18

Tests of the "Beta Model"

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The general stochastic learning models that have been previously studied [1] postulate a probability distribution

$$p_n = \{p_n(1), p_n(2), \cdots, p_n(r)\}$$

over the set of r alternatives available to an organism. This vector gives the probability that each alternative will be chosen on trial n. A transition operator $\boldsymbol{T}$ is postulated such that (i) $\boldsymbol{T}$ does not depend upon n (independence of path); (ii) $\boldsymbol{T}$ depends upon the choice made on trial n and on the outcome; and (iii) $\boldsymbol{p}_{n+1} = \boldsymbol{T}\boldsymbol{p}_n$.

For the most part linear (matrix) operators have been studied—partly because their mathematical properties are comparatively simple, partly because of Estes' stimulus-sampling rationale [3], and partly because the combining-of-classes condition ([1], [2]) leads to a particular type of linear operator. Nonetheless, it is still an open question whether linearity is a tenable assumption or whether one of the possible nonlinear operators will be better able to describe data. The problem, of course, is how to select among all the possible nonlinear operators.

The purpose of this paper is to study some properties of a nonlinear model, called the beta model, and to apply it to three published experiments (Chapter 14 and [6]). The linear model, called the alpha model, and the beta model can both be arrived at from the same general considerations. For this reason, comparisons between these two models are made.

Response-Strength Models

Some learning theorists ([4], [7]) consider response frequencies and their underlying probabilities to be the manifestation in behavior of some latent construct called response strength. Earlier stochastic learning models have

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been criticized for their failure to postulate the source of the response probabilities with which they deal. Recently, the response-strength notion has been formalized [5]; we review it briefly here and then investigate some of its consequences.

Suppose that i is an alternative in the set S, which, in turn, is a subset of the finite set T. Let $P_S(i)$ denote the probability that i is chosen when the choice is confined to S and $P_T(i|S)$, the probability that i is chosen when the choice is confined to T, conditional on its being in S. If $P_{\{i,j\}}(i) \neq 0$ or 1, where i and j are in T, then we postulate

$$P_S(i) = P_T(i|S) .$$

This axiom leads to the conclusion that there exists a ratio scale, v, over the alternatives with the properties

$$P_S(i) = \frac{v(i)}{\sum_{y \in S} v(y)} , \tag{1}$$

and

$$v(i) > 0, \quad \text{for all } i \in T . \tag{2}$$

This v-scale, which in psychophysical problems appears to be closely related to scales that have been studied earlier, is presumed to be the formal counterpart of response strength. For many purposes, this scale may be more useful than the choice probabilities themselves.

In most learning experiments the set of alternatives is fixed, and so one cannot make a direct check of the axiom that leads to the v-scale; however, since this axiom refers to an organism, not to an experiment, it is meaningful to study its consequences for learning. We formulate the learning process as sequential transitions of the v-vector, and let this stochastic process indirectly determine the response probabilities via Equation 1.

Let $\boldsymbol{v}_n = \{v_n(1), v_n(2), \cdots, v_n(r)\}$ denote this vector on trial n and let $\boldsymbol{T}$ now denote a path-independent operator at the level of the v's. The transition equation is

$$\boldsymbol{v}_{n+1} = \boldsymbol{T}\boldsymbol{v}_n . \tag{3}$$

The problem is to restrict $\boldsymbol{T}$. By Equation 2, the v's must be positive; hence, if $\boldsymbol{T}\boldsymbol{v}$ is a distribution of v-values, we must have

$$\boldsymbol{T}\boldsymbol{v} > \boldsymbol{0}, \quad \text{if} \quad \boldsymbol{v} > \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{0}$ is the r-dimensional zero vector. A second condition stems from the fact that v is a ratio scale; therefore, we can multiply all values in the model by any positive constant k. In particular, the equality in Equation 3 should not be affected, so we must have

$$k\boldsymbol{T}\boldsymbol{v}_n = \boldsymbol{T}k\boldsymbol{v}_n, \quad \text{if } k > 0 . \tag{5}$$

This has been called the independence-of-unit condition. If it is not met, then in principle we could determine the unit of the v-scale from learning data. These first two limitations seem basic to the way our problem is formulated. However, Equations 4 and 5 do not narrow down $\boldsymbol{T}$ sufficiently, and, unfortunately, no other conditions seem to follow from the basic choice axiom. Thus, we are forced to make substantive assumptions that are suggested largely by mathematical considerations. First, we note that Equation 5 is one of the two properties that are usually used in defining a linear transformation; the other is

$$\boldsymbol{T}(\boldsymbol{v} + \boldsymbol{v}^*) = \boldsymbol{T}\boldsymbol{v} + \boldsymbol{T}\boldsymbol{v}^* , \quad \text{if } \boldsymbol{v}, \boldsymbol{v}^* > 0 . \tag{6}$$

Often this is called the superposition condition. Even though it is difficult to give an intuitive interpretation for Equation 6 because the addition of two v-vectors does not correspond naturally to any experimental manipulation, we shall impose the condition.

Finally, we shall assume that any positive real number is a possible v-scale value; hence, there is no upper bound to the possible values. This assumption together with Equations 5 and 6 implies that the transformation $\boldsymbol{T}$ must be a matrix operator T, in which case Equation 3 becomes the matrix equation

$$\boldsymbol{v}_{n+1} = T\boldsymbol{v}_n , \tag{7}$$

where $\boldsymbol{v}$ denotes a column vector. Equation 4 implies that T is nonsingular and has nonnegative entries.

The Two-Alternative Alpha Model. For two alternatives, Equation 7 becomes

$$\begin{bmatrix} v_{n+1}(1) \\ v_{n+1}(2) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} v_n(1) \\ v_n(2) \end{bmatrix} . \tag{8}$$

Although, in principle, we could work with Equation 8 in its full generality, in practice there are too many parameters. Each operator has four, and there is usually more than one operator. So we are forced to consider further restrictions.

A question that immediately comes to mind is whether there is any specialization that leads to an operator that is linear in the probabilities. It is not difficult to show (see [5]) that a necessary and sufficient condition is that the column sums of T be equal, i.e.,

$$t_{11} + t_{21} = t_{12} + t_{22} . \tag{9}$$

This specialization is the alpha model. Observe that Equation 9 implies that the sum of the scale values on trial $n + 1$ is simply $t_{11} + t_{21}$ times the sum of the scale values on trial n. This means that, independent of how the total scale value is distributed between the two alternatives, experience on a given trial augments or diminishes that sum by a fixed factor; however, the change in the scale value for a particular alternative is not independent of the distribution over the alternatives. Thus, for example, if alternative 1 is chosen and rewarded on trial n, $v_{n+1}(1)$ depends not only upon $v_n(1)$ but also upon $v_n(2)$ (the propensity to choose alternative 2).

