ANNALS OF MATHEMATICS
Vol. 62, No. 3, November, 1955
Printed in U.S.A.

k-STABILITY OF SYMMETRIC AND OF QUOTA GAMES[1]

BY R. DUNCAN LUCE

(Received November 18, 1954)

1. Introduction

An *n-person game* in characteristic function form [2, 3] is a pair (I_n, m), where I_n is a set of $n \geqq 3$ elements, called the *players*, and m is a real-valued set function defined for all subsets of I_n, called the *characteristic function*, which satisfies

i. $m(0) = 0$,

and ii. if R and S are disjoint subsets of I_n,

$$m(R \cup S) \geqq m(R) + m(S).$$

We shall always take I_n to be the first n integers, i.e., a labeling of the players.

The terms S-equivalence, essential and inessential games, imputations, and coalitions will have their usual meanings [2]; however, we shall let the word "game" mean "essential game" except when it is prefixed by "inessential". The notation used will be standard, except that we shall write $x(T)$ for $\sum_{i \epsilon T} x_i$, where $X = \| x_i \|$ is a real n-tuple and T is a coalition.

Since all of our results are invariant under S-equivalence it will suffice to use one representative characteristic function from each of the equivalence classes; it seems most convenient to use the 0–1 normalization $m(\{i\}) = 0$, $i \epsilon I_n$, and $m(I_n) = 1$.

Any partition $\tau = (T_1, T_2, \cdots T_t)$ of I_n into proper subsets T_i is called a *coalition structure*. The particular coalition structure where there are no non-trivial coalitions, i.e., $[\{1\}, \{2\}, \cdots \{n\}]$, will be denoted by Δ_n. Let k be an integer with $0 \leqq k \leqq n - 2$ and let τ be a coalition structure, then a subset $S \subset I_n$ is called a *k-critical coalition* of τ if there exists a coalition $T \epsilon \tau$ such that $| (S - T) \cup (T - S) | \leqq k$. It is clear that if $T \epsilon \tau$, T is a k-critical coalition of τ for every k.

In an earlier paper [1] we introduced and attempted to justify intuitively the following class of equilibrium notions: A pair (X, τ), where X is an imputation and τ a coalition structure, is said to be *k-stable* if (1) for every k-critical coalition S of τ, $m(S) \leqq x(S)$, and (2) $x_i = 0$ implies $\{i\} \epsilon \tau$. A game is called k-stable if there exists at least one k-stable pair; otherwise it is called k-unstable.

In [1] we presented a few general properties of the notion and we examined the stability conditions for 3- and 4-person constant-sum games and for all simple

[1] This work was supported in part by the Behavioral Models Project, Bureau of Applied Social Research, Columbia University, through funds extended to it by the Office of Naval Research, contract no. Nonr 266(21). This paper may be identified as publication A-175 of the Bureau of Applied Social Research. The final preparation of the paper was executed while the author was a Fellow of the Center for Advanced Study in the Behavioral Sciences, Stanford, California.

games. Here we propose to continue this program and to study the stability of symmetric games and of quota games (Sections 2 and 3). These results are employed in Section 4 to study the stability of games which are both simple and quota. The final section is devoted to a sketchy discussion of 1-stable pairs of the form (X, Δ_n).

2. Symmetric games

One quite general and important class of games which has been studied in the literature is that in which the characteristic function depends only on the size of a coalition, i.e.,

$$m(T) = m(T') \quad \text{if} \quad |T| = |T'|.$$

Such games are called symmetric, and we write $m(T) = m(|T|)$.

THEOREM 1. *A symmetric game with characteristic function $m(i)$ is k-stable if and only if $m(i) \leqq i/n$ for $0 \leqq i \leqq k + 1$.*

PROOF. It is clear that $(\| 1/n \|, \Delta_n)$ is k-stable if the condition is met.

Conversely, suppose (X, τ) is k-stable and that $m(k + 1) > (k + 1)/n$. Consider any positive integer a such that $a(k + 1) \leqq n$. Since we may partition any coalition of $a(k + 1)$ elements into a disjoint coalition of $k + 1$ elements,

$$m[a(k + 1)] \geqq am(k + 1) > a(k + 1)/n.$$

For any $T_i \in \tau$ it is clear that since $0 < |T_i| < n$ and $0 \leqq k \leqq n - 2$ we may write

$$|T_i| = a_i (k + 1) + b_i ,$$

where a_i and b_i are integers such that

$$0 < a_i(k + 1) < n \quad \text{and} \quad -k \leqq b_i \leqq k.$$[2]

We consider three cases:

1. $b_i = 0$. From the condition of k-stability we have $x(T_i) \geqq m(|T_i|) = m[a_i(k + 1)] > a_i(k + 1)/n = |T_i|/n$.

