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A PROBABILISTIC THEORY OF UTILITY[1]

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A model for choices among uncertain alternatives is developed in which preference between pure alternatives and likelihood judgments between events are assumed to be independent probabilistic processes. It is, in some respects, a probabilistic version of utility models of the von Neumann-Morgenstern type. Certain plausible notions about subjective probability are shown to imply a very simple discrimination function between events in which the probability of choice depends only upon differences of subjective probabilities. Similarly, the expected utility hypothesis is shown to imply that preference discrimination depends upon utility differences, and the form of that discrimination function is determined. Questions of empirical verification are discussed.

1. INTRODUCTION

MOST MATHEMATICAL FORMULATIONS of individual decision making – utility theories – have been based upon non-probabilistic preference relations, usually postulated to be weak orders. Few authors have been satisfied with the assumption that preference is transitive, which is easily demonstrated to be at variance with fact. Yet this assumption has been retained as an approximation to reality because of its attractive mathematical properties; for example, only such preference relations can be completely represented by order preserving numerical functions. Nonetheless, a number of people have voiced a desire for a probabilistic theory, mainly, I would judge, so as to be able better to handle empirical data; and several attempts to formulate such

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A draft of this paper was circulated under the same title as a technical report of the Behavioral Models Project, and several of those who saw it have discussed it with me at length. In particular, I should like to thank Dr. Ward Edwards and Professors Eugene Galanter, Paul Lazarsfeld, and Howard Raiffa for their comments and suggestions, many of which are more or less explicitly incorporated into the revision. Also, I am deeply indebted to the referees, especially Professor John S. Chipman, Associate Editor of *Econometrica*, for reading the manuscript very carefully. Their detailed criticisms uncovered certain implicit assumptions concerning subjective probability which are now in the open. One happy byproduct of this rethinking has been an appreciable weakening of some of the basic assumptions, without a corresponding weakening of the results, and some simplication of the proofs.

a theory can be found in the literature: Davidson and Marschak [**3**], Georgescu-Roegen [**5**], Marschak [**11**], and Papandreou *et al.* [**13**, **14**]. Here is another attempt. One pleasing aspect of this theory is that it seems to have conceptual import as well as giving the empiricist a more manageable tool.

The intuitive idea behind the mathematical framework I shall present is this: Pairs of elements (or alternatives or stimuli) are selected from a given set S, and a person is required to choose from each pair the alternative which he views as "superior" according to some given dimension of comparison – a dimension whose choice depends upon the particular empirical context. It may be preference, intensity, size, loudness, importance, etc. It will be convenient to think of the underlying comparative dimension as "strict preference," but it must be kept in mind that this is only one of the many possible interpretations which can be given to the formalism.

If a and b are elements of S, it is postulated that there exists an objective probability $P(a,b)$ that a is judged as strictly preferred to b. One problem that must be faced is the axiomatic formulation of the fact that many dimensions seem to impose something like a linear ordering upon the underlying set S (Section 4). Once this is done, S is specialized to be a set of risky alternatives of the form $a\alpha b$, where this symbol is interpreted to mean that alternative a arises if event α occurs and alternative b if it does not. At this point, my central assumption will be introduced (Section 5), namely, that the activity of deciding which of two pure alternatives is preferred is statistically independent of the activity of discriminating which of two events is the more likely to occur. In the remainder of the paper I investigate some consequences of this assumption when it is coupled with others of a type more or less traditional in utility theory. It should be emphasized that no construction of a utility function is given, as in von Neumann and Morgenstern [**18**] or Savage [**16**], but rather we derive certain necessary consequences under the assumption that one exists. I shall not attempt to outline the results here, since a good deal of terminology is needed; the major theorems are 8, 9, and **11**.

General background references for this paper are Edwards [**4**] and von Neumann and Morgenstern [**18**]; other more specific references will be indicated as they are needed.

2. DISCRIMINATION STRUCTURES

DEFINITION **1**: Let S be a set. A real-valued function P with domain $S \times S$ is said to be a *discrimination structure* on S provided that:

(i) $P(a,b) \geqslant 0$, for every $a,b \in S$,

(ii) $P(a,b) + P(b,a) \leqslant 1$, for every $a,b \in S$, and

(iii) there exist at least two elements $a^*,b^* \in S$, $a^* \neq b^*$, such that $P(a^*,b^*) \neq P(b^*,a^*)$.

The interpretation intended is this: If a person is asked to select from a pair (a,b) of stimuli the one he considers superior according to some specified dimension common to the two stimuli, then $P(a,b)$ is the (objective) probability that he will choose a over b. In psychophysics, such dimensions as loudness and heaviness are employed. Here I shall be dealing with "preference" when S is a set of alternatives and with "likelihood of occurrence" when S is a Boolean algebra of events. It is clear that if preferences are governed by a discrimination structure, then a sample of reports will, in general, exhibit intransitivities; e.g., if $P(a,b) = 0.8$, $P(b,c) = 0.8$ and $P(c,a) = 0.1$, and if selections are independent, then the probability of getting the report that a is preferred to b, b to c, and c to a is 0.064.

The purpose of conditions (i) and (ii) is clear. The conceptual role of (iii) is to prevent the structure from being interpreted as the probability of indifference or any other symmetric relationship; mathematically, it insures that certain pairs of homogeneous equations have only zero solutions.

DEFINITION 2: If P is a discrimination structure on S, then for $a,b, \in S$, $a \gtrsim b$ is defined to hold if $P(a,c) \geqslant P(b,c)$ and $P(c,a) \leqslant P(c,b)$ for every $c \in S$. The relation $\gtrsim$ on S is called the *trace* of P. We write $a \sim b$ if both $a \gtrsim b$ and $b \gtrsim a$, and $a > b$ if $a \gtrsim b$ and not $b \sim a$.

The trace of a discrimination structure can be viewed as a (very incomplete) algebraic representation of the structure; in the case of preference discrimination, it is an inferred preference relation which, ultimately, I shall assume is reflected by an order preserving numerical function – a utility function.

THEOREM 1: *The trace of a discrimination structure is a quasi-order, i.e., it is reflexive and transitive.*

PROOF: Trivial.

Observe that we cannot prove that the trace is anti-symmetric, in which case it would be a partial order, nor that it is connected, in which case it would be a weak order. The principal results of the paper hold only if the trace is assumed to be a weak order.

3. RISK SPACES

Much of modern utility theory treats an underlying space of risky alternatives $a\alpha b$, where this symbol (or something equivalent to it) is usually interpreted to mean that one, but not both, of the alternatives a or b results, the former with probability α and the latter with probability $1-\alpha$. Just what interpretation should be given to the word "probability" is a point of dispute, but if one is attempting to describe choice behavior, it probably has to be some sort of subjective concept. At present, however, the concept of probability need not be introduced at all; instead α can be taken to denote the occurrence of a well specified event, such as whether a given die thrown by a particular mechanism at a specified time will come up six, or whether the word "Britain" will be found in column 5, page 2 of tomorrow's *New York Times*. If α denotes the occurrence of a particular event, let $\bar{\alpha}$ denote its non-occurrence. In these terms, $a\alpha b$ will be interpreted to mean that a results if event α occurs and b if it does not.

DEFINITION 3: Let A be a set (of pure alternatives) having at least two elements and let E be a Boolean algebra (of events) with the null element o. The *risk space* generated by A and E, denoted $S(A,E)$, is defined to be the set such that:

R1. if $a \in A$, then $a \in S(A,E)$ and if $a,b \in S(A,E)$ and $\alpha \in E$, then $a\alpha b \in S(A,E)$,

where for every $a,b \in A$ and $\alpha \in E$,

R2. $a\alpha a = a$,
R3. $aob = b$,
R4. $a\alpha b = b\bar{\alpha}a$.

If $\gtrsim$ is a quasi-ordering of a risk space, then two further properties are distinguished: For every $a,b,c, \in A$ and $\alpha,\beta \in E$,

R5. if $a \gtrsim b$, then $a \gtrsim a\alpha b \gtrsim b$,
R6. $(a\alpha b)\beta c \sim a(\alpha \cap \beta)\,(b\beta c)$.

Observe that if we call $\bar{o} = e$, then in a risk space $aeb = a$ by R3 and R4.

These properties, except for R5, are the direct analogues of properties in Hausner's [**7**] concept of a mixture space, which in turn is very closely related to Stone's [**17**] earlier axiomatization of a barycentric calculus.

The several properties of a risk space seem to be necessitated by the interpretation we have in mind. R2 states that the alternative a is not different from the risky alternative in which a is the outcome whether or not α occurs. R3 states that a risky alternative in which one alternative can never occur is not different from the other alternative. R4 states that the order of writing the components of a risky alternative is immaterial. These are among the least controversial assumptions in utility theory.

The quasi-ordering mentioned in R5 and R6 will, of course, be taken to be the trace of a discrimination structure, and if one thinks of it as "induced preference or indifference," then these two properties seem plausible. R5 says that a risky alternative is never more preferred (in the induced sense of Definition 2) than the more preferred of its two components nor less preferred than the less preferred of its two components. R6 makes a good deal of sense when one analyzes the conditions under which the three alternatives arise: on both sides, a is the outcome if $\alpha \cap \beta$ occurs, b if $\bar{\alpha} \cap \beta$ occurs, and c if $\bar{\beta}$ occurs. Observe that with R2, R6 implies $(a\alpha b)\beta b \sim a(\alpha \cap \beta)b$, which is to say that certain two-stage gambles are equivalent to single-stage ones in which the two alternatives come up under the same conditions. As Professor Marschak [**10**] has pointed out, this is tantamount to supposing that there is "no love or hate" of gambling. From Theorem 6 on, our results depend upon this consequence of R2 and R6.

It should be pointed out that the equality in R2–R4 could be replaced by the indifference of the trace without affecting the conclusions provided the following condition is added:

$$\text{if } a \sim b, \text{ then } a\alpha c \sim b\alpha c, \text{ for every } \alpha \in E \text{ and } c \in S(A,E).$$

4. LINEAR DISCRIMINATION STRUCTURES AND OTHER PRELIMINARY DEFINITIONS

A variety of conditions that a discrimination structure may satisfy will be needed; they are formulated as:

DEFINITION 4: A discrimination structure P on S with trace $\gtrsim$ is said to be

(i) *reflexive* if $P(a,a) = 0$ for all $a \in S$;

(ii) *additive* if there is a constant K, $0 < K \leqslant 1$, such that $P(a,b) + P(b,a) = K$ for all $a,b, \in S$;

(iii) *linear* if $\gtrsim$ is connected, i.e., if for every $a,b \in S$ either $a \gtrsim b$ or $b \gtrsim a$;

(iv) *strictly stochastically transitive* if P is additive and for every $a,b,c \in S$, $P(a,b) > K/2$ and $P(b,c) > K/2$ imply $P(a,c) \geqslant \text{Max}\,[P(a,b),P(b,c)]$; and *strongly stochastically transitive*[2] if the same conclusion holds when one, but not both, of the strict inequalities is weakened to an inequality.

