest $P(a, b) < P(a, b; b)$. If one accepts the above argument for x sufficiently far from b and the discussion stemming from the triangular inequality, then

$$P(a, b) = P(a, b; x) < P(a, b; b),$$

and (8) is replaced by

$$v(a, b) > \begin{cases} v(a)/v(b) & \text{if} \quad v(a) \leq v(b) \\ v(b)/v(a) & \text{if} \quad v(a) \geq v(b). \end{cases}$$

Turning now to the case where $a \mid x \mid b$, then for $v(a) \leq v(x) \leq v(b)$, (8) implies

$$P(a, b; x) = \frac{1}{1 + [v(b, x)/v(a, x)]} = \frac{1}{1 + [v(x)^2/v(a)v(b)]}. \tag{9}$$

Bisection

In the psychophysical method of bisection the subject is required to adjust a variable stimulus until it is "half-way" between two other stimuli, a and b. It is plausible that he selects x so that a and b seem equally similar to it, in which case $x = \overline{ab}$, the midpoint of a and b. Thus, by (9)

$$P(a, b; \overline{ab}) = 1/2 = \frac{1}{1 + [v(\overline{ab})^2/v(a)v(b)]},$$

so

$$v(\overline{ab}) = [v(a)v(b)]^{1/2}.$$

That is to say, the discrimination v-scale value of the midpoint of two stimuli is the geometric mean of their discrimination v-scale values. One needs to convert this to a statement about the physical scale values.

Stevens [8] and Luce [4, 5] have argued that for at least certain classes of continua, the relation between a subjective scale, such as the v-scale, and the physical scale is a power function, i.e., $v(x) = \alpha x^{\beta}$. Thus, if

$$v(\overline{ab}) = [v(a)v(b)]^{1/2},$$

then

$$\alpha(\overline{ab})^{\beta} = [\alpha a^{\beta} \alpha b^{\beta}]^{1/2} = \alpha[(ab)^{1/2}]^{\beta}$$

so

$$\overline{ab} = (ab)^{1/2}.$$

Put in words, the physical scale value of the midpoint must also be the geometric mean of the physical scale values of the two stimuli, or, in a logarithmic transform of the physical scale—the corresponding db scale—it must be their average. It is well known that in general this is not correct [8, 9]. Not only is the subjective midpoint often shifted somewhat above the value just predicted, but its location differs depending upon whether the stimuli are presented in order a, x, b or b, x, a—this fact has been called a hysteresis effect.

Thus far any consideration of the well known fact that subjects exhibit response biases, often called time or space errors depending upon the mode of stimulus presentation, has been completely omitted. Possibly this can be used to explain the midpoint displacement and the hysteresis effects in the bisection method. Response biases will be treated in exactly the same way as in ([5], pp. 30–34).

Two distinct biases may enter. The first is due to the order of presentation of the stimuli; it affects their apparent intensities. Because the scale values can all be changed by a multiplicative constant without affecting (1), one of the biases may be chosen to be 1; let them be r, 1, and s for the first, second, and third presentations, respectively. Assuming that the ascending series is $a < x < b$ and the descending one, $b > y > a$, the intensity scale values are

$$\begin{array}{llll} \text{Ascending:} & v(a)r, & v(x), & v(b)s; \\ \text{Descending:} & v(b)r, & v(y), & v(a)s. \end{array}$$

Thus, according to (8), the similarity scale values are

$$\begin{array}{lrl} \text{Ascending:} & v(a, x) &= v(a)r/v(x), \\ & v(b, x) &= v(x)/v(b)s; \\ \text{Descending:} & v(a, y) &= v(a)s/v(y), \\ & v(b, y) &= v(y)/v(b)r. \end{array}$$

The second bias arises if the subject has a differential tendency to set the middle stimulus nearer either the first or the last one presented. Let these biases be, respectively, 1 and t; so assume that x and y are chosen so that

$$\text{Ascending:}\quad v(a, x) = v(b, x)t,$$

which by previous equations is easily seen to be equivalent to

$$v(x) = v(\overline{ab})(rs/t)^{1/2};$$

and

$$\text{Descending:}\quad v(a, y)t = v(b, y),$$

which is equivalent to

$$v(y) = v(\overline{ab})(rst)^{1/2}.$$

A hysteresis effect exists if and only if $v(x) \neq v(y)$, i.e., if and only if $t \neq 1$; it is of the sort observed, namely, $v(x) > v(y)$, if $t < 1$. Assuming $t = 1$, the bisection point differs from the midpoint provided $rs \neq 1$, and it is above the midpoint, as is generally observed, provided $rs > 1$. With $t \neq 1$, both bisection points are above the midpoint provided $rs > t$ and $> 1/t$. According to this model, the displacement from the midpoint and the hysteresis are independent biasing effects that one should be able to manipulate independently, e.g., by payoffs.

