Econornetrica, Vol. 24, No. 2. (Apr., 1956), pp. 178-191.

SEMIORDERS AND A THEORY OF UTILITY DISCRIMINATION

BY R. DUNCAN LUCE[1]

In the theory of preferences underlying utility theory it is generally assumed that the indifference relation is transitive, and this leads to equivalence classes of indifferent elements or, equally, to indifference curves. It has been pointed out that this assumption is contrary to experience and that utility is not perfectly discriminable, as such a theory necessitates. In this paper intransitive indifference relations are admitted and a class of them are axiomatized. This class is shown to be substantially equivalent to a utility theory in which there are just noticeable difference functions which state for any value of utility the change in utility so that the change is just noticeable. In the case of risk represented by a linear utility function over a mixture space, the precise form of the function is examined in detail.

1. INTRODUCTION

IT IS A commonplace that the modern theories of utility [**7, 12**], and therefore disciplines such as game theory [**5, 12**], statistical decision theory [**5**], and subjective probability theory [**11**] which employ utility theory, assume the existence of binary preference relations which are weak orderings of sets of alternative outcomes. There are various possible axiom systems to characterize a weak order; the one we shall present is not the most elegant, but it is well suited to our purpose. Let $>$ and $\backsim$ be two binary relations on a set S, then we say $(>, \backsim)$ is a *weak ordering* of S if

W1. for every $a, b \in S$, exactly one of $a > b$, $b > a$, or $a \backsim b$ obtains,
W2. $\backsim$ is an equivalence relation,
W3. $>$ is transitive.

From these it follows directly that $>$ is irreflexive and anti-symmetric. In utility theory, $a > b$ is interpreted to mean "a is preferred to b," whereas $a \backsim b$ is interpreted as "indifference between a and b."

There has been some concern over the adequacy of these axioms to represent a person's preference pattern; see, for example, [**10**]. Possibly the knottiest problem is the question of the invariance of a person's preferences over time, and in particular whether they remain invariant during any experiment which purports to discover the preference pattern. A second problem, sometimes considered to be simply an observable correlate of the first one, exists in the assumption that $>$ is transitive. Experiments have been performed, and personal experience is easily adduced, in which $>$ is not transitive. The axiom is not, however, easily sacrificed. It is necessary if one is to have numerical order preserving utility function, and such functions seem indispensable for theories—such as game theory—which rest on preference orderings. Furthermore, there is the important

[1] This work was undertaken and completed when the author was a 1954–55 Fellow at the Center for Advanced Study in the Behavioral Sciences, Stanford, California.

subjective contention that a "rational" preference ordering *should* satisfy the transitivity condition. For example, Savage [**11,** p. 21] writes ". . . when it is explicitly brought to my attention that I have shown a preference for f as compared with g, for g as compared with h, and for h as compared with f, I feel uncomfortable in much the same way that I do when it is brought to my attention that some of my beliefs are logically contradictory." Our quarrel with the axioms of utility theory does not lie here, and we too shall take the attitude that, at least for a normative theory, the preference relation should be transitive.

What appears to have received less attention in these debates is the adequacy of the other major axiom, namely, the condition (W2) that $\backsim$ should be an equivalence relation. Specifically, we feel that there is little defense for supposing that $\backsim$ is transitive. The author who has most repeatedly questioned this assumption is Armstrong [**1, 2, 3, 4**]. For example, on page 122 of [**3**] he writes: "The nontransitiveness of indifference must be recognized and explained on [sic] any theory of choice, and the only explanation that seems to work is based on the imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude."

First, let us consider empirical evidence. It is certainly well known from psychophysics that if "preference" is taken to mean which of two weights a person believes to be heavier after hefting them, and if "adjacent" weights are properly chosen, say a gram difference in a total weight of many grams, then a subject will be indifferent between any two "adjacent" weights. If indifference were transitive, then he would be unable to detect any weight differences, however great, which is patently false. If this example is too far afield from ordinary preferences, consider the following experiment. Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $\left(1 + \frac{i}{100}\right) x$ grams of sugar, $i = 0, 1, \cdots, 400$, where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i + 1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$.

These two examples suggest an important point about the intransitivity of some indifference relations, namely, that it reflects the inability of an instrument to discriminate relatively to an imposed discrimination task. We tacitly, but correctly, assumed that non-human instruments exist which can discriminate between certain weights and certain concentrations when a human being cannot. It might, therefore, be thought that we can always eliminate intransitivities in the indifference relation by providing the subject with more refined measuring tools; however, with respect to certain measurements there are inherent difficulties if present day physics is correct. For example, the Heisenberg uncertainty principle states that there exists an inherent uncertainty in the simultaneous measurement of the position and velocity of a particle, but clearly these uncertainties are not transitive. Thus there seems to be a basic philosophic objection to the assumption that $\backsim$ is transitive. Furthermore, in many preference situations it does not seem justified to assume the existence of a more basic underlying scale which can be detected by more refined instrumentation, and at the same time it is difficult to argue that a person should feel uncomfortable and dissatisfied with his preferences simply because he has reported indifferences which are not transitive. We propose, therefore, to drop this assumption and to replace it by what we feel are intuitively more reasonable conditions.