(v) *transitive* if $a > b$ implies $P(b,a) = 0$;

(vi) *symmetric* if either S is a Boolean algebra and $P(\alpha,\beta) = P(\bar{\beta},\bar{\alpha})$ for every $\alpha,\beta \in S$, or there is a Boolean algebra E and a set A such that $S = S(A,E)$ is a risk space and $P(a\alpha b, a\beta b) = P(a\bar{\beta}b, a\bar{\alpha}b)$ for every $a,b \in A$ and $\alpha \in E$.

LEMMA 1: *A reflexive discrimination structure is transitive.*

PROOF: Suppose that $a > b$. Since P is reflexive, $P(a,a) = 0$; so, by Definition 2, $0 = P(a,a) \geqslant P(b,a)$. Thus, by Definition 1, $P(b,a) = 0$.

The concept of a linear (discrimination) structure is essential for the primary results of this paper, and so it deserves some comment. There is nothing in the definition of a discrimination structure which attempts to capture, even in probabilistic terms, the idea that preference reports tend to weakly order the alternatives. Normatively, one feels that preferences should be transitive, but data rarely live up to that norm. Indeed, the whole reason for worrying at all about a probabilistic utility theory is that preference

[2] Davidson and Marschak [**3**] attribute the term to S. A. Valavanis-Vail.

reports fail either to be transitive or to be consistent over short periods of time or both. But surely it would be folly to ignore the tendency toward transitivity completely. In some way the discrimination structure describing preference reports must be constrained so that, on the average, there is a good deal more transitivity than not; my guess is that this is one of the more delicate restrictions in trying to create a suitable probabilistic utility model. What I am proposing is that P may be so constrained that its trace – which is a transitive relation – permits every pair of elements to be compared.[3]

Before criticizing this assumption, let me put it in another form which may seem more plausible at first glance. This relationship, which is due I think to Professor Marschak, holds only for additive discrimination structures, but in practice this is no real restriction as we shall see later.

LEMMA 2: *For additive discrimination structures, strong stochastic transitivity implies linearity and linearity implies strict stochastic transitivity.*

PROOF: Suppose that P is linear and that $P(a,b) > K/2$ and $P(b,c) > K/2$. Since $P(a,a) = K/2$ for all a in an additive structure, $a \gtrsim b$ and $b \gtrsim c$. If we suppose that $P(a,c) < \text{Max}\,[P(a,b),P(b,c)]$, then either $a < b$ or $b < c$, which contradicts what we have just shown. So the structure is strictly stochastically transitive.

Next, consider any $a,b \in S$ and with no loss of generality suppose that $P(a,b) \geqslant K/2$. Let $c \in S$. If $P(b,c) > K/2$, then strong stochastic transitivity implies $P(a,c) \geqslant P(b,c)$. If $P(b,c) \leqslant K/2$, then we examine two cases: either $P(a,c) \geqslant K/2$, in which case $P(a,c) \geqslant P(b,c)$, or $P(a,c) < K/2$. By additivity, this means $P(c,a) > K/2$, which together with $P(a,b) \geqslant K/2$ implies $P(c,b) \geqslant P(c,a)$, i.e., $P(a,c) \geqslant P(b,c)$. Thus, we can conclude $a \gtrsim b$, so P is linear.

In criticism of the linearity assumption, Professor Howard Raiffa has suggested the following example where it seems doubtful. Let a be a trip from New York to Paris with all expenses for two weeks; b be a trip from New York to Rome under otherwise identical conditions; c be a plus \$20; and d be b plus \$20. It certainly seems reasonable to suppose that $P(a,c) = P(b,d) = 0$ for all people, and it is not clear that we can rule out the possibility that there are some people for whom $P(a,d) > 0$ and $P(b,c) > 0$. For such people, a and b are not comparable since $P(a,d) > P(b,d)$ and $P(a,c) < P(b,c)$. This example rests upon having some, but not all, cases of perfect preference discrimination, i.e., $P(a,c) = 0$, and we shall see later (Section 8) that this is a general source of trouble.

More to the point would be actual data, but there seems to be precious little available. The only experiments I know of are at present unpublished.

[3] In work completed since this paper was written and which will be published in due course, I have shown that an "independence of irrelevant alternatives" axiom is sufficient to insure that an additive discrimination structure is linear.

The first is due to Coombs [2], who obtained preferences from four subjects among 12 shades of gray. His data form an additive structure, and he checked the strong stochastic transitivity assumption. Ignoring for the moment the point Coombs was attempting to demonstrate in the experiment, 580 triples from the four subjects were checked and of these 134 failed to satisfy strong stochastic transitivity. Taken at face value, this looks ominous for the linearity assumption; however, there are two points that must be made. Coombs [1] has devised a model – the unfolding technique – which, roughly, supposes that stimuli lie on a continuum and that the person has an ideal point on that continuum. When reporting preferences among the stimuli, he ranks them higher the smaller their "psychological distance" from his ideal point. The resulting rank order, therefore, is given by folding the scale 180° about the ideal point and reading off the stimuli in order from the fold. Now, if instead of assuming such an algebraic model, one supposes that at any instant the stimuli and the ideal are random variables determined by (unimodal) distributions, then it is easy to see that three stimuli on the same side of the ideal should meet the strong stochastic transitivity assumption, for they are equally affected by shifts in the ideal point; but when two of the three stimuli are on one side and the third is on the other side, it may not be met. His data tend to confirm this hypothesis. It seems plausible that certain families of stimuli – and shades of gray may be one of them – lie on a continuum and that a person may evaluate the stimuli in terms of deviations from a more or less stable ideal point on that continuum. But for other types of stimuli, e.g. money, it does not seem so plausible. Thus, even if the data are taken at face value, it is not clear that other sets of preference judgments will necessarily violate the linearity assumption.

But more important than that, as has been intimated, I have some doubts about the meaningfulness of his analysis. The subjects were presented subsets of four stimuli, not pairs, which they had to rank in order as to preference. This was done to avoid repeated presentations of the same pairs of stimuli. From these orderings, which are the raw data, Coombs had to infer the pairwise probabilities $P(a,b)$. He simply counted the number of times that a was ranked higher than b in all sets where a and b both appeared and divided that by the number of such sets. While this is an obvious estimate of $P(a,b)$, it is not at all clear that it is a suitable one because the probability of any rank order of three or more stimuli is far from uniquely determined by the pairwise probabilities. One can easily generate a half dozen different models as to how a rank order might be constructed by the subject – by equiprobable pairwise comparisons, or by selecting the most preferred stimuli successively from the diminishing subsets, etc. – and the appropriate inference scheme varies widely from model to model. I have, for example, shown that from a hypothetical set of data the estimate of $P(a,b)$ can range from 0.25 to 1.00

depending upon the model assumed. Thus, for the present, I am unwilling to concede that these data clearly force one to reject the linearity assumption.

The second experiment, due to Davidson and Marschak [**3**], is slightly more favorable to that assumption. In design it is familiar: subjects had to choose between money gambles, with no pair of gambles being offered more than once. Clearly, no direct estimate of the probabilities P could be made, and so one might conclude that it could have no relevance to a probabilistic model. The striking fact is that by using certain statistical devices, which I shall not go into here, they were able to make certain weak checks as to whether strong stochastic transitivity was met. For most of their 17 subjects, the data did not give them adequate reason to reject the assumption.

Although the following is really no argument pro or con, it is interesting to note that linearity is generally assumed in the discrimination structures of psychophysics, where that concept first arose and where it has been worked with for many years. Specifically, S is taken to be the positive reals (magnitude of a physical dimension) and $P(a,b)$ is assumed to be strictly increasing in a for each b and strictly decreasing in b for each a. It follows that the trace of P is the natural ordering of the numbers by magnitude.

Since, by definiton, the trace of a linear discrimination structure is a weak ordering, the idea of representing it by an order-preserving numerical function comes to mind. It is interesting that no other assumptions are needed to show that P depends only upon the scale values, or put more formally:

LEMMA 3: *If P is a linear discrimination structure on S and if u is a real-valued function on S which preserves the order of its trace,* i.e., *$u(a) \geqslant u(b)$ if and only if $a \gtrsim b$, then $u(a) = u(a')$ and $u(b) = u(b')$ imply $P(a,b) = P(a',b')$.*

PROOF: Since u is order-preserving, $u(a) = u(a')$ implies $a \sim a'$, and so, by Definition 2, $P(a,b) = P(a',b)$. Similarly, $b \sim b'$, so $P(a',b) = P(a',b')$.

Although mathematical nicety dictates that I should use different symbols for a linear discrimination structure and the real-valued function of two real variables which, by Lemma 3, can be derived from it when an order-preserving function u of its trace is given, I shall nonetheless simply write $P[u(a),u(b)]$.

5. DECOMPOSABLE DISCRIMINATION STRUCTURES

In the remainder of this paper I shall be concerned only with discrimination structures on risk spaces. Since a risky alternative is a composite of pure alternatives and events, it is plausible that people might attempt to decompose decisions as to preferences between risky alternatives into simpler decisions about pure alternatives and about chance events. A particularly transparent example is the decision between $a\alpha b$ and $a\beta b$. The former should be judged preferable to the latter in just two cases: when a is judged preferable to b and α is judged more likely to occur than β; and when b is judged preferable to a and β is judged more likely to occur than α.

We have already assumed that preferences are probabilistically determined, and it seems just as plausible to suppose that the discrimination as to which of two events is more likely to occur will also generally be imperfect. Let us denote by $Q(\alpha,\beta)$ the (objective) probability that the subject judges α more likely to occur than β. If we suppose that this discrimination is statistically independent of the preference discrimination, then $P(a,b)Q(\alpha,\beta)$ gives the probability of the first of the two cases. The probability of the second case is, of course, given by $P(b,a)Q(\beta,\alpha)$, and since the two cases are exclusive, the sum yields the probability of choosing $a\alpha b$ over $a\beta b$. The only hitch in the argument is the assumption that the two discrimination processes are statistically independent. This is certainly not a reasonable assumption if either a or b are risky alternatives which involve events that in some sense depend upon α or β, but it may be otherwise. For the moment, we need only assume that it holds when a and b are pure alternatives, and so are, trivially, independent of α and β. All of this leads to the following definition.