The Two-Alternative Beta Model. The last observation suggests the other model that we examine. We postulate that $v_{n+1}(i)$ depends upon $v_n(i)$, but not upon the scale value of the other alternative. This means that the matrix T in Equation 8 must be diagonal, i.e.,

$$v_{n+1}(1) = t_{11}v_n(1) , \qquad v_{n+1}(2) = t_{22}v_n(2) . \tag{10}$$

From Equation 1 and Equation 10 we have

$$p_{n+1}(1) = \frac{v_{n+1}(1)}{v_{n+1}(1) + v_{n+1}(2)} = \frac{t_{11}v_n(1)}{t_{11}v_n(1) + t_{22}v_n(2)} = \frac{\beta\dfrac{v_n(1)}{v_n(2)}}{\beta\dfrac{v_n(1)}{v_n(2)} + 1}$$

where $\beta = t_{11}/t_{22}$. But

$$\frac{v_n(1)}{v_n(2)} = \frac{p_n(1)}{1 - p_n(1)} ,$$

so

$$p_{n+1}(1) = \frac{\beta p_n(1)}{(\beta - 1)p_n(1) + 1} . \tag{11}$$

Of course,

$$p_{n+1}(2) = 1 - p_{n+1}(1) = \frac{p_n(2)}{p_n(2) + \beta[1 - p_n(2)]} .$$

We observe that, like the alpha model, the beta model can be expressed in terms of path-independent operators acting upon the probabilities. So both models are path-independent at both the level of the v's and the level of the p's, both are linear at the level of the v's, but only the alpha model is also linear at the level of the p's.[2]

For experiments in which one outcome always follows alternative 1 and another outcome always follows alternative 2, the transition law for $p_n = p_n(2)$ is

$$p_{n+1} = \begin{cases} \dfrac{p_n}{p_n + \beta_1(1 - p_n)} & \text{if alternative 1 occurs} \\[2ex] \dfrac{p_n}{p_n + \beta_2(1 - p_n)} & \text{if alternative 2 occurs .} \end{cases} \tag{12}$$

Because p_n is the probability of an error, we anticipate that $\beta_1 > 1$ and $\beta_2 > 1$.

[2] In [5], a somewhat different derivation of the beta model is given, based, in essence, on the condition leading to Equation 10. This condition, without explicitly assuming superposition, then leads to Equation 10. One merit of this approach is that it suggests a third model which differs from the beta model only in that the unboundedness condition is replaced by the condition that the v's are bounded from above. This model is linear and path-independent at the level of the v's, but is not path-independent at the level of the p's.

Estimation of Beta-Model Parameters

To estimate model parameters from a set of data, it is useful—and with three or more parameters, almost essential—to have explicit formulas for properties of the model as functions of its parameters. These functions may then be equated to the corresponding statistics of the data and solved to give estimates of the parameters. Preferably, these expressions should be in closed form, but infinite series are acceptable since tables can be prepared. For the alpha model restricted in various ways, a number of closed expressions are known and several tables have been published. For the beta model, the situation is far less satisfactory because its nonlinearity makes it very difficult to calculate expected values. In fact, for two alternatives with partial reinforcement of each, no computable expression is known for any property of the model. If, however, we are willing to confine our attention to those experiments in which one of the alternatives, say 2, is never rewarded, then a series can be developed for the expected number of trials before the other alternative is chosen. This can be used not only with experiments in which alternative 1 is always rewarded and 2 never rewarded (100 : 0 experiments), but also with experiments in which alternative 1 is rewarded with probability π while alternative 2 is never rewarded (50 : 0 experiments, for example).

Let p_n denote the probability that alternative 2 is chosen (i.e., an error is made) on trial n, and let $\nu + 1$ denote the trial number when alternative 1 is first chosen (i.e., the trial number of the first success). Thus, ν denotes the number of trials before the first success. Because these trials are independent,

$$\Pr(\nu = k) = \prod_{i=1}^{k} p_i(1 - p_{k+1}) \, .$$

Hence,

$$E(\nu) = \sum_{k=1}^{\infty} k \Pr(\nu = k) = \sum_{k=1}^{\infty} k(1 - p_{k+1}) \prod_{i=1}^{k} p_i \, . \tag{13}$$

Let β denote the beta-model parameter of the (nonreward) operator that is always applied when alternative 2 is chosen. By induction on Equation 12 it is clear that for any $n \leqq \nu + 1$,

$$p_{n+1} = \frac{v}{v + \beta^n} \, , \tag{14}$$

where we have defined $v = v_1(2)/v_1(1) = p_1/(1 - p_1)$. Substituting Equation 14 in Equation 13, we obtain

$$E(\nu) = \sum_{k=1}^{\infty} k \left[1 - \frac{v}{v + \beta^k}\right] \prod_{i=1}^{k} \left[\frac{v}{v + \beta^i}\right] \tag{15}$$

$$= \sum_{k=1}^{\infty} k \left[\frac{\beta^k}{v + \beta^k}\right] \frac{v^k}{\prod_{i=1}^{k} (v + \beta^{i-1})}$$

$$= \sum_{k=1}^{\infty} \frac{k\beta^k v^k}{\prod_{j=0}^{k} (v + \beta^j)} = \sum_{k=1}^{\infty} \frac{k\beta^k \left[\dfrac{p_1}{1 - p_1}\right]^k}{\prod_{j=0}^{k} \left[\dfrac{p_1}{1 - p_1} + \beta^j\right]} \, .$$

The final infinite series for the expected number of trials before the first success, which is an example of a function we denote by $L(p, \beta)$, can be computed to any degree of accuracy for any p and β. This is not simple, however, when both parameters are near 1; for example, when $p = 0.9995$ and $\beta = 1.03$, one needs 227 terms to obtain accuracy in the third decimal place. For the experiments that we will analyze, p is very close to 1; hence, we were led to have a table of $L(p, \beta)$ prepared by the Univac computer at the University of Pennsylvania. The table is given at the end of this paper.

With a value of $E(\nu)$, Equation 15 imposes a relationship between p and β, but it does not specify either parameter uniquely. Thus, one must either estimate one of the parameters independently or undertake a trial-and-error exploration of the parameter space using Monte Carlo methods (to match other statistics of the data, such as the total number of errors). For example, in a 100 : 0 experiment, if there are sufficiently many subjects that the choices on trial 1 can be used to estimate p accurately, then the trials to the first success determine the nonreward parameter β_2. This still leaves the parameter β_1 of the reward operator unspecified.

Now observe that if we go to a final trial, N, we can estimate p_N from the observed number of choices on that trial, and if we proceed backwards from that trial to the last error, then only the reward operator will be applied during these trials. Thus, $L(1 - p_N, \beta_1)$ gives the expected number of trials between the last error and the final trial. Matching this expected value to the observed value provides an estimate of β_1.

The method just described does not require that we use the estimated probability on the first and last trials, and in fact we do not. Rather, we used the observed mean learning curve to judge the trial numbers (not necessarily integers) for which the proportion of errors is 0.95 and 0.05. The mean number of trials to the first success and to the last failure from these trials, respectively, was determined, and β_1 and β_2 were thereby estimated from the table of $L(p, \beta)$. The initial probability, p_1, was estimated by applying the inverse of the nonreward operator from the 0.95 trial to the first trial. For example, if the 0.95 trial is 8 and $\hat{\beta}_2 = 1.3$, then $v_8 = 0.95/0.05 = 19$ and so $v_1 = (1.3)^8 \, (19) = 154.98$. Thus,

$$p_1 = \frac{154.98}{155.98} = 0.994 \, .$$

The choice of the points 0.95 and 0.05 is based upon two considerations. First, the probability should not be very far from 1 if the inverse of the nonreward operator can be legitimately applied to estimate the initial probability. Second, if p is very near 1, the mean and variance of the trials to the first success becomes very large, and the estimate of β will not be very stable.