Before turning to this, let us mention one more controversial point in utility theory which has caused some unrest. In applications where a real-valued order-preserving utility function is postulated, a maximization principle is almost always employed which states in effect that a rational being will respond to any finite difference in utility, however small. It is, of course, false that people behave in this manner, but it is argued that this is one of the prices of a simple idealized theory. Nevertheless, some workers have continued to be uncomfortable, for it is clear that the idealized theory cannot exhibit certain possibly important phenomena. Possibly this is not a concern if all the applications are actually treated as normative, though even that is doubtful.

It is not implausible that the phenomenon of imperfect response sensitivity to small changes in utility is closely related to intransitivities of the indifference relation. We propose to examine the interrelation in detail and to show that a comparatively simple theory of utility can be constructed in which a person is not sensitive to all changes in utility. The theory we shall obtain yields a non-statistical analogue of the "just noticeable difference" concept of psychophysics. It is in a sense a much more general notion for it applies to cases where no underlying continuum is assumed to be known, as is always the case in psychophysics, but it is less general in that no statistical assumptions are made concerning the variability of the subject.

In the next section we shall introduce a new set of preference axioms which allow for intransitive indifference relations.[2] We shall use the term semiorder (under the belief that this word has not been used for another concept) for relations satisfying these axioms. The central result of Section 3 will show that in terms of any semiorder it is possible to define a natural weak ordering of the same set. In case the semiorder is a weak order then the induced weak order is identical to the given one. In the following section we assume that the induced weak ordering defines a real-valued order-preserving function. It is shown that

[2] Some months after this paper was submitted for publication, an article by Halphen [4] appeared, and it was brought to my attention by Professor L. J. Savage. In discussing the foundations of probability, Halphen raises much the same arguments as we have against the assumption of transitivity of indifference, or as he calls it "equivalence." He arrives at an axiom system generalizing a partial order which, if the comparability of all pairs of elements were imposed, is extremely similar to but somewhat stronger than ours. In place of our Axiom 3 he has a relatively minor variant, and in place of Axiom 4 he assumes that either part (ii) or part (iii) of Definition 1 (Section 3) holds. We would argue that our Axiom 4 is to be preferred since it does not postulate the existence of an element having such and such properties. He observes that a partial order can be defined in terms of the given order in much the same way as we do in Section 3. Since his interest is in probability theory, he does not pursue our direction of the representation of such orders by numerical functions; rather, he concerns himself with a notion of measure.

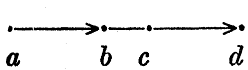
it is possible to define two real-valued nonnegative functions on the same set, which we have called upper and lower j.n.d. (just noticeable difference) functions, which characterize the change in utility necessary for indifference to become preference. In Theorem 2 we establish necessary conditions interrelating these three numerical functions. The third theorem shows these conditions are also sufficient in the following sense: If one is given three functions, defined over the same set and which satisfy the conditions of Theorem 2, then (i) the three functions define a natural semiordering of the set, (ii) the weak order induced by the semiorder is identical to the weak order induced by the utility function, and (iii) the two j.n.d. functions of the induced semiorder are two of the given functions (and the other is, by choice, the utility function of the weak order). In the last section we show that if a semiordered mixture space possesses a linear utility function u satisfying certain weak conditions, then u is a linear transformation of a function ρ which is defined in terms of experimentally realizable operations. The evaluation of ρ at *any* n points requires only n observations. The function ρ is generally defined for semiordered mixture spaces, but it is not necessarily linear or order preserving, so these conditions must be verified to know whether a semiordered mixture space has a linear utility function or not. A complete verification, as for the von Neumann and Morgenstern axioms for weakly ordered mixture spaces, requires an infinity of observations.

2. SEMIORDERS

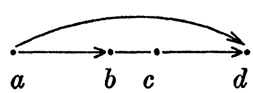
Let S be a set and P and I be two binary relations defined over S. (P, I) is a *semiordering* of S if for every a, b, c, and d in S the following axioms hold:

S1. exactly one of aPb, bPa, or aIb obtains,
S2. aIa.
S3. aPb, bIc, cPd imply aPd,
S4. aPb, bPc, bId imply not both aId and cId.

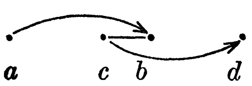
The intuitive grounds for the first two axioms are clear. The last two axioms arise from intuitive considerations such as these: we should like to be able to string out all the elements of S on a line in such a fashion that an indifference interval never spans a preference interval. It is easy to see that we shall need S3 and S4 if we represent them graphically, where an arrow from a to b denotes aPb and a line between them denotes aIb. Axiom S3 is equivalent to:



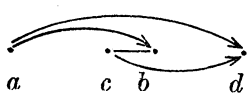
implies



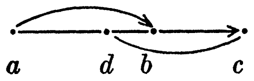
and

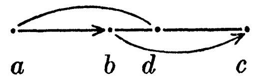


implies

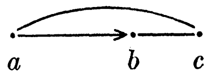


Axiom S4 is equivalent to asserting that neither of the following configurations can ever arise:





It is easily seen that if either of these axioms is violated there will be situations in which an indifference interval includes a preference interval, i.e., a drawing of the form



Thus, S3 and S4 are argued to be necessary axioms; part of our results will show that they, along with S1 and S2, are sufficient to prevent the last diagram from arising.