DEFINITION 5: A discrimination structure P on a risk space $S(A,E)$ will be called *decomposable* if

(i) the elements a^* and b^* (Definition 1, Part (iii)) are in A, and if there exists a real valued function Q on $E \times E$ such that for all $\alpha,\beta \in E$,

(ii) $Q(\alpha,\beta) \geqslant 0$,

(iii) $Q(\alpha,\beta) + Q(\beta,\alpha) \leqslant 1$,

(iv) for all $a,b \in A$,

$$P(a\alpha b,a\beta b) = P(a,b)Q(\alpha,\beta) + P(b,a)Q(\beta,\alpha).$$

This function Q will be called the *core* of P. A decomposable discrimination structure will be denoted by (P,Q,A,E).

The basic intuition lying behind this definition, although not previously formulated in a probabilistic context, has played a considerable role in non-probabilistic theories of utility; see, for example, Ramsey [**15**] and Savage [**16**].

It is simple to give an expression for Q in terms of P, namely:

$$Q(\alpha,\beta) = \frac{P(a\alpha b,a\beta b)P(a,b) - P(a\beta b,a\alpha b)P(b,a)}{P(a,b)^2 - P(b,a)^2},$$

for any $a,b \in A$ such that $P(a,b) \neq P(b,a)$ (in Part (i) of Definition **5** it is assumed that there is at least one such pair). This formula renders it possible to estimate Q from empirical preference data, if the decomposition assumption is correct, and to determine whether or not it is correct by holding α and β fixed while varying a and b (see Section **11**). Of course, the Q so determined is interpreted as the objective probability that α is judged more likely to occur than β.

THEOREM 2: *Let (P,Q,A,E) be a decomposable discrimination structure, then Q is a discrimination structure with $Q(e,o) = 1$ and $Q(o,e) = 0$ and both P and Q are symmetric.*

PROOF: To show that Q is a discrimination structure, it is sufficient (Definitions 1 and 5) to show that $Q(e,o) = 1$ and $Q(o,e) = 0$. If $a,b \in A$, then since $a = aeb$ and $b = aob$, Definition 5 implies

$$P(a,b) = P(aeb,aob) = P(a,b)Q(e,o) + P(b,a)Q(o,e).$$

Rewriting,

$$0 = P(a,b)[Q(e,o) - 1] + P(b,a)Q(o,e).$$

By part (i) of Definition 5, we may choose a and b so that $P(a,b) > 0$, so $Q(e,o) = 1$. Similarly, a and b may be interchanged so that $P(b,a) > 0$, hence $Q(o,e) = 0$.

To show that Q is symmetric (Definition 4), let a^* and b^* be the elements described in part i of Definition 5. By condition R4 of Definition 3,

$$P(a^*\alpha b^*,a^*\beta b^*) = P(b^*\bar{\alpha}a^*,b^*\bar{\beta}a^*),$$

for all $\alpha,\beta \in E$. Apply the decomposition condition to both sides and collect terms

$$P(a^*,b^*)[Q(\bar{\beta},\bar{\alpha}) - Q(\alpha,\beta)] + P(b^*,a^*)[Q(\bar{\alpha},\bar{\beta}) - Q(\beta,\alpha)] = 0.$$

Interchanging the roles of a^* and b^*, we obtain

$$P(b^*,a^*)[Q(\bar{\beta},\bar{\alpha}) - Q(\alpha,\beta)] + P(a^*,b^*)[Q(\bar{\alpha},\bar{\beta}) - Q(\beta,\alpha)] = 0.$$

This pair of equations has no non-trivial solution since the determinant of coefficients, $P(a^*,b^*)^2 - P(b^*,a^*)^2$, is non-zero by the choice of a^* and b^*, so Q is symmetric.

By the decomposition assumption and the symmetry of Q,

$$P(a\alpha b,a\beta b) = P(a,b)Q(\alpha,\beta) + P(b,a)Q(\beta,\alpha) = P(a,b)Q(\bar{\beta},\bar{\alpha}) + P(b,a)Q(\bar{\alpha},\bar{\beta}) = P(a\bar{\beta}b,a\bar{\alpha}b),$$

for every $a,b \in A$ and $\alpha,\beta \in E$; hence P is symmetric.

It is easily seen that if P is a decomposable discrimination structure, its restriction to the set A of pure alternative is itself a discrimination structure, which I shall denote by P_A.

THEOREM 3: *Let (P,Q,A,E) be a decomposable discrimination structure; then either P_A and Q are both additive (the latter with $K = 1$) or P_A is reflexive.*

PROOF: Since $a = a\alpha a$, the decomposition assumption yields

$$P(a,a) = P(a\alpha a,a\beta a) = P(a,a)[Q(\alpha,\beta) + Q(\beta,\alpha)].$$

If P_A is not reflexive, then $P(a',a') > 0$ for some $a' \in A$, so dividing by $P(a',a')$ yields $Q(\alpha,\beta) + Q(\beta,\alpha) = 1$, i.e., Q is additive with $K = 1$. Note that $Q(\alpha,\alpha) = 1/2$. Let $K = 2P(a',a')$. Then for any $b \in A$,

$$K = 2P(a',a') = 2P(boa',boa') = 2Q(o,o)[P(b,a') + P(a',b)] \\ = P(a',b) + P(b,a').$$

Thus,
$$2P(b,b) = 2P(a'ob,a'ob) = P(a',b) + P(b,a') = K.$$

So, for any $a \in A$,

$$K = 2P(b,b) = 2P(aob,aob) = P(a,b) + P(b,a).$$

Therefore, P_A is also additive.

The two previous theorems give several necessary conditions for a discrimination structure to be decomposable; however, to date, no necessary and sufficient conditions are known.

It is interesting to note the relationship between this last theorem and experimental practice. The additive case with $K = 1$ corresponds to forced choice responses, i.e., where a subject is not permitted to report indifference between two alternatives and $P(a,a)$ is taken to be $1/2$ by definition. The alternative procedure is to permit indifference reports. One might be inclined to postulate that a person will always report indifference between an element and itself, i.e., he will yield a reflexive discrimination structure. But in that case, Lemma 1 implies that the structure must be transitive: if $a > b$, then the subject must either report preference for a over b or indifference between them, but never preference for b over a. It is clear that one would have to be quite optimistic to expect this, and experimental practice in psychophysics, where an analogous problem exists, is to use forced choice questions.

THEOREM 4: *Let (P,Q,A,E) be a decomposable discrimination structure. If P is linear, so is Q.*

PROOF: By part (i) of Definition 5, we may choose $a^*,b^* \in A$ such that $P(a^*,b^*) > P(b^*,a^*)$. For any $\alpha, \beta \in E$, $P(a^*\alpha b^*,c) - P(a^*\beta b^*,c)$ and $P(c,a^*\beta b^*) - P(c,a^*\alpha b^*)$ are both nonnegative or both nonpositive for all $c \in S(A,E)$ since P is linear. Without loss of generality, suppose the former is true. Choose $c = a^*\gamma b^*$, where $\gamma \in E$, apply the decomposition assumption, and collect terms:

$$P(a^*,b^*)[Q(\alpha,\gamma) - Q(\beta,\gamma)] - P(b^*,a^*)[Q(\gamma,\beta) - Q(\gamma,\alpha)] \geqslant 0,$$
$$P(a^*,b^*)[Q(\gamma,\beta) - Q(\gamma,\alpha)] - P(b^*,a^*)[Q(\alpha,\gamma) - Q(\beta,\gamma)] \geqslant 0.$$

If P_A is reflexive, then by Lemma 1 it is transitive, so $P(b^*,a^*) = 0$ and $P(a^*,b^*) > 0$. Thus, $Q(\alpha,\gamma) \geqslant Q(\beta,\gamma)$ and $Q(\gamma,\beta) \geqslant Q(\gamma,\alpha)$, for all $\gamma \in E$, and so Q is linear. If P_A is additive, then Q is additive with $K = 1$ (Theorem 3), and so

$$Q(\alpha,\gamma) - Q(\beta,\gamma) = 1 - Q(\gamma,\alpha) - 1 + Q(\gamma,\beta) = Q(\gamma,\beta) - Q(\gamma,\alpha).$$

Substituting,

$$[P(a^*,b^*) - P(b^*,a^*)]\,[Q(\gamma,\beta) - Q(\gamma,\alpha)] \geqslant 0.$$

Since, by the choice of a^* and b^* the first term is positive, the second must be nonnegative, and so Q is linear. By Theorem 3, all cases have been covered.

THEOREM 5: *Let (P,Q,A,E) be a decomposable linear discrimination structure satisfying condition* R5 *(Definition* 3), *and let $\gtrsim$ be the trace of Q. For every $\alpha \in E$, $e \gtrsim \alpha \gtrsim o$ and $e > o$.*

PROOF: Let a^*, $b^* \in A$ be the elements mentioned in part (i) of Definition 5, and let us suppose that $P(a^*,b^*) > P(b^*,a^*)$. If P_A is additive, then $P(a^*,b^*) > K/2 = P(b^*,b^*)$, and if P_A is reflexive, $P(a^*,b^*) > 0 = P(b^*,b^*)$. Since P is linear, this implies $a^* > b^*$.[4] Therefore, by condition R5, $a^* \gtrsim a^*\alpha b^* \gtrsim b^*$, for every $\alpha \in E$, and so by Definition 2,

$$\begin{aligned} 0 \leqslant P(a^*\alpha b^*,a^*\beta b^*) - P(b^*,a^*\beta b^*) &= P(a^*,b^*)[Q(\alpha,\beta) - Q(o,\beta)] \\ &\quad + P(b^*,a^*)[Q(\beta,\alpha) - Q(\beta,o)]. \end{aligned}$$

If P_A is additive, then Q is also, and this reduces to

$$0 \leqslant [P(a^*,b^*) - P(b^*,a^*)]\,[Q(\alpha,\beta) - Q(o,\beta)].$$

Since the first term is positive by choice, $Q(\alpha,\beta) \geqslant Q(o,\beta)$. If P_A is reflexive, then by Lemma 1 it is transitive and so $P(a^*,b^*) > 0$ and $P(b^*,a^*) = 0$; hence the same conclusion follows. The other three inequalities are proved similarly, yielding $e \gtrsim \alpha \gtrsim o$.

By Theorem 2,

$$Q(e,o) = 1 > 1/2 \geqslant Q(o,o),$$

so $e > o$.

The following theorem gives a functional equation for the core which, at the moment, looks to be of limited value, but which will later prove to be very important.