2. $b_i < 0$. We first show that it is always possible to find a set B_i such that

$$B_i \subset -T_i, \qquad |B_i| = |b_i|, \qquad \text{and} \qquad x(B_i) \leqq \frac{[1 - x(T_i)]\,|b_i|}{n - |T_i|}.$$

If this were not the case, then we would have to assume that for the $\binom{n - |T_i|}{|b_i|}$ coalitions B_i meeting the first two conditions

$$x(B_i) > \frac{[1 - x(T_i)]\,|b_i|}{n - |T_i|}.$$

Observe that each $j \in -T_i$ appears in exactly $\binom{n - |T_i| - 1}{|b_i| - 1}$ of these sets, and

[2] It should be noted that this is not the usual division algorithm for integers. There are, in some cases, two possible values for a_i and two corresponding values for b_i.

so if we sum over all of them we obtain

$$\sum_{B_i} x(B_i) = \binom{n - |T_i| - 1}{|b_i| - 1} x(-T_i) = \binom{n - |T_i| - 1}{|b_i| - 1} [1 - x(T_i)]$$
$$> \binom{n - |T_i|}{|b_i|} \frac{[1 - x(T_i)] \, |b_i|}{n - |T_i|} = \binom{n - |T_i| - 1}{|b_i| - 1} [1 - x(T_i)],$$

which contradiction establishes the existence of a B_i meeting the three conditions. Since $|B_i| = |b_i| \leqq k$, $T_i \cup B_i$ is a k-critical coalition of τ and so

$$\begin{aligned} x(T_i) + \frac{[1 - x(T_i)] \, |b_i|}{n - |T_i|} &\geqq x(T_i) + x(B_i) \\ &\geqq m(|T_i| + |b_i|) \\ &= m[a_i(k+1)] \\ &> a_i(k+1)/n. \end{aligned}$$

Thus,

$$\frac{x(T_i)(n - |T_i| - |b_i|) + |b_i|}{n - |T_i|} > a_i(k+1)/n,$$

or

$$\begin{aligned} x(T_i) > \frac{(n - |T_i|)a_i(k+1) - n\,|b_i|}{n(n - |T_i| - |b_i|)} &= \frac{(n - |T_i|)(|T_i| + |b_i|) - n\,|b_i|}{n(n - |T_i| - |b_i|)} \\ &= |T_i|/n. \end{aligned}$$

3. $b_i > 0$. We first show that it is always possible to find a set B_i such that

$$B_i \subset T_i, \qquad |B_i| = b_i, \qquad \text{and} \qquad x(B_i) \geqq \frac{x(T_i)b_i}{|T_i|}.$$

If this were not the case, then we may sum over all $\binom{|T_i|}{b_i}$ sets B_i satisfying the first two conditions, and we obtain

$$\sum_{B_i} x(B_i) = \binom{|T_i| - 1}{b_i - 1} x(T_i) < \binom{|T_i|}{b_i} \frac{x(T_i)b_i}{|T_i|} = \binom{|T_i| - 1}{b_i - 1} x(T_i),$$

which is a contradiction. Observe that for any B_i satisfying the three conditions $T_i - B_i$ is a k-critical coalition of τ, hence

$$\begin{aligned} x(T_i) - x(T_i)b_i/|T_i| &\geqq x(T_i) - x(B_i) \\ &\geqq m[a_i(k+1)] \\ &> a_i(k+1)/n \\ &= (|T_i| - b_i)/n. \end{aligned}$$

Thus, $x(T_i) > |T_i|/n$.

We have therefore shown that for every $T_i \epsilon \tau$, $x(T_i) > | T_i |/n$, and so

$$1 = x(I_n) = \sum_{T_i} x(T_i) > \sum_{T_i} | T_i |/n = 1,$$

which is impossible and so the pair is not k-stable, and the theorem is proved.

In [1] we defined a game to be negative if $m(T) \leqq | T |/n$ for all $T \subset I_n$.

COROLLARY. *A symmetric game is* $(n - 2)$*-stable if and only if it is negative.*

PROOF. The theorem and the definition of a negative game.