This method has the severe drawback that the mean learning curve is fairly flat at these two points, so that the estimated trial numbers are rather sensitive to what smooth curve is passed through the data points. This variability is reflected in a considerable variability of the estimated parameters. It is, therefore, essential that some check be made on the estimates before they are taken too seriously. The one that we use is whether statistics computed from Monte Carlos using the estimates match the corresponding data statistics. A more refined but time-consuming procedure is to determine the theoretical mean curve from the first set of Monte Carlos, use this to reestimate the 0.05 and 0.95 trials, calculate from the data the new mean number of trials to the first success and to the last failure, and then reestimate the parameters. Using these new parameters, a second set of Monte Carlos is run and the whole process is repeated a number of times until convergence is obtained.

We now describe three different experiments reported elsewhere. For each of these experiments we estimate parameter values for the beta model. For two of the experiments, we use these estimates to compute Monte Carlo analogues of the data.[3] Statistics from these computations are then compared with the experimental data and with previously reported statistics of the alpha model.

An Avoidance-Learning Experiment

The Experiment. Solomon and Wynne reported an experiment [6] in which dogs were trained to jump a barrier in a shuttlebox to avoid or escape from a traumatizing electric shock. Prior to the onset of the shock a discriminative stimulus was presented to the animals, and, provided they had learned, they could vault the barrier and consistently avoid shock. The complete sequence of escapes and avoidances for each of the 30 dogs is presented in [1].

Parameter Estimation. By using the technique outlined in the preceding section, parameters were estimated. One point of interest should be made. The initial probability of an avoidance, estimated from the first trial of the experiment, is 0.0. A feature of the beta model is that, unless the initial probability of a success is different from zero, learning will not occur. For this reason, the proportion on the second trial was used to estimate β_2, and then these estimates were used to estimate p_1. The results obtained were $\hat{p}_1 = 0.06$, $\hat{\beta}_1 = 1.2$, $\hat{\beta}_2 = 1.7$. Monte Carlo computations were then made.

Goodness-of-Fit. In Table 1 we record 15 statistics computed from the experimental data and from the beta-model Monte Carlos, along with the statistics previously reported for the alpha model. In Chapter 15 the corresponding statistics for seven other models are given.

[3] All Monte Carlo computations in this study were done at the Computer Center, University of Pennsylvania. For this purpose, a general program for stochastic learning models was developed by Dr. Saul Gorn and Mr. Peter Z. Ingerman.

TABLE 1

Comparison of Several Statistics of the Solomon-Wynne Avoidance-Training Data with the Statistics Obtained from Monte Carlo Computations with the Alpha and Beta Models

Statistic	Real Dogs Mean	Real Dogs S. D.	Alpha Model Stat-Dogs Mean	Alpha Model Stat-Dogs S.D.	Beta Model Stat-Dogs Mean	Beta Model Stat-Dogs S.D.
Trials before first avoidance	4.50	2.25	4.13	2.08	3.93	1.74
Trials before second avoidance	6.47	2.62	6.20	2.06	6.40	1.33
Total number of shocks	7.80	2.52	7.60	2.27	7.80	1.10
Trials before last shock	11.33	4.36	12.53	4.78	13.57	4.17
Number of alternations	5.47	2.72	5.87	2.11	6.50	2.01
Length of longest run of shocks	4.73	2.03	4.33	1.89	4.30	1.29
Trials before first run of four avoidances	9.70	4.14	9.47	3.48	10.13	3.00

A Relearning Experiment

The Experiment. Galanter and Bush (Chapter 14) conducted a T-maze experiment in which rats were rewarded whenever they turned right, and were never rewarded when they turned left. This training period continued at a rate of three trials a day for 48 trials. For the next 48 trials, food reward was always on the left and never on the right. A third and fourth period of 48 trials each were run with food reward on the right again, and then on the left.

Parameter Estimation. Our concern with these data is only the order of magnitude of the two learning parameters; no attempt will be made to fit the data in detail. As pointed out earlier, however, without such a check on the estimates there is some ambiguity about their exact values; nonetheless, certain qualitative conclusions can be made. With the use of two different methods to estimate the mean learning curve, estimates lying between 1.02 and 1.14 were obtained for β_1 from the latter parts of the four periods, and estimates between 1.44 and 2.6 were obtained for β_2 from the initial parts of periods 2, 3, and 4. It is clear, then, that the nonreward parameter is appreciably larger than the reward parameter, as was found with the Solomon and Wynne data.

An Overlearning Experiment

The Experiment. In Chapter 14, a second T-maze experiment is reported. The procedure was identical to the relearning experiment except that the rats were rewarded for turning to the right-hand side of the maze for 144 trials, and then they were run an additional 48 trials with food reward always on the left. The main effect of the overlearning is to increase the final probability in the first period and hence the initial probability of error in the second.

Parameter Estimation. We estimated the beta-model parameters for period 2 with the method used before. The values are $\hat{p}_1=0.996$, $\hat{\beta}_1=1.10$, $\hat{\beta}_2=1.32$. As found in the previous two experiments, $\hat{\beta}_2$ is larger than $\hat{\beta}_1$. However, $\hat{\beta}_2$ in this experiment is smaller than it was in the previous T-maze experiment.

Goodness-of-Fit. As before, Monte Carlo computations were made and various statistics computed. In Table 2, these are compared with corresponding statistics of the data and population values computed from the alpha model. The mean performance curves for the experimental animals and for the Monte Carlo runs are shown in Fig. 1.