THEOREM 6: *Let (P,Q,A,E) be a decomposable discrimination structure satisfying condition* R6 *(Definition* 3). *Let $\alpha,\beta,\gamma \in E$ be such that*

$$P[(a\gamma b)\alpha b,(a\gamma b)\beta b] = P(a\gamma b,b)Q(\alpha,\beta) + P(b,a,\gamma b)Q(\beta,\alpha)$$

for every $a,b \in A$. Then

$$Q(\alpha\cap\gamma,\beta\cap\gamma) = Q(\alpha,\beta)Q(\gamma,o) + Q(\beta,\alpha)Q(o,\gamma).$$

[4] I shall use the same symbol for both the trace of P and of Q. This will not prove to be ambiguous because the symbols, Latin or Greek, for elements in the domain of the relation will make clear which one is intended.

PROOF: Observe that R2 and R6 imply

$$(a\gamma b)ab \sim a(\gamma\cap\alpha)(bab) = a(\gamma\cap\alpha)b.$$

Thus, by Definition 2,

$$P[(a\gamma b)\alpha b,(a\gamma b)\beta b] = P[a(\gamma\cap\alpha)b,a(\gamma\cap\beta)b].$$

Now, apply the hypothesis of the theorem to the left side and then use condition R3 and the decomposition assumption on both sides:

$$\begin{aligned}P(a\gamma b,b)Q(\alpha,\beta) + P(b,a\gamma b)Q(\beta,\alpha) &= P(a\gamma b,aob)Q(\alpha,\beta) + P(aob,a\gamma b)Q(\beta,\alpha)\\ = Q(\alpha,\beta)[P(a,b)Q(\gamma,o) + P(b,a)Q(o,\gamma)] &+ Q(\beta,\alpha)[P(a,b)Q(o,\gamma) + P(b,a)Q(\gamma,o)]\\ &= P(a,b)Q(\gamma\cap\alpha,\gamma\cap\beta) + P(b,a)Q(\gamma\cap\beta,\gamma\cap\alpha).\end{aligned}$$

Collecting terms,

$$\begin{aligned}&P(a,b)[Q(\alpha,\beta)Q(\gamma,o) + Q(\beta,\alpha)Q(o,\gamma) - Q(\gamma\cap\alpha,\gamma\cap\beta)]\\ &+ P(b,a)[Q(\alpha,\beta)Q(o,\gamma) + Q(\beta,\alpha)Q(\gamma,o) - Q(\gamma\cap\beta,\gamma\cap\alpha)] = 0.\end{aligned}$$

Let a^* and b^* be the elements mentioned in part (i) of Definition 5, and consider the two equations obtained by the substitutions $a = a^*$, $b = b^*$ and $a = b^*$, $b = a^*$. Since, by the choice of a^* and b^*, the determinant of coefficients $P(a^*,b^*)^2 - P(b^*,a^*)^2 \neq 0$, the conclusion follows.

6. SUBJECTIVE PROBABILITY AND INDEPENDENCE

A number of authors have recently taken the position that a subjective concept of probability as well as of value, or utility, is needed in any descriptive theory of individual decision making. Without a doubt, Savage's [**16**] formulation is the deepest and most interesting, but, from my point of view, like most of decision theory, it suffers from being an algebraic rather than a probabilistic model. Furthermore, Savage's aim was to construct a foundation for probability theory and so he postulated sufficiently strong axioms to arrive at a subjective probability function possessing all of the formal properties of objective probability. In other words, the role of his theory is to give a decision making, rather than frequency, justification and interpretation of the usual properties. Those who are more concerned with a description of behavior are not sure what properties subjective probabilities have, but many doubt that they are identical to the axioms for objective probability. In particular, there is a good deal of skepticism about finite additivity.

Here I should like to offer another possible interpretation for this intuitive concept. I shall suppose that it refers to a partial numerical description of a discrimination structure over the set of events. Some would say, then, that subjective probability is really a random variable, but I shall employ the term "subjective probability" to refer to a single numerical function which describes the "average" tendency of that random variable. To be more formal, DEFINITION 6: Let Q be a linear discrimination structure on a Boolean algebra E having null element o. Suppose its trace $\gtrsim$ has the properties that $e \gtrsim \alpha \gtrsim o$, for every $\alpha \in E$ and $e > o$. A real-valued function φ with domain E will be called a *subjective probability function* of (Q,E) if

(i) $\varphi(\alpha) \geqslant \varphi(\beta)$ if and only if $\alpha \gtrsim \beta$,
(ii) $\varphi(o) = 0$, and
(iii) $\varphi(\alpha) + \varphi(\bar{\alpha}) = 1$.

It is clear that φ is a mapping of E into the closed unit interval $[0,1]$ and that $\varphi(e) = 1$.

The assumption in part (iii), which is clearly weaker than finite additivity, has been questioned by Edwards [**4**], but many authors have been willing to accept it.

Observe that the conditions of Definition 6 are met by the core of any decomposable linear structure which satisfies condition R5 (Theorems 4 and 5). Hence, if a subjective probability function exists, Lemma 3 implies that discrimination between events depends only upon the subjective probabilities of these elements. Put in the language of one of the referees, this means that for the given person all events must have the same "degree of subjectivity." That this is implausible can be brought out by another example due to Professor Raiffa. Let α denote the event of rain on Wall Street in New York City at some particular time in the future, and β the event of rain both on Wall Street and in Times Square at the same time. No matter what the subjective probabilities $\varphi(\alpha)$ and $\varphi(\beta)$ may be, it seems reasonable to suppose that they are not very different because Times Square is not far from Wall Street. But since event β implies event α, it seems only reasonable always to say that α is more likely than β, in which case $Q(\alpha,\beta) = 1$. Yet, in general, if two events have nearly the same subjective probability and one is not trivially more likely than the other, one expects that $Q(\alpha,\beta) < 1$. Clearly, there is a difficulty somewhere.

Two alternatives seem possible. Either one of the assumptions I have made, such as linearity or decomposability,[5] is seriously in error, or I am mistaken in identifying a scale satisfying the conditions of Definition 6 with the intuitive concept of subjective probability. Assuming that it is possible to make suitable estimates of probabilities of preference, the first alternative can be examined experimentally (see Section 11), but this has yet to be done thoroughly. Should that not be the source of trouble, then it must be the latter, in which case I would suggest that some of the following results indicate that this scale, rather than our intuitions about subjective probability, may be the more useful in decision making models. In any event, I shall tentatively use the expression "subjective probability function" for a

[5] Note that R6, which is controversial, is not assumed.

scale satisfying Definition 6, knowing full well that other authors may decide that it differs too much from the intuitive concept and that therefore some other label should be attached.

Next, we must worry about the idea of independent events. The concept is certainly well defined when a probability measure over the algebra of events is given, but none of us restricts the use of the word "independent" to situations where one knows the probabilities and, moreover, most of us behave some of the time as if certain events are not independent even when it can be demonstrated that they are (e.g., in coin tossing experiments). I am sure that we would all say, and try to act, as if the event of rain tomorrow on Wall Street is independent of whether the next person we see is red-headed; yet few of us know the objective probabilities involved – assuming that they are defined at all. It seems that the subjective notion of independence, which in all likelihood is important in decision making, reflects not so much our knowledge of objective probabilities as our current views about what are the important causal connections in the universe. We see only the most tenuous connections between rain in New York City and the appearance of a red-headed person; but we do see a substantial connection between two tosses of a fair coin, namely the coin, and only the sophisticated can refrain from believing that this connection is relevant to successive outcomes. I raise these trite points simply to suggest that there is a real problem of definition.

In what follows, I shall want independent events to have two properties, one of which will be taken as the definition and the other as an assumption. The choice made here is a matter of expositional convenience, and is exactly the opposite of the choice one will have to make if subjectively independent events are to be isolated experimentally. They need not be isolated, however, to use this theory in the analysis of data.

DEFINITION 7: If φ is a subjective probability function defined on the Boolean algebra E, then α and $\beta \in E$ are *subjectively independent* with respect to φ if and only if $\varphi(\alpha \cap \beta) = \varphi(\alpha)\varphi(\beta)$.

The origins of this definition are clear.

DEFINITION 8: Let (P,Q,A,E) be a decomposable linear discrimination structure satisfying R5 and R6, and suppose that Q has a subjective probability function φ. We say that (P,Q,A,E,φ) is *strongly decomposable* if for every $a,b \in A$ and for every $\alpha,\beta,\gamma \in E$ such that both α and β are subjectively independent of γ with respect to φ,

$$P[(a\gamma b)\alpha b,(a\gamma b)\beta b] = P(a\gamma b,b)Q(\alpha,\beta) + P(b,a\gamma b)Q(\beta,\alpha).$$

It will be noted that the term "strongly decomposable" implies that (P,Q,A,E) is decomposable, linear, and satisfies R5 and R6.

One tends to feel that if subjective independence has been correctly defined, then the property described in Definition 8 should be met; the argument was given prior to Definition 5. Thus, if the later results are offensive, I should be more inclined to change the definition of independence than to give up the assumption of strong decomposability.

THEOREM 7: *If* (P,Q,A,E,φ) *is strongly decomposable, then*

$$Q(\alpha\cap\gamma,\beta\cap\gamma) = Q(\alpha,\beta)Q(\gamma,o) + Q(\beta,\alpha)Q(o,\gamma),$$

for α *and* β *subjectively independent of* γ *with respect to* φ.

PROOF: Theorems 4, 5, and 6.

7. THE FORM OF THE CORE

In any of the utility models that involve the expected utility hypothesis, it is necessary to suppose that the set of events is very large. Von Neumann and Morgenstern do so implicitly by assuming that they can work with any objective probability; Savage is quite explicit about it in his construction of a subjective probability function. Obviously, there is no objection to such an assumption about objective probabilities, and one could hope that it would be equally reasonable for subjective probabilities. For example, one might argue that the organism ought to be designed to react to any conceivable event and so to have a finely graded, if not actually continuous, scale of subjective probability. But equally well, one can argue with some force that, for most practical purposes, organisms really do not need to class events into more than a few categories of likelihood, and certainly the empirical evidence that people cannot use more than a few categories is compelling (see, for example, Miller [**12**]). If this is really so, then all of the traditional utility models, including this one, are inappropriate, and a quite different mathematical analysis is needed. On the assumption, however, that the traditional approach has some merit, we introduce the following restriction which in conjunction with the other restrictions leads to surprisingly detailed results as to the relationship between the core and the subjective probability function.

DEFINITION 9: A subjective probability function φ of (Q,E) will be said to be *dense* if for every $x,y,z \in [0,1]$ there exist $\alpha,\beta,\gamma \in E$ such that

(i) $\varphi(\alpha) = x$, $\varphi(\beta) = y$, and $\varphi(\gamma) = z$, and

(ii) α and β are both independent of γ.