It follows directly from these axioms that I is symmetric, that P is transitive, and that aIb and bPc imply not cPa. It is also trivial to show that a weak order is a semiorder, but not conversely. It is not difficult to establish that the axioms are independent; the only two mildly interesting cases are S3 and S4, and Example 1 below presents a pair of relations in which all save S3 are met, and in Example 2 all save S4.

	a	b	c	d
a	I	P	P	I
b		I	I	P
c			I	P
d				I

Example 1

	a	b	c	d	e	f
a	I	I	P	P	P	I
b		I	I	P	P	I
c			I	I	P	I
d				I	I	I
e					I	I
f						I

Example 2

3. THE WEAK ORDER INDUCED BY A SEMIORDER

If P is an arbitrary relation defined over S, an indifference relation I can always be defined as follows: for $a, b \in S$, aIb if and only if neither aPb nor bPa. Clearly I is symmetric. In what follows we shall always suppose I is defined by exclusion.

Definition 1: The *relation* $(>, \backsim)$ *induced* on S by a given relation (P, I) on S is defined as follows: $a > b$ if either

(i) aPb,

(ii) aIb and there exists $c \in S$ such that aIc and cPb, or

(iii) aIb and there exists $d \in S$ such that aPd and dIb.

If neither $a > b$ nor $b > a$, then $a \backsim b$.

Observe that if (P, I) is a weak ordering of S, then neither condition (ii) nor (iii) can arise because of the transitivity of the indifference relation, so the induced relation is the same as the given one, and in particular it is a weak order.

Our first theorem shows, in part, that this is always the case if (P, I) is a semiorder.

THEOREM 1: *(P, I) is a semiorder if and only if P is transitive and $(>, \backsim)$ is a weak order.*

PROOF: First, let us suppose that P is transitive and $(>, \backsim)$ is a weak order; then we show (P, I) meets the four axioms of a semiorder.

S1. Since I is defined by exclusion, this axiom can be violated only if there exist $a, b \in S$ such that aPb and bPa. In this case D1 implies $a > b$ and $b > a$, which contradicts W1.[3]

S2. aIa, for if aPa, then (D1) $a > a$, which contradicts W2.

S3. Suppose aPb, bIc, and cPd. If dPa, then by the transitivity of P, dPa and aPb imply dPb. But cPd and dPb imply cPb, which is impossible since cIb. Next, suppose aId. If aIc, then cPd, dIa, and cIa imply (D1) $c > a$. But aPb, bIc, and aIc imply (D1) $a > c$, which is impossible. If cPa, then by the transitivity of P, cPa and aPb imply cPb, which by S1 contradicts cIb. If aPc, then by the transitivity of P, aPc and cPd imply aPd, which contradicts the assumption aId. Thus, we must conclude aPd.

S4. Suppose aPb, bPc, bId, aId, and dIc. By D1, bPc, cId, and bId imply $b > d$, but dIb, dIa, and aPb imply $d > b$, which is a contradiction by W1.

(In the above portion of the proof the assumption that P is transitive is necessary as can be seen by the following example:

	a	b	c	d
a	I	I	P	P
b		I	P	I
c			I	P
d				I

The relation P is not transitive since bPc, cPd, and bId, and so it is not a semiorder, but the relation $(>, \backsim)$ induced by D1 is easily seen to be a weak ordering of $\{a, b, c, d\}$.)

Conversely, if (P, I) is a semiorder then we know that P is transitive and we proceed to check that $(>, \backsim)$ is a weak ordering of S.

W1. It is sufficient to show that $a > b$ implies not $b > a$. Suppose, on the contrary, $a > b$ and $b > a$; then, by D1 and S1, aIb. We may distinguish four cases: There exist elements c and d such that

(i) aIc, cPb, bId, dPa. dPa, aIc, cPb imply (S3) dPb, which by S1 contradicts dIb.

(ii) aIc, cPb, bPd, dIa. cPb, bPd, aIb imply (S4) not both cIa and aId, which is contrary (S1) to assumption.

(iii) aPc, cIb, bId, dPa. dPa, aPc, aIb imply (S4) not both dIb and bIc, which is contrary (S1) to assumption.

(iv) aPc, cIb, bPd, dIa. aPc, cIb, bPd imply (S3) aPd, which by S1 contradicts aId.

[3] D1 refers to Definition 1, W1 to Axiom W1 of Section 1, etc.

W2. By D1, $\backsim$ is symmetric. It is reflexive for if $a > a$, then by D1 either aPa, which is impossible (S2), or there exists $c \in S$ such that aIa, aIc, cPa, which is impossible (S1). It is therefore sufficient to show $\backsim$ is transitive. Suppose, with no loss of generality, $a \backsim b$, $b \backsim c$, $a > c$. By D1, $a \backsim b$ and $b \backsim c$ imply aIb and bIc, whereas $a > c$ implies either aPc or aIc. If aPc, then aPc, bIc, aIb imply (D1) $a > b$, which contradicts (W1) $a \backsim b$. Thus, aIc and, by D1, either there exists $d \in S$ such that aId and dPc or there exists $e \in S$ such that aPe and eIc. Suppose the former. If dIb, then dIb, dPc, bIc imply (D1) $b > c$, which contradicts (W1) $b \backsim c$. If dPb, then dPb, dIa, aIb imply (D1) $a > b$, which contradicts (W1) $a \backsim b$. If bPd, then bPd, dId, dPc imply (S3) bPc, which contradicts (S1) bIc. A similar argument applies if e exists, so $\backsim$ is transitive.