3. Quota games

Shapley [4] defines the following class of games: A game is called a *quota game* if there exists a real n-tuple $Q = \| q_i \|$, called the *quota*, such that

i. $q(I_n) = 1$,

and ii. $m(\{i, j\}) = q_i + q_j$, $i, j \epsilon I_n$, $i \neq j$.

A player i such that $q_i < 0$ is called *weak*. Since $m(\{i, j\}) \geqq 0$, there is at most one weak player, and when n is odd there is no weak player. For suppose, without loss of generality, n is a weak player and n is odd, then

$$\begin{aligned} m(-\{n\}) &= m(\{1, 2, \cdots, n-1\}) \geqq m(\{1, 2\}) + \cdots + m(\{n-2, n-1\}) \\ &= \sum_{i=1}^{n-1} q_i = 1 - q_n > 1, \end{aligned}$$

which is impossible.

THEOREM 2. *A quota game is* 1*-stable if and only if there is no weak player.*

PROOF. If there is no weak player, then clearly the pair (Q, Δ_n) is 1-stable.

Conversely, suppose there is a weak player, which by relabeling we may take to be n, and let (X, τ) be a 1-stable pair. Label the coalitions $T_1, \cdots, T_t$ of τ so that $n \epsilon T_t$. For any $T_i \epsilon \tau$, the 1-stability requirements implies $m(T_i) \leqq x(T_i)$. Now, if $| T_i |$ is even, then T_i can be partitioned into $| T_i |/2$ non-overlapping two element coalitions, each of which has the value $m(\{i, j\}) = q_i + q_j$. Thus, $x(T_i) \geqq m(T_i) \geqq q(T_i)$. If $| T_i | > 1$ and odd, then for every $k \epsilon T_i$, $| T_i - \{k\}|$ is even, and so by the same argument $x(T_i - \{k\}) \geqq q(T_i - \{k\})$. Summing over all $k \epsilon T_i$,

$$\sum_{k \epsilon T_i} x(T_i - \{k\}) = (| T_i | - 1)x(T_i) \geqq \sum_{k \epsilon T_i} q(T_i - \{k\}) = (| T_i | - 1)q(T_i)$$

hence $x(T_i) \geqq q(T_i)$. If $| T_i | = 1$, let $T_i = \{i\}$, and then for any $k \epsilon -\{i\}$, $\{i, k\}$ is a 1-critical coalition and so

$$x_i + x_k \geqq m(\{i, k\}) = q_i + q_k .$$

Summing over all $k \epsilon -\{i\}$,

$$(n - 2)x_i + x(I_n) \geqq (n - 2)q_i + q(I_n).$$

But $x(I_n) = 1 = q(I_n)$, so with $n \geqq 3$, $x_i \geqq q_i$. Since these inequalities hold for all $T_i \epsilon \tau$ and since $x(I_n) = q(I_n)$, the equalities

$$\begin{aligned} x(T_i) &= q(T_i) = m(T_i), && \text{if } | T_i | \text{ is even} \\ x(T_i) &= q(T_i), && \text{if } | T_i | \text{ is odd} \end{aligned}$$

must hold.

Next we show that if n is weak and $n \in T_t$, then $|T_t|$ is even. Suppose, on the contrary, $|T_t|$ is odd. If $|T_t| > 1$, then by the partitioning argument $m(T_t) \geqq m(T_t - \{n\}) \geqq q(T_t - \{n\})$. But we know that $x(T_t) = q(T_t)$, and since n is weak, $q_n < 0$, so $m(T_t) \geqq q(T_t) - q_n > x(T_t)$ which violates the 1-stability assumption. If $|T_t| = 1$, then $T_t = \{n\}$ and we have shown above that $x_n = q_n < 0$, which is impossible. Thus $|T_t|$ is even.

It is clear that in $-T_t$ there is at least one k such that $q_k \geqq x_k$. Consider the 1-critical coalition $T_t \cup \{k\}$. Since $|T_t|$ is even, so is $|(T_t \cup \{k\}) - \{n\}|$, and so we may partition that coalition into non-overlapping two element coalitions:

$$m(T_t \cup \{k\}) \geqq m[(T_t \cup \{k\}) - \{n\}] \geqq q(T_t) + q_k - q_n .$$

But $q_n < 0$ and $q_k \geqq x_k$, so

$$m(T_t \cup \{k\}) > x(T_t) + x_k + 0 = x(T_t \cup \{k\}),$$

which violates the assumption that (X, τ) is 1-stable. Thus, we must conclude that there is no weak player.

COROLLARY. *All quota games with an odd number of players are 1-stable.*

PROOF. The theorem coupled with the observation that when n is odd there is no weak player.