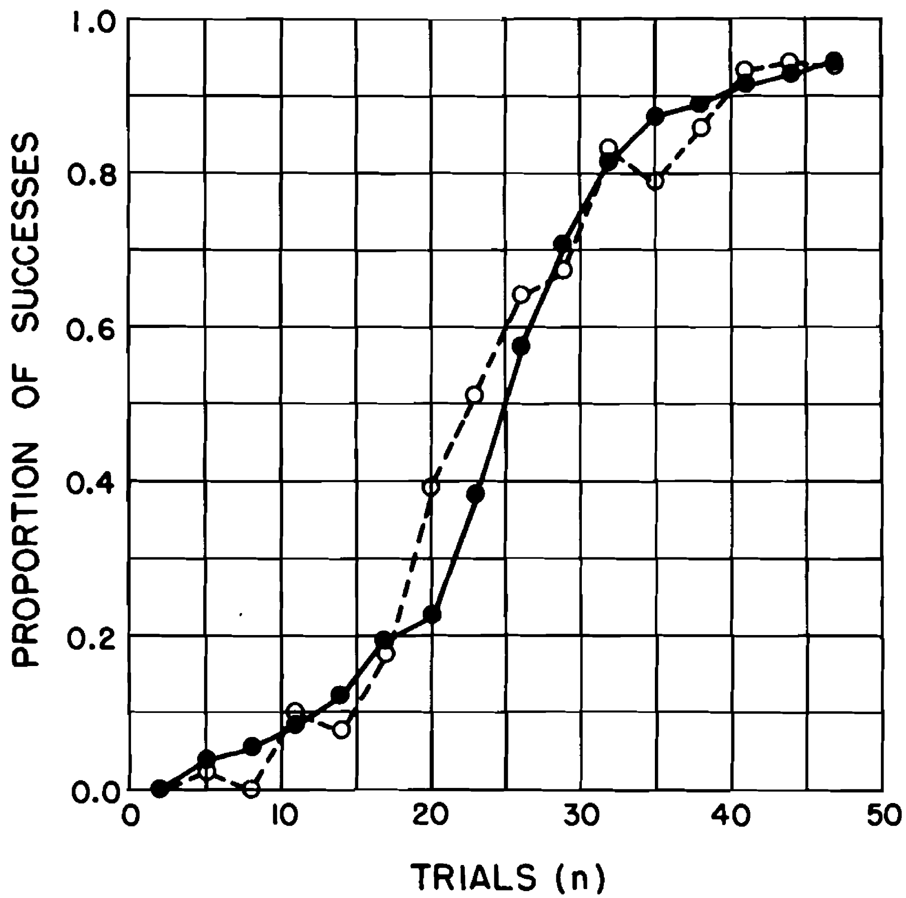

FIGURE 1. Period 2 of the overlearning experiment showing the average response frequencies in three-trial blocks for the experimental animals (filled circles) and for the beta-model Monte Carlo analogs (open circles).

TABLE 2

Comparison of Several Statistics of the Overlearning Experiment with Statistics Obtained for the Two Models

Statistic	Real Rats	Alpha Model (Expected Values)	Beta Model (Stat-Rats)
Mean total errors	24.68	24.62	24.40
Variance of total errors	5.01	26.57	3.33
Mean trials before first success	13.32	12.53	14.24
Mean number of error runs	6.11	7.03	6.24
Mean error runs of length 1	3.11	3.63	3.32
Mean error runs of length 2	0.47	1.13	0.80
Mean error runs of length 3	0.53	0.54	0.48
Mean error runs of length 4	0.32	0.32	0.20
Mean error runs of length 5	0.42	0.21	0.20

Discussion

One of the more interesting results of the beta-model analyses just presented is that the estimates of the nonreward parameter, β_2, are uniformly larger than the corresponding estimates of the reward parameter, β_1. The alpha-model analyses, on the other hand, lead to the opposite conclusion about the relative effects of reward and nonreward for the first and third experiments described. Therefore, it is evident that one's inferences about the relative effectiveness of reward and nonreward (or avoidance and escape) are "model-bound." If such inferences could be made by using a nonparametric technique which makes no assumptions other than those embodied in a large class of models (including the alpha and beta models), then evidence in support of one model or the other would be obtained. Unfortunately, we have not found a satisfactory technique for this purpose; we must rely on other evidence if we wish to decide which model is the more satisfactory.

We compared the alpha and beta models by analyzing in detail two experiments. The alpha model is in very close agreement with the avoidance-learning data on all properties examined; the beta-model figures are likewise very close to the data, except for the variance of total shocks. Thus, the alpha model has a slight edge on the beta model for these data. On the other hand, the beta model gives a decidedly more satisfactory description of the data on retraining after overlearning. With this experiment, the alpha model appears to be in serious trouble, particularly in predicting the variance of the total number of errors.

The variance of total errors is a very useful statistic for discriminating between the two models. As was pointed out to us by S. Sternberg, this is a consequence of the different roles played by reward and nonreward in the two analyses. When reward is less effective than nonreward, the process has "negative feedback": if an animal receives a large number of rewards during the early trials, his probability of error remains high and so he will make few rewarded responses during the later trials. Similarly, if he makes many errors early, his probability of error decreases to a low value, and so few errors are made later. This effect tends to equalize the total errors made by different animals. On the other hand, when reward is more effective than nonreward, "positive feedback" exists and so one would expect a large variance of the total number of errors.

If reward and nonreward are assumed to have equal effects, each model predicts a specified variance of total errors; these predictions can serve as baselines in pursuing the argument given in the previous paragraph. Formulas for the mean and variance of total errors for the equal-alpha model are well known (see Chapter 10). For the relearning period of the overlearning experiment, we observed a mean of 24.7. Equating this to the expected value and taking $p_1 = 1$, we estimate α to be 0.96. The variance is then computed to be 9.9. For the equal-beta model,[4] using the observed mean and $p_1 = 0.996$, the value previously obtained for the beta model, we estimate β to be 1.25. This leads to a computed variance of 4.5. Both of these computed variances are consistent with our arguments about how the relative effects of reward and nonreward alter the variance. The unequal-alpha model with reward more effective led to a variance of 26.6, compared with the 9.9 figure for the equal-alpha model. The unequal-beta model with nonreward more effective gave a variance of 3.3, compared with the 4.5 value for the equal-beta model. (All computations fixed the mean total errors at the observed value of 24.7.)

A desirable property of any learning model is that the event parameters should be independent of experimental variables such as the number of trials of previous training. This property has been termed "event invariance" or "parameter invariance" ([1], Chapter 14). As noted in Chapter 14, the alpha-model analyses of the two learning experiments described above do not exhibit this property. Likewise, the beta-model analyses of these same data fail to support the hypothesis of parameter invariance in that model. The data on relearning after overtraining lead to a nonreward parameter that is less effective than that obtained from the data on relearning after moderate training.

Additional evidence for lack of parameter invariance in the beta model is found by examining the data from the first period of the overlearning experiment. Proceeding backwards from the 0.95 point on the learning curve (trial 31), we estimated β_1 to be 1.20. But, when we moved backwards from the end of the first period, using the estimate $\hat{p}_1 = 0.996$ obtained from the

[4] Major simplifications in the beta model result from the special assumption $\beta_1 = \beta_2 = \beta$, which implies that reward and nonreward have equal effects. The probability of an error on trial n has the fixed value

$$p_n = p_1/[p_1 + (1 - p_1)\beta^{n-1}] ,$$

where $\beta > 1$. Defining a random variable x_n that has the value 1 when an error occurs on trial n and the value 0 otherwise, we obtain for the total number of errors $u_1 = \sum x_n$ (all summations in this note are from 1 to ∞). The expected value is

$$E(u_1) = \sum E(x_n) = \sum p_n = \sum\{p_1/[p_1 + (1 - p_1)\beta^{n-1}]\} .$$

If we replace the sum with an integral from 1 to ∞, we obtain the approximation $E(u_1) \cong -\log(1 - p_1)/\log\beta$. The variance is

$$\operatorname{var}(u_1) = \sum \operatorname{var}(x_n) = \sum p_n(1 - p_n) = \sum\{p_1(1 - p_1)\beta^{n-1}/[p_1 + (1 - p_1)\beta^{n-1}]^2\} .$$

The integral approximation is $\operatorname{var}(u_1) \cong p_1/\log\beta$.

beginning of the second period, we obtained $\hat{\beta}_1 = 1.015$. Thus, reward seems to have much less effect during the late trials of overlearning than it does anywhere else in the data.