W3. $>$ is transitive: Suppose $a > b$ and $b > c$. By D1, four cases can arise. If aPb and bPc, then (S3) aPc and so (D1), $a > c$. If aPb and bIc, then either aPc or aIc. If aPc, then (D1) $a > c$, so we suppose aIc. By D1, aIc, bIc, aPb imply $a > c$. If aIb and bPc, a similar argument holds. If aIb and bIc, then by D1 we need only consider the cases cPa and cIa. By D1 we know that there exist two elements d and e in one of four possible arrangements:

(i) aId, dPb, bIe, ePc. If cPa, then cPa, aId, dPb imply (S3) cPb, which contradicts (S1) bIc. So aIc. Suppose ePa, then dPb, bIe, ePa imply (S3) dPa, which contradicts (S1) aId. Suppose aPe, then aPe, ePc, bIe imply (S4) not both aIb and bIc, which is contrary (S1) to hypothesis. Thus (S1), aIe, but aIc, aIe, ePc imply (D1) $a > c$.

(ii) aId, dPb, bPe, eIc. If cPa, then cPa, aId, dPb imply (S3) cPb, which contradicts (S1) bIc. So aIc. Suppose ePa, then ePa, aIb, bPe imply (S3) ePe, which is impossible (S2). Suppose eIa, then eIa, bPe, aIb imply (D1) $b > a$, which contradicts (W1) $a > b$. Thus (S1), aPe, and aPe, eIc, aIc imply (D1) $a > c$.

(iii) aPd, dIb, bIe, ePc. If cPa, then ePc, cIc, cPa imply (S3) ePa, but ePa, aPd, aIb imply (S4) not both eIb and bId, which is contrary (S1) to assumption. So aIc. Suppose cPd, then cPd, bIc, bId imply (D1) $c > b$, which contradicts (W1) $b > c$. Suppose dPc, then aPd, dPc, dIb imply (S4) not both aIb and bIc, which is contrary (S1) to assumption. Thus (S1), cId, and cId, aPd, aIc imply (D1) $a > c$.

(iv) aPd, dIb, bPe, eIc. If cPa, then cPa, aIa, aPd imply (S3) cPd, but cPd, dIb, cIb imply (D1) $c > b$, which contradicts (W1) $b > c$. So aIc. aPd, dIb, bPe imply (S3) aPe, and aPe, eIc, aIc imply (D1) $a > c$.

4. UTILITY DISCRIMINATION

Let $(>, \backsim)$ be a weak ordering of a set S; then any real-valued function u defined over S will be called an *order preserving* function of $(S, >, \backsim)$ provided that $u(a) > u(b)$ if and only if $a > b$. Examples can be given to show that such a function does not always exist [**7**]; however, certain sufficient conditions for their existence are known [**6, 7, 12**].

DEFINITION 2: Let (P, I) be a semiordering of S and suppose u is an order preserving function of $(S, >, \backsim)$, where $(>, \backsim)$ is the weak ordering induced by (P, I) according to D1. We define the *upper j.n.d.* function as

$$\bar{\delta}(a) = \sup_{\substack{b \\ aIb}} [u(b) - u(a)]$$

and the *lower j.n.d.* function as

$$\underline{\delta}(a) = \sup_{\substack{b \\ aIb}} [u(a) - u(b)].$$

Since aIa, it is obvious that these functions are nonnegative.

DEFINITION 3: Let (P, I) be a semiordering of S and u a real-valued function defined over S; then we say u is a *utility function* of (S, P, I) if

U1. u is an order preserving function of $(S, >, \backsim)$, where $(>, \backsim)$ is the weak ordering of S induced by (P, I), and

U2. for any $a \in S$ there exist $b, c \in S$ such that $u(b) = u(a) + \bar{\delta}(a)$ and aIb, $u(c) = u(a) - \underline{\delta}(a)$ and aIc, where $\bar{\delta}$ and $\underline{\delta}$ are defined in D2.

THEOREM 2: *Let (P, I) be a semiordering of S such that there is a utility function u of (S, P, I); then*

(i) *aIb if and only if $u(b) - \underline{\delta}(b) \leqslant u(a) \leqslant u(b) + \bar{\delta}(b)$ and aPb if and only if $u(a) > u(b) + \bar{\delta}(b)$;*

(ii) *$u(a) \leqslant u(b) + \bar{\delta}(b)$ if and only if $u(a) \leqslant u(b) + \underline{\delta}(a)$;*

(iii) *if $u(a) < u(b)$ then either $u(a) + \bar{\delta}(a) < u(b) + \bar{\delta}(b)$ or $u(a) + \underline{\delta}(b) < u(b) + \underline{\delta}(a)$.*

PROOF: (i) If aIb then the conclusion is obvious from D2. Conversely, we consider three cases:

1. $u(a) = u(b)$. By U1, $a \backsim b$ and so (D1) aIb.

2. $u(a) > u(b)$. By U2 there exists $c \in S$ such that $u(c) = u(b) + \bar{\delta}(b)$ and cIb, so (using the hypothesis), $u(c) \geqslant u(a)$. Since $u(a) > u(b)$, $a > b$ by U1, and so from D1 we know the only interesting case is aPb. Now, suppose cPa, then cPa, aIa, aPb imply (S3) cPb, which contradicts (S1) cIb. So cIa. But aIc, cIb, aPb imply (D1) $a > c$, and by U1 this implies $u(a) > u(c)$, contrary to what we have shown.