THEOREM 8: *Let* (P,Q,A,E,φ) *be a strongly decomposable discrimination structure such that* φ *is dense. Then either there exists a positive* ε *such that, for* P_A *additive,*

$$Q(\alpha,\beta) = \begin{cases} \frac{1}{2}\left\{1 + [\varphi(\alpha) - \varphi(\beta)]^{\varepsilon}\right\}, & \text{if } \alpha \gtrsim \beta, \\ \frac{1}{2}\left\{1 - [\varphi(\beta) - \varphi(\alpha)]^{\varepsilon}\right\}, & \text{if } \alpha \gtrsim \beta, \end{cases}$$

and, for P_A reflexive,

$$Q(\alpha,\beta) = \begin{cases} [\varphi(\alpha) - \varphi(\beta)]^{\varepsilon}, & \text{if } \alpha \gtrsim \beta, \\ 0 & \text{, if } \alpha \not\gtrsim \beta; \end{cases}$$

or Q is the limit of these expressions as $\varepsilon \to 0$; or Q is the limit as $\varepsilon \to \infty$.

PROOF: Observe that if x, $y \in [0,1]$ and if $\alpha,\beta \in E$ are such that $x = \varphi(\alpha)$ and $y = \varphi(\beta)$, then by Lemma 3 we may write $Q(\alpha,\beta) = Q[\varphi(\alpha),\varphi(\beta)] = Q(x,y)$. Now, suppose x, y, $z \in [0,1]$. Since φ is dense, we can rewrite the functional equation of Theorem 7 as

$$Q(xz,yz) = Q(x,y)Q(z,0) + Q(y,x)Q(0,z).$$

Let us define Q' on $[-1,1]$ as follows:

$$Q'(x) = \begin{cases} Q(x,0), & \text{if } x \in [0,1], \\ Q(0,-x), & \text{if } x \in [-1,0]. \end{cases}$$

We now show that $Q(x,y) = Q'(x-y)$. Suppose $x \geqslant y$, then $y/x \leqslant 1$, so

$$Q(x,y) = Q(1x,\frac{y}{x}x) = Q(1,y/x)Q(x,0) + Q(y/x,1)Q(0,x).$$

Also,

$$Q'(x-y) = Q(x-y,0) = Q[(1-y/x)x,0x] = Q(1-y/x,0)Q(x,0) + Q(0,1-y/x)Q(0,x).$$

But by property (iii) of a subjective probability function and the symmetry of Q (Theorem 2),

$$Q(x,y) = Q[\varphi(\alpha),\varphi(\beta)] = Q(\alpha,\beta) = Q(\bar{\beta},\bar{\alpha}) = Q[\varphi(\bar{\beta}),\varphi(\bar{\alpha})] = Q(1-y,1-x),$$

and the assertion is proved. If $x < y$, a similar argument holds. Thus, we may rewrite the functional equation as

$$Q'(xy) = Q'(x)Q'(y) + Q'(-x)Q'(-y),$$

for $x,y \in [-1,1]$. By Theorem 2,

$$Q'(1) = Q(1,0) = Q(e,o) = 1,$$

and

$$Q'(-1) = Q(0,1) = Q(o,e) = 0.$$

Define $R(x) = Q'(x) - Q'(-x)$ and $S(x) = Q'(x) + Q'(-x)$.

$$R(xy) = Q'(xy) - Q'(-xy) = Q'(x)Q'(y) + Q'(-x)Q'(-y) - Q'(-x)Q'(y) - Q'(x)Q'(-y) = [Q'(x) - Q'(-x)]\,[Q'(y) - Q'(-y)] = R(x)R(y).$$

Furthermore, $R(1) = 1$, $R(0) = 0$, $R(-1) = -1$, as is easily verified. If P_A is additive, then by Theorem 3 so is Q (with $K = 1$); so $S(x) = 1$. If P_A is reflexive, then an argument similar to that used for R shows that $S(xy) = S(x)S(y)$. Also $S(x) = S(-x)$ and $S(1) = 1$. $S(0) = 0$ follows from the fact that we can choose a^* and $b^* \in A$ such that $P(a^*,b^*) > 0$; so

$$\begin{aligned} 0 &= P(a^*,a^*) \\ &= P(a^*eb^*,a^*eb^*) \\ &= P(a^*,b^*)Q(e,e); \end{aligned}$$

and hence, $Q(e,e) = 0$.

From this it follows that Q is transitive ($Q'(x) = 0$ for $x \leqslant 0$) in the reflexive case, for if $x = 0$ it follows immediately and if $x < 0$ and $Q'(x) > 0$, then for $a \in E$ such that $\varphi(a) = —x$,

$$Q(o,a) = Q(0,—x) = Q'(x) > 0 = Q(e,e) = Q'(0) = Q(o,o).$$

But since Q is linear (Theorem 4), it follows that $a < o$, which is impossible by Theorem 5.

Now, suppose that Q' is a continuous function; then so are R and S, and it is well known that the functional equation for R with the given initial points is solved by

$$R(x) = \begin{cases} x^{\varepsilon} & \text{, if } x \in [0,1], \\ —|x|^{\varepsilon}, & \text{if } x \in [—1,0], \end{cases}$$

for some $\varepsilon > 0$. Similarly, if P_A is reflexive,

$$S(x) = |x|^{\delta},\ x \in [—1,1],$$

for some $\delta > 0$. The expressions for Q' are obtained by noting that $Q' = (R + S)/2$ and that, in the reflexive case, $\delta = \varepsilon$ because Q' is transitive.

Next, we consider the discontinuous cases. We observe that if $x \geqslant 0$,

$$R(x) = \begin{cases} 2Q'(x) — 1, & \text{if } P_A \text{ is additive,} \\ Q'(x) & \text{, if } P_A \text{ is reflexive.} \end{cases}$$

Thus, R and Q have discontinuities at exactly the same places. Furthermore, R is nondecreasing for $x \geqslant 0$ because, by the definition of the subjective probability function, Q is nondecreasing; therefore, the discontinuities are jumps. Suppose there is one at $x \in (0,1)$; then for any y, $x < y < 1$, and $z = x/y$, we know

$$R(x) = R(yz) = R(y)R(z),$$

so there must be a discontinuity at either y or z. Since $x < 1$, this means there are a non-denumerable number of jumps, which is impossible because R is bounded. Thus, the only discontinuities can be at —1, 0, and 1.

Suppose that 1 is a discontinuity, then there exists a $\delta > 0$ such that for any $x \in [0,1]$, $R(x) \leqslant 1—\delta$. But, for any $y \in [0,1]$ and any n, however large, we can find an $x \in [0,1]$ such that $y = x^n$; hence by induction

$$R(y) = R(x^n) = R(x)^n \leqslant (1—\delta)^n.$$

Thus, $R(y) = 0$ for $y \in [0,1]$. We observe that $R(—y) = —R(y)$, so

$$\begin{aligned} R(1) &= 1, \\ R(y) &= 0, \ -1 < y < 1, \\ R(-1) &= -1. \end{aligned}$$

This is easily seen to be the case when $\varepsilon \to \infty$. This also covers the case of a discontinuity at -1. If there is a discontinuity at 0, then a very similar argument establishes that Q must be of the form that is obtained by letting $\varepsilon \to 0$.

COROLLARY: *If the conditions of the theorem are met and if Q is continuous, then*

$$\varphi(a) = \begin{cases} \frac{1}{2}\left\{ 1 + [Q(a,\bar{a}) - Q(\bar{a},a)]^{1/\varepsilon} \right\}, \text{ if } a \gtrsim \bar{a}, \\ \frac{1}{2}\left\{ 1 - [Q(\bar{a},a) - Q(a,\bar{a})]^{1/\varepsilon} \right\}, \text{ if } a \gtrsim \bar{a}. \end{cases}$$

PROOF: Trivial.

It is clear that no inference can be drawn about the subjective probability function from data on the core in either of the discontinuous cases. The first of the discontinuous cores, $\varepsilon \to 0$, represents the perfect discrimination that has generally been assumed in utility models. The second is, I suspect, of absolutely no empirical interest; it represents nearly total failure to discriminate: only pairs in which one event is subjectively certain to occur and the other is subjectively certain not to occur elicit a significant response. Such a person surely would be confined.

8. AN IMPOSSIBILITY THEOREM

I shall not yet discuss how Theorem 8 may play a role in experimental studies of preference behavior (see Section 11), but I should like to devote a few words to its plausibility. A number of people who have seen it have been dismayed that such assumptions as I have made determine the discrimination function up to a single parameter, and they have been – rightly, I think – skeptical of its applicability to empirical materials. But, interestingly enough, much of the objection to this theory has not been directed against the form of the function per se, but against the much weaker result that discrimination depends only upon the subjective probabilities of the two events. These criticisms have already been presented (following Definition 6), but I stress them again because there may be a tendency to focus attention upon Definitions 7 (subjective independence), 8 (strong decomposability), and 9 (the denseness of the subjective probability function), whereas the heart of many objections appears to be concerned with earlier assumptions than these.

Two points that I know of can be directed against the form of the function independently of the objection I have just mentioned. The first is, I think, the less forceful. Suppose that each event a has an objective probability

$p(\alpha)$ and that the subject has had some experience, i.e., independent Bernoulli trials, with events which he knows have the same (unknown) objective probability. Then, we might suppose that he decides whether α or β is more likely on the basis of which has had the greater percentage of successes in the past, and so $Q(\alpha,\beta)$ would be nothing but the objective probability that in $n(\alpha)$ trials of an event α with probability $p(\alpha)$ of success there is a larger percentage of successes than in $n(\beta)$ trials of event β with probability $p(\beta)$ of success. It is clear that, even if we can select $n(\alpha)$ for all $\alpha \in E$ such that Q is of the form of the model, for an arbitrary choice of the $n(\alpha)$ this procedure will not fit the model. Furthermore, the only situation for which I have been able to choose $n(\alpha)$ (as a function of $p(\alpha)$) to yield a fit is with $n(\alpha) = 1$, in which case $\varphi(\alpha) = p(\alpha)$ and the discrimination function in the additive case is the straight line from (—1,0) to (1,1), i.e., $\varepsilon = 1$, which is certainly not like much discrimination data I have seen.

The second criticism takes the form of an impossibility theorem, for which we need another definition.

Definition 10: A decomposable linear discrimination structure having a trace $\gtrsim$ that satisfies condition R5 is said to satisfy the *Archimedean condition* provided that for every $a,b,c \in A$ such that $a > b > c$, there exists an $\alpha \in E$, $e > \alpha > o$, such that $b \sim a\alpha c$.