Strong Stochastic Transitivity

For a reference stimulus x, the condition of strong stochastic transitivity is:

if $P(a, b; x) > \frac{1}{2}$ and $P(b, c; x) > \frac{1}{2}$,

then $P(a, c; x) \geq P(a, b; x), P(b, c; x)$.

Because the choice axiom has been assumed for a fixed reference stimulus, it is known ([5], p. 19) that this condition is satisfied.

Coombs [1, 2] in discussing preference data has argued that, at least in some cases, the choice between two stimuli really is determined by their similarity to some subjectively ideal stimulus on the continuum being judged, each subject having his own ideal. Thus, the present model for the method of triads, rather than the corresponding simple choice model, should apply to such data. Furthermore, Coombs has argued that if the subject fails to hold the ideal fixed, then apparent violations of strong stochastic transitivity can be expected to occur. This idea will now be examined in terms of the present model.

For the sake of simplicity, suppose that the variations of the ideal x are sufficiently small relative to the separations between the stimuli so that the order relations between the stimuli and the ideal are unchanged. As Coombs has pointed out, there are two inherently different cases. A unilateral triple is a set of judged stimuli, $\{a, b, c\}$, which are all on one side of x, e.g., $a \mid b \mid c \mid x$. It is not difficult to show that variations in x, subject to the requirement that the order relations not change, cannot affect the strong stochastic transitivity property in this case.

Bilateral triples are of the form $a \mid x \mid b \mid c$. Suppose that $v(a) > v(x)$, $v(y)$, $v(z) > v(b) > v(c)$. The first case considered here is what Coombs has called a bilateral adjacent triple:

$$P(a, b; x) > 1/2, \qquad P(b, c; y) > 1/2, \quad \text{and} \quad P(a, c; z) > 1/2.$$

Substituting

$$P(a, b; x) = \frac{1}{1 + [v(a)v(b)/v(x)^2]}, \quad \text{etc.,}$$

these three conditions are equivalent, respectively, to

$$v(x) > v(\overline{ab}), \qquad v(c) < v(b), \quad \text{and} \quad v(z) > v(\overline{ac}).$$

Violations of strong stochastic transitivity can occur in two ways:

$$P(a, c; z) < P(b, c; y), \text{ which is equivalent to } v(z) < v(\overline{ab}),$$

and

$$P(a, c; z) \qquad < P(a, b; x), \text{ which is equivalent to } v(z) < v(x)\left[\frac{v(c)}{v(b)}\right]^{1/2}.$$

The first violation appears to be easy to obtain provided the ideal is in the neighborhood of the midpoint $\overline{ab}$, for the only requirements are $v(x) > v(\overline{ab}) > v(z) > v(\overline{ac})$. The second appears much less likely to occur if c and b are not too close, for it requires a considerable shift in the ideals x and y. Coombs has found the first violation common, and the second much more rare in his data (personal communication).

The second case is that of a bilateral split triple

$$P(b, a; x) > 1/2, \qquad P(a, c; z) > 1/2, \quad \text{and} \quad P(b, c; y) > 1/2,$$

which are equivalent to $v(x) < v(\overline{ab})$, $v(z) > v(\overline{ac})$, and $v(c) < v(b)$. The possible violations are:

$$P(b, c; y) < P(b, a; x), \text{ which is equivalent to } v(x) < v(\overline{ac}),$$

and

$$P(b, c; y) < P(a, c; z), \text{ which is equivalent to } v(z) > v(\overline{ab}).$$

Thus, for the first to occur, the ideal must be located in the neighborhood of the $\overline{ac}$ midpoint and for the second it must be in the neighborhood of the $\overline{ab}$ midpoint. It is not obvious why this cannot happen, yet such violations are very rare in Coombs' data.

One may conclude, nonetheless, that the present theory is entirely consistent with Coombs' idea of violations in strong stochastic transitivity for certain types of data without, in fact, forcing one to reject the choice axiom.

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