3. $u(a) < u(b)$. The proof closely parallels case 2.

The second part of (i) follows immediately.

(ii) Suppose $u(a) \leqslant u(b) + \bar{\delta}(b)$. If $u(a) \leqslant u(b)$ then since $\underline{\delta}(a)$ is nonnegative the conclusion follows. If $u(a) > u(b)$, then by part (i), aIb. But $\underline{\delta}(a) = \sup_{\substack{c \\ cIa}} [u(a) - u(c)] \geqslant u(a) - u(b)$, and the conclusion follows. The converse is similar.

(iii) Suppose $u(a) < u(b)$; then by U1, $b > a$. If bPa, then by part (i), $u(a) + \bar{\delta}(a) < u(b) \leqslant u(b) + \bar{\delta}(b)$ and $u(b) - \underline{\delta}(b) > u(a) \geqslant u(a) - \underline{\delta}(a)$. If bIa, then by D1 either there exists $c \in S$ such that aIc and bPc or there exists $d \in S$ such that bId and dPa. Suppose the former obtains. Using part (i) twice, $u(c) <$

$u(b) - \underline{\delta}(b) \leqslant u(a)$. But aIc, so by part (i), $u(a) - \underline{\delta}(a) \leqslant u(c) < u(b) - \underline{\delta}(b)$, or, rewriting, $u(a) + \underline{\delta}(b) < u(b) + \underline{\delta}(a)$. If d exists, then the first statement of (iii) follows by a parallel argument.

COROLLARY: *If u is a utility function of (S, P, I), if $\bar{\delta}(a) = \underline{\delta}(a) = \delta(a)$ for every $a \in S$, and if δ is a monotonic function of u, then δ is a constant which is nonzero only if* $\sup u(a) = \infty$ *and* $\inf u(a) = -\infty$.

PROOF: Assume $\delta(a) \not\equiv 0$, and choose any $a \in S$ such that $\delta(a) > 0$. Let us suppose δ is monotone nondecreasing. Define b by $u(a) = u(b) + \delta(a)$. Since $u(a) > u(b)$, $\delta(a) \geqslant \delta(b)$ by the monotonicity assumption. But by case (ii) of the theorem, $u(a) \leqslant u(b) + \delta(b)$, so $\delta(a) \leqslant \delta(b)$, hence $\delta(a) = \delta(b)$. Because δ is monotonic, it is constant in the interval from $u(b)$ to $u(a)$. Beginning at b we may show in the same manner that δ is constant and equal to $\delta(a)$ from $u(b) - \delta(a)$ to $u(b)$. The proof proceeds inductively in steps of $\delta(a)$ for decreasing u. Clearly $\inf u(a) = -\infty$. For increasing u, choose $b, c \in S$ such that $u(a) + \delta(a) = u(b)$ and $u(c) = u(b) - \delta(b)$. Since δ is monotonic nondecreasing, $\delta(c) \leqslant \delta(b)$ and $u(c) \leqslant u(a)$. By part (ii) of the theorem, $u(c) + \delta(b) = u(b)$ implies $u(b) \leqslant u(c) + \delta(c)$, so $\delta(b) \leqslant \delta(c)$, i.e., $\delta(b) = \delta(c)$. But since $u(c) \leqslant u(a)$, the first part of the proof implies $\delta(b) = \delta(a)$. We may proceed inductively in steps of $\delta(a)$ to show δ is a constant for all values of u. Clearly, $\sup u(a) = \infty$.

The next theorem, a partial converse to Theorem 2, shows that under quite broad conditions no relations among u, $\bar{\delta}$, and $\underline{\delta}$ other than conditions (ii) and (iii) of Theorem 2 can be found.

THEOREM 3: *Let u be a real-valued function and let $\bar{\delta}$ and $\underline{\delta}$ be nonnegative real-valued functions defined over a set S. If these functions satisfy condition* (ii), *Theorem* 2, *then part* (i), *Theorem* 2, *defines a semiordering (P, I) of S. If, in addition, condition* (iii) *is met and if for every $a \in S$, $u(a) + \bar{\delta}(a)$, $u(a) - \underline{\delta}(a) \in u(S)$, then*

(i) *the weak order induced by (P, I) according to* D1 *is identical to the weak order induced by u according to* U1,

(ii) *the upper and lower j.n.d. functions induced by (P, I) and u according to* D2 *are $\bar{\delta}$ and $\underline{\delta}$ respectively, and*

(iii) *u is a utility function of (S, P, I).*

PROOF: From part (i), Theorem 2, we define (P, I) as follows: for any $a, b \in S$ with, say, $u(a) \geqslant u(b)$, aPb if $u(b) + \bar{\delta}(b) < u(a)$, and aIb if $u(b) + \bar{\delta}(b) \geqslant u(a)$.

First, we show (P, I) is a semiordering of S.

S1. This axiom follows immediately from the definition.

S2. For any a, $u(a) + \bar{\delta}(a) \geqslant u(a)$ since $\bar{\delta}$ is nonnegative, so aIa.