Theorem 9: *The following conditions are incompatible*:

(i) *(P,Q,A,E,φ) is strongly decomposable and φ is dense,*

(ii) *P satisfies the Archimedean condition,*

(iii) *Q is continuous (see Theorem 8),*

(iv) *there are three elements $a,b,c, \in A$ which are perfectly discriminated in the sense that $P(a,b) = P(b,c) = P(a,c) = C$, where $C = 1$ in the reflexive case and K in the additive case.*

Proof: It is sufficient to prove that if (i) and (ii) are met and if a and b are perfectly discriminated, then either Q is discontinuous or $a \gtrsim c \gtrsim b$, for all $c \in A$. For, by part (iv), $P(b,c) = C > P(c,c)$; so $b > c$; so Q must be discontinuous, which contradicts part (iii).

To prove the subassertion, suppose $c > a > b$. By Definition 2, $P(c,b) \geqslant P(a,b) = C$, so $P(c,b) = C$ and $P(b,c) = 0$ by Definitions 1 and 4. By the Archimedean condition, there exists an $\alpha \in E$ with $e > \alpha > o$ such that $a \sim c\alpha b$. Thus,

$$P(a,b) = C = P(c\alpha b,cob) = P(c,b)Q(\alpha,o) + P(b,c)Q(o,\alpha) = CQ'[\varphi(\alpha)].$$

So $Q'[\varphi(\alpha)] = 1$. Since $e > \alpha > o$, $1 > \varphi(\alpha) > 0$, and so by Theorem 8 Q is discontinuous.

The dilemma is clear: One certainly can expect to find cases where three or more pure alternatives are perfectly discriminated and, at the same time, where there is imperfect discrimination among gambles and events. For example, if the pure alternatives are sums of money spaced by more than a few cents, perfect discrimination may always be expected. Thus, the difficulty must be found in the other three assumptions. The new one, the Archimedean condition, might be considered offensive, especially since it has not been needed up to now. Certainly, it has been criticised in the literature. But if we accept the expected utility hypothesis (see next section), which together with the density of φ implies the Archimedean condition, there is little point in worrying about the Archimedean condition as such.

Since the expected utility hypothesis has been so basic to all thought in this area, one is not likely to give it up easily, and so is led to focus on the density of φ. Possibly the idealization – for that is surely what it is – that to every real number in the unit interval there corresponds an event with that number as its subjective probability is too gross. This brings us back to the empirical observation mentioned at the beginning of Section 7 which suggests that subjective scales may be discrete. Indeed, the observations suggest that there may be only a finite number of scale values. Work in this direction is needed.

Without trying to deny the force of the above criticisms, it should also be pointed out that for some situations we may not be too far from the truth. The form of the discrimination function has been derived rather indirectly from axioms that have a history of serious study, and for $\varepsilon < 1$ it is, fortunately, not unlike a lot of discrimination data that have been obtained in psychophysical experiments. For example, although an adequate rationale has yet to be given, it has been customary to fit discrimination data by the integral of a normal curve, and the fits have been fairly good. It is not difficult to see that the continuous cores ($\varepsilon < 1$) in the additive case are not very different from the integral of a normal, and, although I have not checked it, I would be surprised if there exist discrimination data that could decide between the two curves.

9. UTILITY FUNCTIONS AND SENSATION SCALES

It is clear that if there is a numerical function that preserves the order of the trace of a linear discrimination structure, then any strictly increasing function of u is also order preserving. Thus, if no further specifications are imposed, there is little if any point in introducing numerical representations of the trace. Historically, two quite distinct traditions for restricting the class of admissible scales have developed. Utility theorists, assuming weak orders over mixture spaces, rather than linear discrimination structures over risk spaces, have concerned themselves with the existence of linear utility functions (see Definition 11 below). The reasons are largely pragmatic: it has been practically impossible to devise mathematically interesting decision theories if expected utility values do not represent the utility of risky alternatives. The von Neumann axiom system was devised to give an intuitive justification for restricting one's attention to linear utility functions; the criticisms of it are well known and need not be entered into here.

At the same time, a much older tradition for selecting order preserving functions exists in psychophysics. It traces back to the middle of the last century when Fechner postulated the equality of just noticeable differences as the defining property of subjective sensation. In more recent work, one postulates a linear structure P on the positive real line, and the idea – which we will specify more precisely in Definition **12** – is to find those scales, if any, which render P dependent only upon scale differences. This is often summarized by saying "equally often noticed differences are equal." It is argued, largely on philosophical grounds, that a subjective sensation scale must have the property that the probability of detecting a difference on that scale depends upon the difference and not upon its location on the scale. While from the start and still today this definition has been extremely controversial, it has played a remarkably important and useful role in the development of psychophysics.

When S is taken to be the positive reals and P is assumed to be strictly monotonic in its two variables, conditions are known for the existence of such scales (which are unique up to a linear transformation) and their analytic form has been given. The reader should be warned that the traditional mathematical formulation of this problem is in error; the correct formulation and solution may be found in [**9**].

I know of no attempt to link these two distinct traditions.[6] Presumably this is because, on the one hand, utility theorists have concentrated mainly on algebraic models where the psychological condition could not be easily imposed; and, on the other hand, the idea of a risk space seems to make no sense in traditional sensory psychology, so linear utility functions were not meaningful there. Once a general probability model is postulated, however, the psychological condition is meaningful for all linear discrimination structures, and so in the domain of risk spaces there are two, apparently conflicting, criteria of selection. The point of this section is to establish that for an interesting class of linear discrimination structures on risk spaces there is no conflict: the two concepts are the same.

DEFINITION **11**: Let P be a linear discrimination structure on a risk space $S(A,E)$ and φ a mapping of E into $[0,1]$ such that $\varphi(o) = 0$. A real valued

[6] This observation is true only to this extent: At about the same time as I was working on this material, Professor Marschak independently suggested that Fechner's idea might be useful in utility theory; see [**3**]. He did not, however, explore the relationship between that notion and the expected utility hypothesis.

function u on $S(A,E)$ is called a *linear utility function* (with respect to φ) if

(i) u preserves the order of the trace of P, and

(ii) for every $a,b \in A$ and $\alpha \in E$,

$$u(a\alpha b) = \varphi(\alpha)u(a) + \varphi(\bar{\alpha})u(b).$$

This notion, often referred to as the expected utility hypothesis, has played an extremely important role in the development of utility theory. It is not usual to restrict a and b to pure alternatives; however, it is clear that they cannot be arbitrary risky alternatives, since by R2 and R6 we have $(a\alpha b)\alpha b \sim a\alpha b$, and so by two applications of (ii) on the left and one on the right we could conclude $\varphi(\alpha) = 0$ for all α. When objective probabilities are used, it is implicit that condition (ii) holds when a and b only involve events independent of α. Although I could make the same assumption here, using subjective rather than objective independence, it is sufficient for my purpose to restrict a and b to A.

LEMMA 4: *If u is a linear utility function with respect to φ, then for all $\alpha \in E$, $\varphi(\alpha) + \varphi(\bar{\alpha}) = 1$.*

PROOF: By R2, $a\alpha a = a$ for all a, and therefore $u(a) = u(a\alpha a) = u(a)[\varphi(\alpha) + \varphi(\bar{\alpha})]$.

This means, of course, that the φ of Definition 11 is compatible with Definition 6 of a subjective probability function.

Note that given R2 of Definition 3, then Definition 11 implies that R3, R4, and R5 must hold with indifference substituted for equality.

DEFINITION 12: Let P be a linear discrimination structure and u a real-valued function which preserves the order of its trace. u is said to be a (Fechnerian) *sensation scale* if there exists a real-valued function P' of one real variable such that for all $a,b \in S$, $P(a,b) = P'[u(a)—u(b)]$.

We note that part of Theorem 8 asserts that, under the given conditions, the subjective probability function forms a sensation scale. Under somewhat different conditions, a linear utility function also forms a sensation scale of P_A.

THEOREM 10: *Let (P,Q,A,E) be a decomposable linear structure which satisfies the Archimedean condition. If there is a subjective probability function φ and a linear utility function u with respect to φ, then u is a sensation scale of P_A.*

PROOF: It is sufficient to show that $P(a,b) = P(c,d)$ for any $a,b,c,d \in A$ such that $u(a) - u(b) = u(c) - u(d)$. Suppose, with no loss of generality, that $u(b) \leqslant u(d)$. We consider the case where $u(a)—u(b) > 0$; the other case is similar. Thus, $u(b) < u(a) \leqslant u(c)$; so, by the Archimedean condition or R3, there exists an $\alpha \in E$ such that $a \sim b\alpha c$. Hence, $P(a,b) = P(b\alpha c,b)$. Using the assumed relations among the elements and the linearity of u,

$$\begin{aligned} u(d) &= u(c) - u(a) + u(b) = u(c) - u(bac) + u(b) \\ &= \varphi(\alpha)u(c) + [1-\varphi(\alpha)]u(b) = u(cab), \end{aligned}$$

so $d \sim cab$. Thus, $P(c,d) = P(c,cab)$. But by Theorem 2, P is symmetric, and so

$$P(a,b) = P(bac,bec) = P(boc,b\bar{\alpha}c) = P(c,cab) = P(c,d).$$

10. THE FORM OF P_A

THEOREM 11: *Let (P,Q,A,E,φ) be a strongly decomposable discrimination structure such that φ is dense and suppose Q is continuous with parameter ε. If there is a linear utility function u with respect to φ, and if $u(a^*) = 1$ and $u(b^*) = 0$, then for $a,b \in A$,*

$$P(a,b) = \begin{cases} \frac{1}{2}\left\{ K + [P(a^*,b^*)-P(b^*,a^*)]\,[u(a)-u(b)]^{\varepsilon} \right\}, & \text{if } a \gtrsim b, \\ \frac{1}{2}\left\{ K - [P(a^*,b^*)-P(b^*,a^*)]\,[u(b)-u(a)]^{\varepsilon} \right\}, & \text{if } a < b, \end{cases}$$

if P_A is additive, and

$$P(a,b) = \begin{cases} P(a^*,b^*)[u(a)-u(b)]^{\varepsilon}, & \text{if } a \gtrsim b, \\ 0, & \text{if } a < b, \end{cases}$$

if P_A is reflexive.

PROOF: By the density of φ and the linearity of u with respect to φ, it follows that P satisfies the Archimedean condition. For any $a \in A$, there are three cases:

(i) $a > a^* > b^*$, so $a^* \sim a\alpha b^*$ and $u(a) = 1/\varphi(\alpha)$;
(ii) $a^* \gtrsim a \gtrsim b^*$, so $a \sim a^*\alpha b^*$ and $u(a) = \varphi(\alpha)$; and
(iii) $a^* > b^* > a$, so $b^* \sim a^*\alpha a$ and $u(a) = -\varphi(\alpha)/[1-\varphi(\alpha)]$.