In summary, we have uncovered two pieces of evidence against the beta model from the three experiments analyzed: (a) underestimates of the variance of total errors, and (b) lack of parameter invariance. There are two reasons, however, why we feel that these apparent weaknesses of the model need not be taken too seriously. The first has to do with the experiments themselves. It is reported in Chapter 14 that the data from the two T-maze experiments, both of which had three trials a day, exhibited a very significant daily recovery effect, at least for the last 48 trials. The extent to which this phenomenon affects the parameter estimates and the various measures of goodness-of-fit is not known, but we would not be surprised if it were quite serious. The second reason for tempering the evidence against the beta model is our implicit assumption of a single unique value of the initial probability for each period of each experiment. Unlike the alpha model, the beta model is extremely sensitive to p_1 in the neighborhood of 1 or 0. Therefore, a distribution of p_1 with very small spread might have a strong effect on subsequent analyses. Furthermore, we know that the model implies a non-zero-variance distribution of p's at the end of a training period, and therefore at the beginning of the following period. One might hope, therefore, that the apparent evidence against the beta model would disappear when both the experiments and the analyses are refined.

Alternatively, however, the more refined experiments and analyses may continue to exhibit a lack of parameter invariance. In particular, the tail of the learning curve in an overtraining experiment may be considerably flatter than predicted by the beta model with parameters estimated from other regions. (It should be noted that with parameter values of the order of 1.1, the beta model would predict an initial probability of 0.999,999 at the beginning of period 2 of the relearning experiment.) If this is the case, then it will be necessary to devise models that exhibit more reduction in the effect of experience as the probability of choice approaches 0 or 1.

This study represents the first detailed inquiry into the adequacy of the beta model. More such studies are needed before a final evaluation can be made. To facilitate the analyses, further mathematical work on model properties and related estimation problems is needed.

Table of $L(p, \beta)$

The following five-page table of the function

$$L(p, \beta) = \sum_{k=1}^{\infty} \frac{k\beta^k\left[\dfrac{p}{1-p}\right]^k}{\prod_{j=0}^{k}\left[\dfrac{p}{1-p} + \beta^j\right]} ,$$

was prepared by the Computer Center, University of Pennsylvania. We are indebted to Dr. Saul Gorn, Director of the Center, and to Mr. Peter Ingerman, who wrote the program.

The references to this chapter follow the table.

p \ β	1.000	1.010	1.020	1.030	1.032	1.034	1.036	1.038	1.040	1.042	1.044	1.046	1.048	1.050
.9995	1999.000	260.453	162.129	121.295	115.738	110.734	106.199	102.070	98.291	94.819	91.616	88.651	85.898	83.333
.9994	1665.667	244.650	153.709	115.520	110.301	105.597	101.331	97.442	93.882	90.607	87.584	84.785	82.183	79.759
.9993	1427.571	231.534	146.673	110.680	105.744	101.290	97.247	93.559	90.180	87.071	84.199	81.537	79.062	76.755
.9992	1249.000	220.370	140.647	106.523	101.827	97.587	93.735	90.220	86.996	84.028	81.285	78.741	76.375	74.168
.9991	1110.111	210.685	135.387	102.886	98.400	94.345	90.660	87.294	84.206	81.361	78.730	76.290	74.019	71.899
.9990	999.000	202.158	130.730	99.658	95.356	91.466	87.928	84.694	81.726	78.990	76.459	74.109	71.922	69.880
.9989	908.091	194.560	126.558	96.759	92.623	88.879	85.473	82.358	79.496	76.858	74.416	72.148	70.036	68.064
.9988	832.333	187.725	122.786	94.133	90.145	86.534	83.246	80.237	77.473	74.923	72.561	70.368	68.324	66.414
.9987	768.231	181.525	119.348	91.733	87.881	84.390	81.210	78.299	75.623	73.152	70.864	68.738	66.756	64.904
.9986	713.286	175.862	116.193	89.527	85.799	82.418	79.336	76.514	73.919	71.522	69.302	67.237	65.312	63.513
.9985	665.667	170.660	113.281	87.487	83.872	80.593	77.603	74.863	72.342	70.013	67.854	65.847	63.975	62.224
.9984	624.000	165.855	110.581	85.591	82.082	78.896	75.990	73.327	70.874	68.609	66.507	64.553	62.730	61.024
.9983	587.235	161.397	108.065	83.821	80.410	77.312	74.484	71.891	69.503	67.296	65.248	63.343	61.565	59.901
.9982	554.556	157.245	105.712	82.163	78.843	75.827	73.072	70.545	68.217	66.065	64.067	62.208	60.472	58.848
.9981	525.316	153.363	103.503	80.605	77.370	74.430	71.744	69.279	67.007	64.906	62.955	61.139	59.443	57.856
.9980	499.000	149.723	101.424	79.135	75.980	73.112	70.490	68.083	65.864	63.811	61.905	60.130	58.471	56.918
.9975	399.000	134.381	92.572	72.845	70.030	67.465	65.117	62.957	60.963	59.114	57.396	55.793	54.295	52.890
.9970	332.333	122.469	85.585	67.845	65.294	62.967	60.832	58.866	57.047	55.360	53.790	52.323	50.950	49.662
.9965	284.714	112.856	79.864	63.722	61.386	59.251	57.290	55.481	53.806	52.250	50.800	49.445	48.175	46.983
.9960	249.000	104.879	75.054	60.234	58.077	56.102	54.286	52.609	51.054	49.608	48.259	46.997	45.814	44.701
.9955	221.222	98.120	70.929	57.225	55.220	53.381	51.689	50.123	48.671	47.319	46.057	44.875	43.766	42.722
.9950	199.000	92.297	67.335	54.590	52.716	50.995	49.409	47.941	46.578	45.307	44.120	43.007	41.962	40.978
.9940	165.667	82.728	61.341	50.163	48.504	46.978	45.568	44.261	43.044	41.909	40.846	39.849	38.911	38.027
.9930	141.857	75.142	56.502	46.557	45.069	43.697	42.428	41.249	40.150	39.123	38.160	37.256	36.404	35.600
.9920	124.000	68.949	52.486	43.539	42.191	40.946	39.792	38.718	37.716	36.778	35.898	35.070	34.289	33.551
.9910	110.111	63.775	49.082	40.963	39.731	38.592	37.534	36.549	35.628	34.765	33.954	33.191	32.470	31.788
.9900	99.000	59.377	46.150	38.728	37.595	36.546	35.570	34.660	33.809	33.010	32.258	31.550	30.881	30.247
.9850	65.667	44.427	35.871	30.766	29.967	29.222	28.526	27.873	27.259	26.680	26.133	25.616	25.125	24.658
.9800	49.000	35.643	29.556	25.755	25.149	24.581	24.048	23.546	23.072	22.624	22.198	21.795	21.410	21.044
.9750	39.000	29.797	25.211	22.242	21.761	21.310	20.884	20.481	20.100	19.738	19.394	19.066	18.754	18.455
.9700	32.333	25.603	22.011	19.614	19.221	18.851	18.500	18.168	17.853	17.553	17.267	16.994	16.733	16.483
.9650	27.571	22.438	19.542	17.561	17.232	16.922	16.627	16.348	16.081	15.827	15.585	15.353	15.131	14.918
.9600	24.000	19.958	17.572	15.904	15.626	15.361	15.109	14.870	14.641	14.423	14.214	14.014	13.822	13.638
.9550	21.222	17.961	15.961	14.536	14.296	14.067	13.850	13.642	13.444	13.254	13.072	12.897	12.729	12.568
.9500	19.000	16.317	14.616	13.384	13.175	12.975	12.785	12.603	12.429	12.262	12.102	11.947	11.799	11.657