S3. Suppose aPb, bIc, and cPd. From aPb and cPd we know $u(b) + \bar{\delta}(b) < u(a)$ and $u(d) + \bar{\delta}(d) < u(c)$. If $u(c) \leqslant u(b)$, then $u(d) + \bar{\delta}(d) < u(c) \leqslant u(b) < u(a) - \bar{\delta}(b) \leqslant u(a)$, so aPd by definition. If $u(c) \geqslant u(b)$, then bIc implies $u(b) + \bar{\delta}(b) \geqslant u(c)$, and so $u(d) + \bar{\delta}(d) < u(c) \leqslant u(b) + \bar{\delta}(b) < u(a)$, hence aPd.

S4. Suppose aPb, bPc, bId, aId, and cId. Let us suppose $u(d) \geqslant u(b)$. If $u(c) \leqslant u(d)$, then since cId, $u(c) + \bar{\delta}(c) \geqslant u(d) \geqslant u(b)$, but bPc implies $u(b) > u(c) + \bar{\delta}(c)$. This contradiction implies $u(c) > u(d)$, which in turn implies $u(c) > u(d) \geqslant u(b)$. However, bPc implies $u(b) \geqslant u(c)$, which contradiction forces us to assume $u(d) < u(b)$. Since aPb, $u(b) + \bar{\delta}(b) < u(a)$, which by condition (ii), Theorem 2, implies $u(b) + \underline{\delta}(a) < u(a)$. Rewriting, $u(a) - \underline{\delta}(a) > u(b) > u(d)$. If we suppose $u(d) \geqslant u(a)$, then we have $u(a) - \underline{\delta}(a) > u(a)$, which is impossible since $\underline{\delta}$ is nonnegative, so $u(d) < u(a)$. This with aId implies $u(d) + \bar{\delta}(d) \geqslant u(a)$, so by condition (ii), Theorem 2, $u(d) \geqslant u(a) - \underline{\delta}(a)$, which is in contradiction with what we have shown above. So Axiom S4 is met.

(i) Next, let $(>, \backsim)$ be the weak ordering of S induced by (P, I) and let $(\cdot>, \cdot\backsim)$ be that induced by u. We show that these are the same. Suppose $a > b$, then by D1 there are three cases to consider:

1. aPb, whence by definition $u(a) > u(b) + \bar{\delta}(b) \geqslant u(b)$, since $\bar{\delta}$ is nonnegative; so by U1 $a \cdot> b$.

2. aIb and there exists an element c such that aIc and cPb. We need only consider the case $u(a) \leqslant u(b)$. If $u(a) > u(c)$, then $u(b) \geqslant u(a) > u(c)$, which is impossible by definition since cPb; so $u(a) \leqslant u(c)$. Since aIc, $u(a) + \bar{\delta}(a) \geqslant u(c)$, by definition; so by condition (ii), Theorem 2, $u(a) + \underline{\delta}(c) \geqslant u(c)$. But cPb implies, by definition, $u(b) + \bar{\delta}(b) < u(c)$; so, using Condition (ii), Theorem 2, $u(c) - \underline{\delta}(c) > u(b) \geqslant u(a)$, which is in contradiction to our above conclusion. Thus $a \cdot> b$.

3. aIb and there exists an element d such that aPd and dIb. Again we need only consider the case $u(a) \leqslant u(b)$. If $u(d) > u(b)$, then $u(d) > u(b) \geqslant u(a)$ which is impossible since aPd; so $u(d) \leqslant u(b)$. But since dIb, $u(d) + \bar{\delta}(d) \geqslant u(b)$ and since aPd, $u(d) + \bar{\delta}(d) < u(a) \leqslant u(b)$, a contradiction. Thus, $a \cdot> b$.

Conversely, suppose $a \cdot> b$, i.e., $u(a) > u(b)$. We may suppose aIb, for if aPb then $a > b$. Since $u(a) > u(b)$, condition (iii), Theorem 2, implies two cases, namely:

1. $u(b) + \bar{\delta}(b) < u(a) + \bar{\delta}(a)$. By assumption there exists $c \in S$ such that $u(c) = u(a) + \bar{\delta}(a)$ and so by definition cIa. By hypothesis, $u(c) = u(a) + \bar{\delta}(a) > u(b) + \bar{\delta}(b)$; so cPb. By D1, aIb, cIa, cPb imply $a > b$.

2. $u(b) + \underline{\delta}(a) < u(a) + \underline{\delta}(b)$. Choose d such that $u(d) = u(b) - \underline{\delta}(b) < u(a) - \underline{\delta}(a)$ and it follows that dIb. Applying Condition (ii), Theorem 2, $u(d) + \bar{\delta}(d) < u(a)$; so aPd. By D1, aIb, dIb, aPd imply $a > b$.

Thus $a > b$ if and only if $a \cdot> b$, and since $\backsim$ and $\cdot\backsim$ are defined by exclusion, the two weak orderings are identical.

(ii) Let $\bar{\delta}$ be the upper j.n.d. of (P, I). If $u(b) \geqslant u(a)$, aIb implies $u(b) - u(a) \leqslant \bar{\delta}(a)$, and

$$\bar{\delta}(a) = \sup_{\substack{b \\ aIb}} [u(b) - u(a)] = \sup_{\substack{b \\ aIb \\ u(b) \geqslant u(a)}} [u(b) - u(a)] \leqslant \bar{\delta}(a).$$

However, by assumption there exists $c \in S$ such that $u(c) = u(a) + \tilde{\delta}(a)$. Since, by definition, aIc,

$$\bar{\delta}(a) = \sup_{\substack{b \\ aIb}} [u(b) - u(a)] \geqslant u(c) - u(a) = \tilde{\delta}(a),$$

and so $\bar{\delta} = \tilde{\delta}$. A similar argument shows that $\underset{\sim}{\delta} = \underline{\delta}$.