A game (I_n, m) is called *constant-sum* [2, 3] if for every $S \subset I_n$, $m(S) + m(-S) = 1$.

THEOREM 3. *Let (I_n, m) be a k-stable quota game and let (X, τ) be a k-stable pair. If n is odd or if n is even and $k \geqq 2$, then $X = Q$. If n is even and $k = 1$, then either $X = Q$ or $|T|$ is even and $m(T) = q(T) = x(T)$ for every $T \in \tau$. There are quota games (both constant-sum and non-constant-sum) with n even and $k = 1$ in which $X \neq Q$.*

PROOF. Suppose (X, τ), where $\tau = (T_1, \cdots, T_t)$, is 1-stable and that for some r, $x_r \neq q_r$. From the proof of Theorem 2 we know that for each $T_i \in \tau$, $x(T_i) = q(T_i)$. It follows, therefore, that in some T_i, say T_t, there exist r and s such that $x_r > q_r$ and $x_s < q_s$. Now suppose that for $i \neq t$, $|T_i|$ is odd, then $T_i \cup \{s\}$ has an even number of elements and is 1-critical, so

$$x(T_i \cup \{s\}) \geqq m(T_i \cup \{s\}) \geqq q(T_i \cup \{s\}) = q(T_i) + q_s > x(T_i \cup \{s\})$$

which is impossible. Thus, $|T_i|$ is even. If n is even, then so is $|T_t|$. Suppose n, and therefore $|T_t|$, is odd. Since we know that if $T_t = \{r\}$, $q_r = x_r$, it follows that $|T_t| > 1$. Since $|T_t - \{r\}|$ is even, $m(T_t - \{r\}) \geqq q(T_t - \{r\}) > x(T_t - \{r\})$, which is impossible. Thus, if (X, τ) is 1-stable either $X = Q$ or $|T|$ is even for $T \in \tau$. Since any k-stable pair is also 1-stable, the conclusion holds for k-stable pairs. If $|T|$ is even we know from the proof of Theorem 2 that $m(T) = q(T) = x(T)$.

Next, let us assume that n is even and $k \geqq 2$, and suppose (X, τ) is k-stable and $X \neq Q$. Thus there exists $r \in T_i$, for some i, such that $x_r > q_r$, and for any $j \neq i$, there exists $s \in T_j$ such that $x_s \leqq q_s$. Consider $(T_i - \{r\}) \cup \{s\}$ which is k-critical for $k \geqq 2$ and which has an even number of elements since T_i does. Thus,

$$m[(T_i - \{r\}) \cup \{s\}] \geqq q(T_i) - q_r + q_s > x[(T_i - \{r\}) \cup \{s\}],$$

which is impossible. Thus, $X = Q$.

The following is an example of a non-constant-sum symmetric quota game in which (X, τ) is 1-stable and $X \neq Q$:

$$n = 6, \quad q_i = 1/6, \quad m(2) = 4/12, \quad m(3) = 5/12, \quad m(4) = 8/12, \quad m(5) = m(6) = 1.$$

It is easy to show that

$$(\| 1/12, 3/12, 2/12, 2/12, 2/12, 2/12 \|, [\{1, 2\}, \{3, 4\}, \{5, 6\}])$$

is 1-stable. This example is modified into a constant-sum example if $m(3)$ is altered as follows:

$$m(\{1, 3, 4\}) = m(\{1, 5, 6\}) = 5/12$$
$$m(\{2, 5, 6\}) = m(\{2, 3, 4\}) = 7/12$$
$$m(\{i, j, k\}) = 6/12 \quad \text{for all other } i, j, k.$$

The same pair is 1-stable.

THEOREM 4. *Let (I_n, m) be a quota game without a weak player. A sufficient, but not a necessary, condition for the game to be k-stable, $k \geqq 2$, is that*

$$(1) \qquad m(T) \leqq q(T)$$

holds for all T such that $|T| \leqq k + 1$. A necessary, but not a sufficient, condition for it to be k-stable is that Equation 1 holds for all T such that $|T| \leqq k$. Suppose the game is k-stable and there exists a set T such that $|T| = k + 1$ and Equation 1 is violated, and let τ be the coalition structure of any k-stable pair. If T_i is any coalition of τ which intersects T, then $|T_i \cap T| = 1$ and $|T_i|$ is even.