p \ β	1.052	1.054	1.056	1.058	1.060	1.062	1.064	1.066	1.068	1.070	1.075	1.080	1.085	1.090
.9995	80.938	78.696	76.591	74.612	72 746	70.984	69.318	67.739	66.241	64.817	61.547	58.635	56.023	53.667
.9994	77.493	75.371	73.378	71.503	69.735	68.065	66.484	64.986	63.564	62.212	59.106	56.337	53.852	51.609
.9993	74.598	72.576	70.677	68.889	67.203	65.609	64.101	62.671	61.312	60.021	57.051	54.403	52.025	49.876
.9992	72.104	70.168	68.349	66.637	65.020	63.493	62.046	60.674	59.371	58.131	55.279	52.734	50.447	48.380
.9991	69.916	68.056	66.307	64.660	63.105	61.635	60.242	58.921	57.666	56.471	53.722	51.268	49.062	47.066
.9990	67.969	66.176	64.489	62.900	61.400	9.981	58.636	57.360	56.147	54.993	52.336	49.962	47.827	45.895
.9989	66.217	64.484	62.853	61.316	59.865	58.491	57.189	55.954	54.779	53.661	51.086	48.785	46.713	44.839
.9988	64.626	62.947	61.367	59.877	58.469	57.137	55.875	54.676	53.536	52.450	49.950	47.714	45.701	43.878
.9987	63.169	61.539	60.005	58.558	57.191	55.897	54.670	53.505	52.396	51.341	48.908	46.732	44.772	42.997
.9986	61.826	60.242	58.750	57.343	56.013	54.754	53.559	52.425	51.346	50.317	47.948	45.826	43.915	42.183
.9985	60.582	59.040	57.587	56.217	54.921	53.694	52.529	51.423	50.371	49.368	47.056	44.986	43.120	41.429
.9984	59.424	57.920	56.504	55.167	53.903	52.706	51.570	50.490	49.463	48.484	46.226	44.203	42.379	40.725
.9983	58.341	56.873	55.491	54.186	52.951	51.782	50.672	49.617	48.613	47.656	45.448	43.469	41.685	40.066
.9982	57.323	55.890	54.539	53.264	52.057	50.914	49.828	48.796	47.814	46.878	44.717	42.780	41.032	39.446
.9981	56.366	54.964	53.643	52.395	51.215	50.096	49.033	48.023	47.062	46.145	44.028	42.130	40.417	38.862
.9980	55.461	54.089	52.796	51.574	50.418	49.322	48.282	47.292	46.350	45.451	43.377	41.515	39.835	38.309
.9975	51.569	50.326	49.152	48.042	46.991	45.994	45.046	44.144	43.284	42.464	40.568	38.864	37.324*	35.924
.9970	48.450	47.308	46.228	45.207	44.239	43.320	42.446	41.613	40.819	40.061	38.307	36.729	35.300	34.000
.9965	45.859	44.800	43.798	42.850	41.950	41.095	40.281	39.506	38.766	38.059	36.422	34.948	33.611	32.393
.9960	43.653	42.663	41.727	40.839	39.997	39.196	38.434	37.707	37.013	36.349	34.811	33.424	32.166	31.018
.9955	41.737	40.807	39.927	39.092	38.299	37.545	36.827	36.141	35.486	34.860	33.408	32.096	30.905	29.818
.9950	40.050	39.172	38.341	37.552	36.802	36.088	35.408	34.759	34.139	33.545	32.167	30.922	29.790	28.756
.9940	37.191	36.400	35.650	34.938	34.261	33.615	32.999	32.411	31.848	31.310	30.057	28.923	27.891	26.946
.9930	34.839	34.118	33.434	32.784	32.165	31.574	31.011	30.472	29.956	29.462	28.311	27.268	26.317	25.445
.9920	32.853	32.190	31.561	30.962	30.391	29.847	29.326	28.829	28.352	27.895	26.829	25.862	24.979	24.169
.9910	31.142	30.529	29.946	29.391	28.861	28.355	27.872	27.409	26.965	26.540	25.547	24.644	23.819	23.062
.9900	29.646	29.075	28.532	28.015	27.521	27.048	26.597	26.164	25.749	25.351	24.421	23.574	22.799	22.087
.9850	24.214	23.791	23.386	23.000	22.630	22.276	21.936	21.610	21.296	20.994	20.286	19.638	19.043	18.493
.9800	20.695	20.361	20.041	19.735	19.441	19.159	18.888	18.626	18.375	18.133	17.563	17.039	16.556	16.108
.9750	18.170	17.896	17.634	17.382	17.140	16.907	16.683	16.466	16.258	16.056	15.582	15.144	14.739	14.362
.9700	16.243	16.014	15.793	15.581	15.376	15.179	14.989	14.806	14.629	14.458	14.053	13.679	13.331	13.008
.9650	14.713	14.516	14.327	14.145	13.969	13.800	13.636	13.478	13.325	13.176	12.826	12.501	12.198	11.915
.9600	13.460	13.290	13.125	12.967	12.814	12.666	12.523	12.384	12.250	12.120	11.812	11.526	11.259	11.009
.9550	12.412	12.262	12.118	11.978	11.843	11.713	11.586	11.464	11.345	11.230	10.957	10.702	10.465	10.242
.9500	11.519	11.386	11.258	11.134	11.014	10.898	10.785	10.676	10.570	10.467	10.223	9.995	9.781	9.581