(iii) Obvious.

5. LINEAR UTILITY FUNCTIONS ON SEMIORDERED MIXTURE SPACES

Since much of utility theory is applied to situations involving decision making under risk, it is of some interest to examine the linear utility functions of semiordered mixture spaces. Our result, as expressed in Theorem 4, does not take the usual form of a set of axioms which insure the existence of a linear utility function, but rather it presents a function which is definable in terms of experimentally meaningful operations for almost all semiordered mixture spaces and which is a linear utility function if one exists.

A *mixture space* [**9**] is a set M satisfying the following axioms:

M1. if $a, b \in M$ and if α is a real number, $0 \leqslant \alpha \leqslant 1$, then $a\alpha b \in M$,
M2. $a\alpha b = b(1 - \alpha)a$,
M3. $a\alpha(b\beta c) = \left(a \dfrac{\alpha}{\alpha + \beta - \alpha\beta} b\right)(\alpha + \beta - \alpha\beta)c$, if $\alpha + \beta - \alpha\beta \neq 0$,
M4. $a\alpha a = a$,
M5. If $a\alpha c = b\alpha c$ for some α, $0 < \alpha \leqslant 1$, then $a = b$.

A real valued function u defined over a semiordered mixture space (M, P, I) is called a *linear utility function* if it satisfies axioms U1 and U2 of Section 4 and

U3. for any $a, b \in M$ and α, $0 \leqslant \alpha \leqslant 1$, $u(a\alpha b) = \alpha u(a) + (1 - \alpha)u(b)$.

For brevity we introduce the following notation: If (M, P, I) is a semiordered mixture space and $a, b \in M$ and aPb, then

$$\bar{\alpha}(a, b) = \sup \{\alpha \mid (a\alpha b)Ib\}$$

$$\underline{\alpha}(a, b) = \sup \{\alpha \mid (b\alpha a)Ia\}.$$

THEOREM 4: *Let (M, P, I) be a semiordered mixture space having elements $a^*, b^*, c \in M$ such that a^*Pc and cPb^*. There exists a linear utility function u of (M, P, I) such that $\bar{\delta}(b^*) > 0$ and $\underline{\delta}(a^*) > 0$ if and only if*

(i) *for any $a \in M$ such that aPb^*, $\bar{\alpha}(a, b^*) > 0$,*

(ii) *for any $a \in M$ such that a^*Pa, $\underline{\alpha}(a^*, a) > 0$, and*

(iii) *the function*

$$\rho(a) = \begin{cases} \dfrac{\bar{\alpha}(a^*, b^*)}{\bar{\alpha}(a, b^*)}, & \text{if} \quad aPb^* \\[2ex] 1 - \dfrac{\underline{\alpha}(a^*, b^*)}{\underline{\alpha}(a^*, a)}, & \text{if} \quad a^*Pa \end{cases}$$

is single valued, linear, and order preserving.

The function u is a linear transformation of ρ.

Proof: *Necessity.* Consider any $a \epsilon M$, then either aPb^* or a^*Pa. For suppose aIc, then a^*Pc, cPb^*, and aIc imply (S4) either aPb^* or a^*Pa. If aPc, then (S3) aPb^*, and if cPa, then (S3) a^*Pa. So we need only consider these two cases.

By applying the appropriate linear transformation to u, there is no loss of generality in assuming $u(a^*) = 1$ and $u(b^*) = 0$.

(i) Suppose aPb^*. Let $A = \{\alpha \mid 0 \leqslant \alpha \leqslant 1, b^*I(a\alpha b^*)\}$. By Theorem 2, $\alpha \epsilon A$ if and only if $0 \leqslant \alpha \leqslant 1$ and $u(a\alpha b^*) \leqslant u(b^*) + \bar{\delta}(b^*)$. Using the linearity of u and the fact $u(b^*) = 0$, we obtain $\alpha u(a) \leqslant \bar{\delta}(b^*)$. Thus, $\bar{\alpha}(a, b^*) = \min \{1, \bar{\delta}(b^*)/u(a)\}$. Suppose $\bar{\delta}(b^*)/u(a) > 1$, then $u(a) < \bar{\delta}(b^*) = u(b^*) + \bar{\delta}(b^*)$, and so by Theorem 2, aIb^*, which is contrary to assumption. Thus, $\bar{\alpha}(a, b^*) = \bar{\delta}(b^*)/u(a)$. Since aPb^*, U1 and D1 imply $u(a) > u(b^*) = 0$, and by hypothesis $\bar{\delta}(b^*) > 0$; so $\bar{\alpha}(a, b^*) > 0$.

(ii) Suppose a^*Pa. In a similar fashion we can show

$$\underline{\alpha}(a^*, a) = \underline{\delta}(a^*)/(1 - u(a)) > 0$$

using the hypothesis $\underline{\delta}(a^*) > 0$.