For any $a,b \in A$ there are six possible cases of location relative to a^* and b^*. We shall prove the theorem only when P_A is additive and for only one of these cases; the other cases are similar. Suppose $a > a^* \gtrsim b \gtrsim b^*$. By the Archimedean condition or R3, there exist $\alpha,\beta \in E$ such that $b \sim a\beta b^*$ and $a^* \sim a\alpha b^*$. By the decomposability assumption,

$$P(a,b) = P(aeb^*,a\beta b^*) = P(a,b^*)Q(e,\beta) + P(b^*,a)Q(\beta,e).$$

By Theorem 8 and the additivity of P_A, this can be rewritten as

$$P(a,b) = \tfrac{1}{2}\{K + [P(a,b^*)-P(b^*,a)][1-\varphi(\beta)]^{\varepsilon}\}.$$

But, $$\begin{aligned} P(a^*,b^*) - P(b^*,a^*) &= P(a\alpha b^*,aob^*)-P(aob^*,a\alpha b^*) \\ = [P(a,b^*)-P(b^*,a)]\,[Q(\alpha,o)-Q(o,\alpha)] &= [P(a,b^*)-P(b^*,a)]\varphi(\alpha)^{\varepsilon}. \end{aligned}$$

Thus, $$P(a,b) = \tfrac{1}{2}\left\{ K + \left[P(a^*,b^*)-P(b^*,a^*) \right] \left[\frac{1}{\varphi(\alpha)} - \frac{\varphi(\beta)}{\varphi(\alpha)} \right]^{\varepsilon} \right\}.$$

But $u(a) = 1/\varphi(\alpha)$ and $u(b) = u(a\beta b^*) = \varphi(\beta)u(a) = \varphi(\beta)/\varphi(\alpha)$, and the result follows.

COROLLARY: *If the conditions of the theorem are met, then*

$$u(a) = \begin{cases} \left[\dfrac{P(a,b^*) - P(b^*,a)}{P(a^*,b^*) - P(b^*,a^*)}\right]^{1/\varepsilon}, & \text{if } a > b^*. \\[2ex] 1 - \left[\dfrac{P(a^*,a) - P(a,a^*)}{P(a^*,b^*) - P(b^*,a^*)}\right]^{1/\varepsilon}, & \text{if } a < b^*. \end{cases}$$

PROOF: Trivial.

The similarity in form between P_A and Q should be noted. In fact, if $P(a^*,b^*) = 1$ and, in the additive case, $K = 1$, then the dependence of P upon u is identical to that of Q upon φ.

By making some added assumptions, including the finite additivity of subjective probability, one can give sufficient conditions for the existence of a linear utility function. This I shall not include for it is little different from the traditional constructions; the main result is not that, but rather the relationship between a linear utility function and the discrimination structure on A.

11. PROPOSED EMPIRICAL PROCEDURES

Beyond question, the most difficult problem in confronting this model with data is in obtaining the data. Some of the probabilities $P(a,b)$ must be estimated, and no one is certain how this should be done. Since we are dealing with prefeıences, an arbitrarily selected pair of people cannot be expected to have the same probabilities; and since no method is known for selecting in advance people having the same probabilities, a population study does not seem feasible. To study a single individual, two methods are a priori possible: To identify a number of pairs of alternatives that have the same probability and to use selections among these to estimate the probability. But again no one has suggested how such pairs can be identified without knowing the very probabilities one is trying to estimate. Alternatively, one can ask the subject to make the same preference decision time after time. Even when these instances are well spaced and intermixed with a lot of other decisions, there is considerable a prioıi doubt that suitable estimates can be obtained. One senses that, although a may be chosen over b with probab-ility $P(a,b)$ the first time the choice is made, once having made it the subject remembers it more often than not and his desire for consistency makes him select it again. I do not want to do more than mention these problems here, for they are largely matters of suitable experimental design. In what follows I shall suppose that good forced choice data exist involving a total of N decisions for each (a,b) pair.

Given such estimates, it is a straightforward matter to check the linearity condition (Definition 4).

It is somewhat more complicated to check the decomposability assumption (Definition 5) and to estimate Q. The following discussion of this problem is due entirely to Professor Robert R. Bush.[7] First, observe that in the additive case (with $K = 1$) the expression, given after Definition 5, for Q simplifies to

$$Q(\alpha,\beta) = \frac{P(a,b) + P(\alpha ab,\alpha\beta b) - 1}{2P(a,b) - 1},$$

for all a,b such that $P(a,b) \neq \frac{1}{2}$. If we choose our symbols so that $P(a,b) > 0$, there are two quite distinct cases depending upon whether $P(a,b) = 1$ (as it would for money) or not.

Case 1. $P(a,b) = 1$. Thus, $Q(\alpha,\beta) = P(\alpha ab,\alpha\beta b)$. Suppose that there are k different (a,b) pairs and that of the N observations on the ith pair X_i are preferences for αab over $\alpha\beta b$. Then Bush proposes that Q be estimated by

$$\hat{Q} = \frac{1}{Nk} \sum_{i=1}^{k} X_i.$$

To test the null hypothesis that $Q_i = \hat{Q}$, $i = 1,2, \ldots, k$, we have for each i,

$$\chi_i^2 = \frac{(X_i - NQ_i)^2}{NQ_i(1 - Q_i)} = \frac{(X_i - N\hat{Q})^2}{N\hat{Q}(1 - \hat{Q})}.$$

Summing,

$$\chi^2 = \sum_{i=1}^{k} \chi_i^2 = \frac{\sum_{i=1}^{k} (X_i - N\hat{Q})^2}{N\hat{Q}(1 - \hat{Q})}.$$

This is not exactly the test one desires since it fails to take into account that $\hat{Q}$ is an estimate of the true Q; however, the exact test appears to be very complicated.

Case 2. $0 < P(a,b) < 1$. For the ith (a,b) pair, use the observed data to obtain estimates $\hat{u}_i$ of the numerator and v_i of the denominator of the expression for Q. A least-squares estimate based upon $\sum_{i=1}^{k} (\hat{u}_i - Q\hat{v}_i)^2$ is easily seen to yield

$$\hat{Q} = \frac{\sum_{i=1}^{k} \hat{u}_i \hat{v}_i}{\sum_{i=1}^{k} \hat{v}_i^2}.$$

[7] Personal communication.

It seems preferable to use this estimate rather than the one based upon the more obvious $\sum_{i=1}^{k} \left(Q - \frac{\hat{u}_i}{\hat{v}_i}\right)^2$ because the variance is smaller; in particular, the expression is defined when $\hat{v}_i = 0$, as can happen with data.

The same estimate can be arrived at as follows: Since the X_i's are binomially distributed, they are approximately normally distributed for reasonably large sample sizes; hence, so are u_i and v_i. Thus, the maximum-likelihood estimate of Q can be computed, and it is the same as the above least-squares estimate. Independence of a and b can be tested by the likelihood ratio test.

Should either the linearity or decomposability assumption be rejected, then none of the rest of the theory is applicable and we are blocked. If, however, they are acceptable, then we can try to verify the derived forms of the discrimination functions. One way this might be done is as follows: By linearity, the (finite) set of experimental alternatives can be ordered; let a^* be the most preferred and b^* the least. Combining Theorem 11 and its corollary gives an equation involving only the P's and ε, which I shall write for the case where $P(a^*,b^*) = 1$, $K = 1$, and $a \gtrsim b$:

$$[2P(a,b) - 1]^{1/\varepsilon} = [P(a,b^*) - P(b^*,a)]^{1/\varepsilon} + [P(b,b^*) - P(b^*,b)]^{1/\varepsilon}.$$

The P's are all known and it is a matter of choosing ε to fit these equations. In all likelihood, the easiest way to choose ε is numerically so that it approximately minimizes the sum of the squared errors over all (a,b) pairs.

That being done, Theorem 8 and its corollary leads to the prediction

$$[2Q(\alpha,\beta) - 1]^{1/\varepsilon} + [2Q(\beta,\gamma) - 1]^{1/\varepsilon} = [2Q(\alpha,\gamma) - 1]^{1/\varepsilon},$$

which can be used as an independent check. If the fit is deemed acceptable, then φ and u can be calculated by the corollaries of Theorems 8 and 10, or by other similar equations.

I am currently running such an experiment, and it will be analyzed along these lines; it will be reported elsewhere.

12. REPRESENTATIONS OF LINEAR DISCRIMINATION STRUCTURES BY SEMIORDERS

This, and the final section, are peripheral to my main topic; they are included largely to establish bridges to several other papers.

Elsewhere [**8**] I have presented a purely algebraic model for utility discrimination, and it seems appropriate to indicate the connection between it and discrimination structures. The standard psychophysical device for simplifying a discrimination structure is to choose a probability cut off, usually 0.75, and to say that stimuli which are not discriminated more than

75 per cent of the time are one "just noticeable difference" (jnd) or less apart. While this is usually cast in the numerical framework of the psychophysical model, there is no need to restrict it in that way. So, we suppose that P is a discrimination structure on a set S and that k is a number, $\frac{1}{2} \leqslant k < 1$, and we define the pair of binary relations (L_k,I_k) on S as

$$aL_kb \text{ if } P(a,b) > k,$$
$$aI_kb \text{ if } P(a,b) \leqslant k \text{ and } P(b,a) \leqslant k.$$

It will be recalled that in [**8**] the following concept was introduced:

DEFINITION 13: A pair of binary relations (L,I) on a set S is called a *semiorder* if for every $a,b,c,d \in S$ the following axioms are met:

S1. exactly one of aLb, bLa, or aIb obtains,
S2. aIa,
S3. aLb, bIc, cLd imply aLd,
S4. aLb, bLc, bId imply not both aId and dIc.

THEOREM 12: *The relations (L_k, I_k) of a linear discrimination structure form a semiorder.*

PROOF: S1. By part (ii) of Definition 1 and $k \geqslant 1/2$.
S2. By part (ii) of Definition 1, $P(a,a) \leqslant 1/2 \leqslant k$, and so aI_ka.
S3. Suppose aL_kb, bI_kc, cL_kd. We show $a \gtrsim c$. Suppose $c > a$.

Then by Definition 2, $P(c,b) \geqslant P(a,b)$; but by the definition of I_k, $k \geqslant P(c,b)$ and $P(a,b) > k$, which is a contradiction. Thus, $a \gtrsim c$. Then by Definition 2, $P(a,d) \geqslant P(c,d) > k$, so aL_kd.

S4. Suppose aL_kb, bL_kc, bI_kd. Since $P(a,b) > k \geqslant P(b,b)$, linearity implies $a > b$. In like manner, $b > c$. As above, one shows that either $a \gtrsim d \gtrsim b$ or $b \gtrsim d \gtrsim c$. If the former, $P(d,c) \geqslant P(b,c) > k$, and it is not so that dI_kc. If the latter, then it is not so that aI_kd.

We observe that if a linear discrimination structure is reflexive (and so transitive), then the range of k may be extended to $0 < k < 1$.