PROOF. Sufficiency: It is obvious that (Q, Δ_n) is k-stable if equation 1 holds for all T such that $|T| \leqq k + 1$. To show that this condition is not necessary consider the following game:

$$n = 6, \qquad q_i = 1/6$$
$$m(S) = |S|/6 \quad \text{for all } S \text{ such that } |S| = 2, 3, 4, 6 \text{ except } \{1, 3, 5\} \text{ and } \{2, 4, 6\}$$
$$= 1 \quad \text{for all } S \text{ such that } |S| = 5$$
$$m(\{1, 3, 5\}) = 4/6$$
$$m(\{2, 4, 6\}) = 2/6.$$

Let $\tau = [\{1, 2\}, \{3, 4\}, \{5, 6\}]$ and then it is easy to see that (Q, τ) is 2-stable since any 2-critical coalition contains at most four elements and if it contains $\{1, 3, 5\}$ it must contain four. But Equation 1 is violated for $T = \{1, 3, 5\}$.

Necessity: If the game is k-stable, $k \geqq 2$, then by Theorem 3 any k-stable pair is of the form (Q, τ). Suppose that T is a set such that $|T| \leqq k + 1$ and $m(T) > q(T)$. Let T_i be any coalition of τ which intersects T and suppose $|T_i \cap T| > 1$. First, $T_i - T \neq 0$, since if $T_i \subset T$, then $|T - T_i| \leqq k$ because $|T| \leqq k + 1$. In that case T is a k-critical coalition of τ and the hypothesis of k-stability is violated. Second, $T_i \cup T \neq I_n$, for suppose on the contrary $T_i \cup T = I_n$, then $|T_i - T|$ is odd, for if it were even then

$$\begin{aligned} 1 = m(I_n) &= m(T \cup [T_i - T]) \\ &\geqq m(T) + m(T_i - T) \\ &> q(T) + q(T_i - T) \\ &= q(I_n) = 1, \end{aligned}$$

which is impossible. But if $|T_i - T|$ is odd, then for any $s \in T_i - T$, $|(T_i - T) - \{s\}|$ is even and $I_n - \{s\}$ is a k-critical coalition of τ since

$$\begin{aligned} &|[T_i - (I_n - \{s\})] \cup [(I_n - \{s\}) - T_i]| \\ &\qquad = |\{s\} \cup (T - T_i)| \\ &\qquad \leqq 1 + (k - 1) = k. \end{aligned}$$

Observe,

$$\begin{aligned} m(I_n - \{s\}) &= m(T \cup [(T_i - T) - \{s\}]) \\ &\geqq m(T) + m[(T_i - T) - \{s\}] \\ &> q(T) + q[(T_i - T) - \{s\}] \\ &= q(I_n - \{s\}), \end{aligned}$$

which violates the k-stability condition. Thus we know that $T_i - T \neq 0$ and $T_i \cup T \neq I_n$. By choosing an element $s \in T_i - T$ or in $-(T_i \cup T)$, we may make $|(T_i - T) \cup \{s\}|$ even. Observe that since $|T_i \cap T| > 1$,

$$|(T \cup T_i \cup \{s\}) - T_i| \leqq |(T \cup \{s\}) - (T \cap T_i)| \leqq k,$$

and so $T \cup T_i \cup \{s\}$ is a k-critical coalition of τ. But,

$$\begin{aligned} m(T \cup T_i \cup \{s\}) &= m[T \cup (T_i - T) \cup \{s\}] \\ &\geqq m(T) + m[(T_i - T) \cup \{s\}] \\ &> q(T) + q[(T_i - T) \cup \{s\}] \\ &= q(T \cup T_i \cup \{s\}), \end{aligned}$$

which violates the k-stability assumption, and so we must conclude that $| T_i \cap T | = 1$.

Next, suppose that $| T_i |$ is odd, then $| T_i - T |$ is even and $T \cup T_i$ is k-critical, so

$$m(T \cup T_i) \geqq m(T) + m(T_i - T) > q(T \cup T_i),$$

which is impossible, so $| T_i |$ is even. Finally, if we suppose $| T | \leqq k$, then for $s \in -(T \cup T_i)$

$$| (T \cup T_i \cup \{s\}) - T_i | = | (T \cup \{s\}) - (T \cap T_i) | \leqq k,$$

so $T \cup T_i \cup \{s\}$ is k-critical. Since $| T_i |$ is even, $| (T_i - T) \cup \{s\} |$ is even, and the same argument as above leads to a contradiction. Thus, we must conclude that equation 1 holds for all T with $| T | \leqq k$, and if it fails for some T with $| T | = k + 1$, then for any $T_i \in \tau$ such that $T_i \cap T \neq 0$, $| T_i \cap T | = 1$ and $| T_i |$ is even.