* Interpolated value

β / p	1.095	1.100	1.105	1.110	1.115	1.120	1.130	1.140	1.150	1.160	1.170	1.180	1.190	1.200
.9995	51.528	49.578	47.791	46.149	44.632	43.228	40.707	38.507	36.570	34.849	33.309	31.923	30.668	29.526
.9994	49.572	47.713	46.010	44.442	42.995	41.654	39.245	37.142	35.289	33.641	32.167	30.839	29.636	28.540
.9993	47.924	46.142	44.508	43.004	41.615	40.327	38.013	35.992	34.208	32.623	31.203	29.924	28.764	27.708
.9992	46.502	44.786	43.212	41.763	40.424	39.182	36.949	34.997	33.275	31.743	30.371	29.133	28.011	26.989
.9991	45.252	43.594	42.073	40.671	39.376	38.174	36.013	34.123	32.454	30.969	29.638	28.437	27.349	26.356
.9990	44.137	42.531	41.057	39.698	38.441	37.275	35.178	33.342	31.721	30.278	28.984	27.816	26.757	25.791
.9989	43.132	41.573	40.140	38.820	37.598	36.465	34.425	32.638	31.060	29.654	28.393	27.255	26.223	25.281
.9988	42.218	40.700	39.306	38.021	36.831	35.727	33.738	31.997	30.457	29.086	27.855	26.744	25.736	24.816
.9987	41.380	39.900	38.541	37.287	36.127	35.049	33.109	31.408	29.904	28.564	27.361	26.275	25.289	24.389
.9986	40.606	39.162	37.839	36.610	35.477	34.424	32.527	30.865	29.394	28.082	26.905	25.841	24.876	23.994
.9985	39.887	38.476	37.179	35.982	34.873	33.843	31.987	30.360	28.919	27.634	26.481	25.439	24.492	23.628
.9984	39.217	37.837	36.567	35.395	34.310	33.301	31.483	29.888	28.476	27.216	26.085	25.062	24.133	23.285
.9983	38.590	37.238	35.994	34.846	33.782	32.794	31.011	29.446	28.061	26.825	25.714	24.710	23.797	22.964
.9982	38.000	36.675	35.455	34.329	33.286	32.316	30.567	29.031	27.670	26.456	25.364	24.378	23.481	22.662
.9981	37.443	36.143	34.947	33.842	32.818	31.865	30.147	28.638	27.301	26.108	25.035	24.064	23.182	22.376
.9980	36.917	35.641	34.466	33.381	32.374	31.439	29.750	28.267	26.952	25.778	24.722	23.767	22.899	22.105
.9975	34.644	33.470	32.388	31.387	30.459	29.595	28.034	26.660	25.441	24.351	23.370	22.482	21.673	20.934
.9970	32.810	31.718	30.710	29.777	28.911	28.104	26.645	25.360	24.218	23.196	22.275	21.440	20.680	19.984
.9965	31.278	30.253	29.307	28.431	27.616	26.857	25.482	24.270	23.192	22.227	21.356	20.566	19.846	19.186
.9960	29.966	28.998	28.104	27.276	26.505	25.787	24.484	23.335	22.311	21.394	20.565	19.814	19.128	18.500
.9955	28.821	27.903	27.054	26.267	25.535	24.851	23.612	22.516	21.540	20.664	19.873	19.155	18.499	17.898
.9950	27.807	26.933	26.124	25.373	24.674	24.022	22.837	21.790	20.856	20.017	19.259	18.570	17.940	17·363
.9940	26.078	25.278	24.536	23.847	23.204	22.604	21.513	20.547	19.683	18.205	18.205	17.566	16.982	16.445
.9930	24.644	23.903	23.216	22.578	21.982	21.424	20.410	19.511	18.706	17.981	17.325	16.728	16.181	15.678
.9920	23.422	22.732	22.092	21.496	20.939	20.418	19.469	18.625	17.870	17.189	16.572	16.010	15.494	15.020
.9910	22.363	21.716	21.115	20.556	20.032	19.542	18.649	17.854	17.142	16.499	15.915	15.383	14.895	14.446
.9900	21.430	20.821	20.255	19.727	19.233	18.770	17.925	17.173	16.498	15.888	15.334	14.828	14.364	13.937
.9850	17.983	17.509	17.067	16.654	16.266	15.901	15.232	14.634	14.095	13.605	13.159	12.751	12.375	12.028
.9800	15.691	15.303	14.940	14.599	14.278	13.976	13.421	12.922	12.471	12.060	11.684	11.339	11.021	10.726
.9750	14.011	13.683	13.375	13.085	12.812	12.555	12.080	11.652	11.264	10.909	10.584	10.285	10.009	9.752
.9700	12.705	12.422	12.155	11.904	11.668	11.443	11.030	10.655	10.315	10.003	9.717	9.453	9.209	8.982
.9650	11.651	11.402	11.168	10.947	10.739	10.541	10.175	9.843	9.540	9.263	9.008	8.772	8.553	8.349
.9600	10.775	10.554	10.347	10.150	9.964	9.788	9.460	9.163	8.891	8.642	8.411	8.198	8.000	7.816
.9550	10.032	9.835	9.648	9.472	9.305	9.146	8.851	8.582	8.336	8.109	7.900	7.706	7.526	7.357
.9500	9.392	9.214	9.045	8.886	8.734	8.590	8.322	8.077	7.853	7.646	7.455	7.277	7.112	6.957

β / p	1.210	1.220	1.230	1.240	1.250	1.260	1.270	1.280	1.290	1.300	1.320	1.340	1.360	1.380
.9995	28.481	27.523	26.639	25.822	25.064	24.358	23.700	23.084	22.507	21.965	20.972	20.086	19.290	18.569
.9994	27.538	26.618	25.769	24.985	24.256	23.578	22.946	22.354	21.798	21.277	20.322	19.469	18.702	18.008
.9993	26.742	25.854	25.036	24.278	23.575	22.920	22.309	21.737	21.200	20.696	19.773	18.947	18.205	17.534
.9992	26.054	25.194	24.401	23.667	22.985	22.351	21.758	21.203	20.683	20.194	19.298	18.497	17.776	17.124
.9991	25.448	24.613	23.842	23.129	22.466	21.849	21.273	20.733	20.227	19.751	18.879	18.099	17.398	16.762
.9990	24.907	24.094	23.343	22.648	22.003	21.402	20.840	20.314	19.820	19.356	18.505	17.744	17.059	16.439
.9989	24.418	23.625	22.893	22.214	21.584	20.997	20.448	19.935	19.452	18.999	18.167	17.424	16.754	16.148
.9988	23.973	23.198	22.482	21.819	21.203	20.628	20.092	19.589	19.117	18.673	17.860	17.131	16.475	15.881
.9987	23.564	22.806	22.105	21.456	20.852	20.290	19.764	19.272	18.809	18.374	17.577	16.863	16.220	15.637
.9986	23.186	22.443	21.756	21.120	20.528	19.977	19.461	18.978	18.524	18.098	17.315	16.614	15.983	15.411
.9985	22.835	22.106	21.432	20.808	19.227	19.685	19.179	18.705	18.260	17.840	17.072	16.383	15.763	15.200
.9984	22.507	21.791	21.129	20.516	19.945	19.413	18.916	18.450	18.012	17.600	16.844	16.167	15.557	15.004
.9983	22.200	21.496	20.845	20.242	19.681	19.158	18.669	18.211	17.780	17.375	16.631	15.965	15.364	14.819
.9982	21.910	21.218	20.578	19.985	19.433	18.918	18.437	17.985	17.562	17.162	16.430	15.774	15.182	14.646
.9981	21.636	20.955	20.326	19.741	19.198	18.691	18.217	17.773	17.355	16.962	16.240	15.594	15.010	14.481
.9980	21.377	20.707	20.086	19.511	18.976	18.476	18.009	17.571	17.159	16.772	16.060	15.423	14.847	14.326
.9975	20.255	19.629	19.050	18.512	18.012	17.544	17.107	16.697	16.311	15.948	15.280	14.681	14.141	13.651
.9970	19.344	18.755	18.208	17.701	17.229	16.787	16.374	15.986	15.621	15.278	14.646	14.079	13.567	13.101
.9965	18.580	18.020	17.501	17.020	16.571	16.151	15.758	15.389	15.042	14.714	14.112	13.572	13.083	12.639
.9960	17.922	17.388	16.893	16.433	16.004	15.603	15.227	14.874	14.542	14.228	13.652	13.134	12.666	12.240
.9955	17.344	16.833	16.359	15.918	15.506	15.122	14.761	14.422	14.103	13.802	13.248	12.750	12.299	11.889
.9950	16.831	16.340	15.884	15.460	15.064	14.693	14.346	14.020	13.712	13.422	12.888	12.408	11.973	11.577
.9940	15.951	15.493	15.068	14.673	14.303	13.958	13.633	13.328	13.041	12.769	12.269	11.819	11.411	11.039
.9930	15.214	14.785	14.386	14.014	13.667	13.341	13.036	12.749	12.478	12.222	11.750	11.324	10.939	10.587
.9920	14.583	14.177	13.800	13.449	13.120	12.812	12.523	12.251	11.994	11.751	11.303	10.899	10.533	10.199
.9910	14.031	13.646	13.288	12.954	12.642	12.349	12.074	11.815	11.570	11.339	10.912	10.527	10.177	9.858
.9900	13.542	13.175	12.834	12.516	12.218	11.938	11.676	11.428	11.194	10.973	10.565	10.196	9.861	9.555
.9850	11.706	11.406	11.127	10.865	10.620	10.390	10.173	9.968	9.775	9.591	9.252	8.944	8.664	8.408
.9800	10.453	10.197	9.959	9.735	9.525	9.328	9.141	8.965	8.798	8.640	8.346	8.080	7.837	7.614
.9750	9.513	9.291	9.082	8.886	8.702	8.528	8.364	8.209	8.061	7.922	7.662	7.426	7.210	7.012
.9700	8.770	8.572	8.386	8.211	8.047	7.892	7.745	7.606	7.474	7.349	7.116	6.903	6.709	6.530
.9650	8.159	7.980	7.813	7.656	7.507	7.367	7.235	7.109	6.989	6.875	6.664	6.471	6.293	6.130
.9600	7.643	7.481	7.329	7.186	7.051	6.923	6.802	6.687	6.578	6.474	6.280	6.103	5.940	5.790
.9550	7.200	7.052	6.913	6.781	6.657	6.540	6.429	6.323	6.222	6.127	5.948	5.784	5.634	5.495
.9500	6.812	6.676	6.548	6.427	6.313	6.204	6.101	6.004	5.910	5.822	5.656	5.504	5.364	5.235