(iii) If aPb^*, then

$$\rho(a) = \frac{\bar{\alpha}(a^*, b^*)}{\bar{\alpha}(a, b^*)} = \frac{\bar{\delta}(b^*)u(a)}{u(a^*)\bar{\delta}(b^*)} = u(a).$$

If a^*Pa, then

$$\rho(a) = 1 - \frac{\underline{\alpha}(a^*, b^*)}{\underline{\alpha}(a^*, a)} = 1 - \frac{\underline{\delta}(a^*)(1 - u(a))}{(1 - u(b^*))\underline{\delta}(a^*)} = u(a).$$

Thus, ρ is single valued, linear, and order preserving since u is. Inverting our original linear transformation, it follows that any linear utility function meeting the hypotheses is a linear transformation of ρ.

Sufficiency. Obvious.

It may be worth noting that by arguments similar to those used in the proof of the theorem one can show:

If aPb^*,

$$\underline{\delta}(a) = \underline{\alpha}(a, b^*)\rho(a) = \frac{\underline{\alpha}(a, b^*)\bar{\alpha}(a^*, b^*)}{\bar{\alpha}(a, b^*)},$$

and if a^*Pa,

$$\bar{\delta}(a) = \bar{\alpha}(a^*, a)(1 - \rho(a)) = \frac{\bar{\alpha}(a^*, a)\underline{\alpha}(a^*, b^*)}{\underline{\alpha}(a^*, a)}.$$

In conclusion, a few remarks concerning the possible applications of the above theorem may not be amiss. It should be obvious that the theorem does not follow the now classic pattern set by von Neumann and Morgenstern [**12**] of a set of axioms which, if they are met by the preference relation on a mixture space, insure the existence of a linear utility function. That type of theorem has the virtue of justifying the use of linear utility functions in theories of rational decision making, provided the axioms are intuitively acceptable as a definition of rationality. There is no question that it would be desirable to have an analogous set of axioms and the corresponding theorem for semiorders, but at present this remains an open problem.

The complete verification that a linear utility function exists, i.e., that the ρ so constructed is linear and order preserving, is just as impossible here as in the classical case—an infinity of observations and calculations being required in both cases. Partial verification is practical, however, and in practice one would probably determine ρ for a fairly large number of interrelated points, e.g., if a and b are examined, then $a(\frac{1}{4})b$, $a(\frac{1}{2})b$, and $a(\frac{3}{4})b$ might also be examined. All of these points can then be checked both for linearity and the preservation of order. If these data check, then there is some hope that a linear utility function actually exists. It is, of course, clear that for sophisticated experimental applications two major extensions of the theory are needed. First, there should be developed a statistical theory of preference orderings so that neither linearity nor preservation of order need hold exactly, and second, a finitistic theory of semiordered risk situations should be created for which it is possible to verify *all* requirements of the model. An example of the latter for weakly ordered preferences has been given by Davidson and Suppes [**6**].

A final point on experimentation should be made. It seems extremely doubtful that laboratory experiments of the type so far reported in the literature, which for the most part have involved simple gambling situations, can utilize our model. These designs have been carefully cultivated and abstracted from actual social situations in such a manner that the preference relation is almost certainly a weak order because it reflects the natural ordering of money by amounts. It is hard to imagine anyone in such a laboratory experiment who will be indifferent between \$1 and \$0 α \$1 for any $\alpha > 0$. One senses, however, that in society there are many decision situations with which we wish to deal in which the indifference relation is more complex than an equivalence relation. If this intuition is correct, it should be a challenge to the experimenter to try to reproduce these situations as simple laboratory experiments where the nature and origin of the intransitivities can be carefully examined.

Columbia University

REFERENCES

[**1**] Armstrong, W. E.: "The Determinateness of the Utility Function," *Economic Journal*, 49 (1939), pp. 453–467.

[**2**] ———: "Uncertainty and the Utility Function," *Economic Journal*, 58 (1948), pp. 1–10.

[**3**] ———: "A Note on the Theory of Consumer's Behavior," *Oxford Economic Papers*, 2 (1950), pp. 119–122.

[**4**] ———: "Utility and the Theory of Welfare," *Oxford Economic Papers*, 3 (1951), pp. 259–271.

[5] Blackwell, David and M. A. Girshick: *Theory of Games and Statistical Decisions*, John Wiley and Sons, New York (1954).

[6] Davidson, Donald and Patrick Suppes: *Finitistic Rational Choice Structures*, Report 3, Stanford Value Theory Project and Department of Philosophy, Stanford (1953), 38 pp.

[7] Debreu, Gerard: "Representation of a Preference Ordering by a Numerical Function," *Decision Processes* (R. M. Thrall, C. H. Coombs, and R. L. Davis, Editors), John Wiley and Sons, New York (1954), pp. 159–165.

[8] Halphen, Etienne: "La notion de vraisemblance," publication de l'Institut de Statistique de l'Université de Paris, 4 (1955), pp. 41–92.

[9] Hausner, Melvin: "Multidimensional Utilities," *Decision Processes* (R. M. Thrall, C. H. Coombs, and R. L. Davis, Editors), John Wiley and Sons, New York (1954), pp. 167–180.

[10] May, K. O.: "Intransitivity, Utility, and the Aggregation of Preference Patterns," *Econometrica*, 22, (1954), pp. 1–13.

[11] Savage, L. J.: *The Foundations of Statistics*, John Wiley and Sons, New York (1954).

[12] von Neumann, John and Oskar Morgenstern: *Theory of Games and Economic Behavior*, Princeton University Press, Princeton (1947).