Given a pair of relations (L,I), another pair of relations $(>',\sim')$ can be defined as follows:

$a >'b$ if either (i) aLb, (ii) aIb and there exists c such that aIc and cLb, or (iii) aIb and there exists d such that aLd and dIb.

$a \sim'b$ if neither $a >'b$ nor $b >'a$.

In [**8**] it was shown that if (L,I) is a semiorder, then $(>', \sim')$ is a weak order, which we speak of as the weak order induced by the semiorder.

THEOREM 13: *Suppose P is a linear structure on S, k is a number, $1/2 \leqslant k < 1$, $\gtrsim$ is the trace of P, and $\gtrsim_k$ is the weak order induced by the semiorder (L_k,I_k). Then $>_k$ implies $>$.*

PROOF: If $a >_k b$, then either

(i) $aL_k b$, so $P(a,b) > k \geqslant P(a,a)$, and so $a > b$; or
(ii) $aI_k b$, $aI_k c$, and $cL_k b$, so $P(c,b) > k \geqslant P(c,a)$, and so $a > b$; or
(iii) $aI_k b$, $aL_k d$, and $dI_k b$, so $P(a,d) > k \geqslant P(b,d)$, and so $a > b$.

Some of the preceding results have their analogue for semiorders. For example, let (L,I) be a semiorder on a risk space; then it is said to be symmetric if $(a\alpha b)I(a\beta b)$ implies $(a\bar{\beta}b)I(a\bar{\alpha}b)$. It is easy to show that the semiorder induced by a symmetric linear discrimination structure on a risk space is a symmetric semiorder. Recall that in proving Theorem 10 we really only used the symmetry of the discrimination structure to show that a linear utility function is a sensation scale, and so one might expect some sort of an analogue for semiorders. Using methods similar to those in [**8**] one can show that the jnd functions of a linear utility function are constant provided that the linear utility function is unbounded. When there are bounds, the jnd functions must drop to zero near them, and so it is difficult to phrase the result cleanly. For such reasons, it is generally better to work directly with a discrimination structure rather than with one of its induced semiorders.

13. INTERPERSONAL COMPARISONS OF UTILITY

A well known conceptual problem which plagues many of the attempted applications of the utility notion is this: when can or should a utility increment for one person be considered equal to an increment for another? This is known as the problem of interpersonal comparisons of utility. Since we will confine our attention to linear utility functions (which are unique up to a linear transformation), the problem amounts to selecting a general unit in terms of which to measure utility. Since utility differences, not absolute utility, are important, the zero of the scale is of no matter.

In a context of the social welfare function problem and under certain very special assumptions, Goodman and Markowitz [**6**] have suggested that a plausible measure of the intensity that one alternative is preferred to another is the number of just discriminably different alternatives that can be introduced between them. This, they suggest, can sometimes be taken as a solution to the interpersonal comparison problem, since we can equate just noticeable differences – jnd's – between two people and then say an interval spanning more jnd's is larger than one spanning fewer. Surrounding this proposal are all sorts of subtle problems: Is it a normative or descriptive solution? If the latter, does it accord with empirical observations on the comparisons that people do make? From what population of alternatives does one choose those used to determine the number of jnd's in an interval? In particular, are gambles among alternatives admissible? Etc.? I do not propose to enter into a discussion here of these thorny issues, but I do want to point out that utility theory has to exhibit certain very special features before the idea makes any sense at all.

First of all, to treat the jnd as a unit in any way, one must be assured that, for a particular individual, jnd's are equal throughout his utility scale. This means, in effect, that one must show that the utility function under consideration is a sensation scale. We already know that this is so for the linear utility functions of certain decomposable linear structures (Theorem 10).

Second, it is well known that the jnd is a statistical notion in the sense that one chooses a probability cut-off k, $1/2 \leqslant k < 1$, and says that a and b are within one jnd if $P(a,b) \leqslant k$ and $P(b,a) \leqslant k$. Clearly, k is merely a technical artifact of our calculations that has no inherent meaning in the theory, and therefore any comparisons that we make involving jnd's should be quite independent of its choice. The purpose of this section, then, is to determine for the present model the conditions under which such comparisons would be independent of this artifact. As we will see, they are so special that one suspects that if the present theory is at all near the truth, then equating jnd's is not a possible solution to the interpersonal comparison problem.

Suppose that an arbitrary probability cut-off k, $1/2 < k < 1$, is selected and that (P,Q,A,E,φ) is a strongly decomposable discrimination structure with a dense φ, a continuous Q, and a linear utility function u. Let (L_k,I_k) be the induced semiorder (see Section 12). We wish to find an expression for the jnd functions of u corresponding to the induced semiorder. These are constant except at the bounds of the utility function.

Let $a^*,b^* \in A$ be the elements described in Definition 1, and suppose $a^* > b^*$. We may take the linear transformation of the utility function so that $u(a^*) = 1$ and $u(b^*) = 0$. By the remarks following Theorem 4 of [8] we know that the jnd function δ can be expressed as

$$\delta = \bar{a}(a^*,b^*) = \sup[\varphi(a) \mid (a^*b^*)I_k b^*].$$

Since P is symmetric (Theorem 2), $(a^*ab^*)I_k b^*$ if and only if

$$k \geqslant P(a^*ab^*,b^*) = P(a^*,b^*)Q'[\varphi(a)] + P(b^*,a^*)Q'[-\varphi(a)].$$

In the additive case, this reduces to

$$k \geqslant \tfrac{1}{2}\left\{ K + [Pa^*,b^*) - P(b^*,a^*)]\varphi(a)^{\varepsilon} \right\},$$

and in the reflexive case to

$$k \geqslant P(a^*,b^*)\varphi(a)^{\varepsilon}.$$

Solving,

$$\delta = \left[\frac{2k - K}{P(a^*,b^*) - P(b^*,a^*)}\right]^{1/\varepsilon},$$

if P_A is additive, and

$$\delta = \left[\frac{k}{P(a^*,b^*)}\right]^{1/\varepsilon},$$

if P_A is reflexive.

Now, suppose two different people express preferences among alternatives of the same risk space under the same conditions, and that both people yield strongly decomposable discrimination structures having linear utility functions, dense φ's, and continuous cores. Distinguish them by subscripts 1 and 2. Since the same experimental conditions obtain, we may suppose that both structures are additive or both are reflexive. For each person, the same probability cut-off k is used, but a^* and b^* may differ. Although, a priori, the values of K may also differ, in practice forced answers probably would be used in which case $K = 1$. This we will assume. Then the ratio of the jnd's is given by

$$\frac{\delta_1}{\delta_2} = \frac{[P(a^*_2,b^*_2) - P(b^*_2,a^*_2)]^{1/\varepsilon_2}}{[P(a^*_1,b^*_1) - P(b^*_1,a^*_1)]^{1/\varepsilon_1}} (2k-1)^{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_2}},$$

if P_A is additive, and

$$\frac{\delta_1}{\delta_2} = \frac{[P(a^*_2,b^*_2)]^{1/\varepsilon_2}}{[P(a^*_1,b^*_1)]^{1/\varepsilon_1}} (k)^{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_2}},$$

if P_A is reflexive.

These are independent of k if and only if $\varepsilon_1 = \varepsilon_2$. From our earlier discussion it seems doubtful that this would be the case, in which case we would have to reject the equating of jnd's as a suitable resolution of the interpersonal comparison problem. On the other hand, it is clear how to save the idea. Different but related cut-offs, k_1 and k_2, could be used so as to make the ratio independent of the cut-offs, namely,

$$(2k_1-1)^{\varepsilon_2} = c(2k_2-1)^{\varepsilon_1}$$

in the additive case and

$$k_1^{\varepsilon_2} = ck_2^{\varepsilon_1},$$

where c is a positive constant (probably $c = 1$ is the best choice).

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REFERENCES

[1] COOMBS, C. H.: "A Theory of Psychological Scaling," Bulletin No. 34, Engineering Research Institute, University of Michigan, 1952.

[2] ———, "Inconsistency of Preferences as a Measure of Psychological Distance," University of Michigan, 1956 (mimeographed).

[3] DAVIDSON, DONALD AND JACOB MARSCHAK: "Experimental Tests of Stochastic Decision Theory," Cowles Foundation Discussion Paper No. 22, Yale University, 1957 (mimeographed).

[4] EDWARDS, WARD: "The Theory of Decision Making," *Psychological Bulletin*, 51 (1954), 380-417.

[5] GEORGESCU-ROEGEN, N.: "The Pure Theory of Consumer's Behavior," *Quarterly Journal of Economics*, 50 (1936), 545-593.

[6] GOODMAN, L. A. AND HARRY MARKOWITZ: "Social Welfare Functions Based on Individual Rankings," *The American Journal of Sociology*, 58 (1952), 257-262.

[7] HAUSNER, MELVIN: "Multidimensional Utilities," *Decision Processes* (R. M. Thrall, C. H. Coombs, and R. L. Davis, eds.), John Wiley and Sons, New York, 1954, 167-180.

[8] LUCE, R. D.: "Semiorders and a Theory of Utility Discrimination," *Econometrica*, 24 (1956), 178-191.

[9] LUCE, R. D., AND WARD EDWARDS: "Derivation of Subjective Scales from Just Noticeable Differences," *Psychological Review*, in press.

[10] MARSCHAK, JACOB: "Rational Behavior, Uncertain Prospects, and Measurable Utility," *Econometrica*, 18 (1950), 111-141.

[11] ———, "Norms and Habits of Decision Making under Certainty," *Mathematical Models of Human Behavior*, Dunlap and Associates, Stamford, Conn., 1955, 45-54.

[12] MILLER, G. A.: ,,"The Magical Number Seven, Plus or Minus Two: Some Limits on our Capacity for Processing Information," *Psychological Review*, 63 (1956), 81-97.

[13] PAPANDREOU, A. G.: "An Experimental Test of an Axiom in the Theory of Choice," *Econometrica*, 21 (1953), 477 (abstract).

[14] ———, O. H. SAUERLENDER, O. H. BROWNLEE, LEONID HURWICZ, AND WILLIAM FRANKLIN: "A Test of a Proposition in the Theory of Choice," University of Minnesota, 1954 (mimeographed).

[15] RAMSEY, F. P.: *The Foundations of Mathematics*, Harcourt, Brace and Co., New York, 1931.

[16] SAVAGE, L. J.: *The Foundations of Statistics*, John Wiley and Sons, New York, 1954.

[17] STONE, M. H.: "Postulates for the Barycentric Calculus," *Annali di Mathematica Pura ed Applicata*, 29 (1949), 25-30.

[18] VON NEUMANN, JOHN, AND OSKAR MORGENSTERN: *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1947.