p \ β	1.400	1.420	1.440	1.460	1.480	1.500	1.550	1.600	1.650	1.700	1.800	1.900	2.000	2.500
.9995	17.915	17.317	16.769	16.264	15.798	15.366	14.412	13.603	12.909	12.306	11.307	10.513	9.864	7.829
.9994	17.377	16.801	16.273	15.786	15.336	14.919	13.998	13.217	12.547	11.964	10.998	10.230	9.602	7.630
.9993	16.923	16.365	15.853	15.382	14.946	14.542	13.649	12.892	12.241	11.675	10.737	9.991	9.381	7.463
.9992	16.530	15.988	15.491	15.032	14.608	14.215	13.346	12.609	11.976	11.425	10.511	9.784	9.189	7.317
.9991	16.184	15.656	15.171	14.724	14.311	13.928	13.080	12.361	11.742	11.204	10.312	9.601	9.020	7.189
.9990	15.875	15.359	14.885	14.449	14.045	13.670	12.842	12.139	11.534	11.007	10.134	9.438	8.869	7.075
.9989	15.596	15.091	14.627	14.200	13.805	13.438	12.626	11.938	11.345	10.829	9.973	9.290	8.732	6.971
.9988	15.341	14.846	14.392	13.973	13.585	13.226	12.430	11.754	11.173	10.666	9.826	9.156	8.607	6.876
.9987	15.106	14.621	14.175	13.764	13.384	13.031	12.250	11.586	11.014	10.517	9.691	9.032	8.492	6.789
.9986	14.890	14.413	13.975	13.571	13.197	12.851	12.083	11.430	10.868	10.379	9.566	8.917	8.386	6.709
.9985	14.688	14.220	13.789	13.392	13.024	12.683	11.927	11.285	10.732	10.250	9.450	8.811	8.287	6.634
.9984	14.500	14.039	13.615	13.224	12.862	12.526	11.782	11.149	10.605	10.130	9.341	8.711	8.195	6.564
.9983	14.323	13.869	13.452	13.066	12.710	12.379	11.646	11.022	10.485	10.017	9.239	8.617	8.108	6.498
.9982	14.157	13.709	13.298	12.918	12.567	12.240	11.517	10.902	10.372	9.910	9.143	8.529	8.027	6.436
.9981	13.999	13.558	13.152	12.778	12.431	12.109	11.396	10.789	10.266	9.810	9.052	8.446	7.949	6.377
.9980	13.850	13.415	13.014	12.645	12.303	11.985	11.281	10.681	10.165	9.714	8.966	8.367	7.876	6.322
.9975	13.203	12.793	12.416	12.068	11.745	11.446	10.781	10.215	9.726	9.300	8.591	8.023	7.557	6.080
.9970	12.677	12.288	11.929	11.598	11.292	11.007	10.374	9.835	9.369	8.963	8.286	7.743	7.298	5.883
.9965	12.233	11.862	11.519	11.203	10.909	10.637	10.031	9.514	9.068	8.678	8.028	7.507	7.078	5.716
.9960	11.851	11.494	11.165	10.861	10.579	10.317	9.734	9.237	8.807	8.432	7.805	7.302	6.889	5.572
.9955	11.515	11.171	10.854	10.561	10.289	10.036	9.474	8.994	8.578	8.215	7.609	7.122	6.722	5.445
.9950	11.215	10.883	10.577	10.293	10.030	9.786	9.241	8.776	8.374	8.022	7.434	6.962	6.573	5.331
.9940	10.699	10.387	10.099	9.832	9.584	9.354	8.841	8.402	8.021	7.689	7.132	6.684	6.315	5.135
.9930	10.266	9.970	9.697	9.444	9.210	8.991	8.503	8.086	7.725	7.408	6.878	6.451	6.098	4.970
.9920	9.893	9.611	9.351	9.110	8.887	8.678	8.213	7.814	7.469	7.166	6.658	6.249	5.911	4.826
.9910	9.566	9.297	9.048	8.817	8.603	8.403	7.958	7.576	7.244	6.953	6.465	6.071	5.746	4.700
.9900	9.275	9.017	8.778	8.557	8.351	8.159	7.730	7.363	7.043	6.763	6.293	5.913	5.599	4.588
.9850	8.172	7.955	7.754	7.567	7.393	7.230	6.866	6.553	6.280	6.040	5.636	5.308	5.037	4.157
.9800	7.409	7.219	7.043	6.880	6.727	6.584	6.264	5.988	5.747	5.534	5.176	4.884	4.642	3.853
.9750	6.829	6.660	6.503	6.357	6.220	6.092	5.805	5.557	5.340	5.148	4.823	4.559	4.339	3.619
.9700	6.365	6.212	6.070	5.937	5.813	5.697	5.436	5.210	5.011	4.836	4.539	4.296	4.093	3.428
.9650	5.980	5.840	5.710	5.588	5.475	5.368	5.123	4.920	4.737	4.575	4.300	4.075	3.887	3.268
.9600	5.651	5.523	5.403	5.291	5.186	5.087	4.865	4.672	4.502	4.352	4.096	3.886	3.710	3.130
.9550	5.367	5.247	5.136	5.032	4.934	4.843	4.636	4.456	4.297	4.157	3.917	3.720	3.556	3.009
.9500	5.116	5.004	4.901	4.803	4.712	4.627	4.433	4.265	4.116	3.984	3.759	3.573	3.418	2.901

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