The necessary condition is not sufficient as the following example shows. Let n and k be both even or both odd and let (I_n, m) be any symmetric quota game with $m(k + 1) > (k + 1)/n$ and $m(i) \leqq i/n$, for $i \leqq k$. It is easy to see that in a symmetric quota game, $q_i = 1/n$, so $m(k + 1) > (k + 1)q_i$. Thus, the game satisfies only the necessary condition, and by Theorem 1 it is k-unstable.

COROLLARY 1. *The necessary condition is sufficient if either n and k are of opposite parity or if $k > (n - 2)/2$.*

PROOF. If n and k have opposite parity, then $n - k - 1$ is even, so if T is any set such that $| T | = k + 1$, $| -T |$ is even, thus $m(-T) \geqq q(-T)$. It therefore follows that

$$m(T) \leqq 1 - m(-T) \leqq 1 - q(-T) = q(T),$$

and so the sufficient condition holds.

Suppose Equation 1 does not hold for T where $| T | = k + 1$. In this case we know that for any $T_i \in \tau$ such that $T_i \cap T \neq 0$, $| T_i \cap T | = 1$ and $| T_i |$ is even, thus there are at least $k + 1$ non-overlapping sets each having at least two elements, so $n \geqq 2(k + 1)$, or $k \leqq (n - 2)/2$. Thus, if $k > (n - 2)/2$, the necessary condition is also sufficient.

COROLLARY 2. *Any quota game with an odd number of players is 2-stable.*

PROOF. Since a quota game with an odd number of players has no weak player, the first part of Corollary 1 implies the result.

4. Simple quota games

Following the definition of von Neumann and Morgenstern [3] for zero-sum games, we defined [1] a game to be *simple* if $m(S) = 0$ or 1 for every $S \subset I_n$. Those coalitions S with $m(S) = 0$ are called *losing* and those with $m(S) = 1$ *winning.*

THEOREM 5. *Let (I_n, m) be a simple game. It is non-constant-sum and quota if and only if there exists an element r such that any coalition properly including $\{r\}$*

is winning, and all other coalitions are losing. It is constant-sum and quota if and only if either:

i. *it is the 4-person game:*

$$m(\{i, j\}) = 1, \quad \text{for} \quad i, j \neq r,$$

$$m(\{i, r\}) = 0, \quad \text{for} \quad i \neq r,$$

(this game will be called the exceptional simple quota game); or

ii. *there exists an element r such that* $-\{r\}$ *and any coalition properly including* $\{r\}$ *are winning, and all other coalitions are losing.*

Proof. It is not difficult to see that the simple games so defined are quota games by taking $q_r = 1$, $q_i = 0$, $i \neq r$, in the non-exceptional cases and by letting $q_r = -1/2$, $q_i = 1/2$, $i \neq r$, in the exceptional 4-person case.

Conversely, suppose (I_n, m) is both a simple and a quota game. Suppose there exists a weak player, which without loss of generality we may take to be n. It is clear that for $i \neq n$, $q_i \geqq 0$ and that there exists some $r \neq n$ such that $q_r > 0$. For any $i \neq r, n$, $m(\{i, r\}) = q_i + q_k > 0$, so $\{i, r\}$ is winning. If we suppose that in addition to r, there is a j with $q_j > 0$, then any set $\{k, j\}$ must also be winning. If $n \geqq 5$, then we may choose i, j, k, r all different and different from n. But both $\{i, r\}$ and $\{k, j\}$ are winning, which is impossible. Thus, if there is a weak player then either $n \leqq 4$ or $q_r = 1$, $q_i = 0$, for $i \neq r, n$. In the latter case, $q_n = 1 - q(I_n - \{n\}) = 0$, which contradicts the assumption that n is weak. For $n = 4$, the same argument applies as above except if $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are all winning. In this case, $q_1 + q_2 = q_1 + q_3 = q_2 + q_3 = 1$, so $q_1 = q_2 = q_3 = 1/2$ and $q_4 = -1/2$. Thus $\{i, 4\}$, $i \neq 4$, are losing coalitions. For $n = 3$, the fact that $\{1, 2\}$ is winning implies $q_1 + q_2 = 1$, which implies $q_3 = 0$ and so there is no weak player.

We may now suppose the game has no weak player. By a repetition of the first argument of the proof we may show that there exists an element r such that any coalition properly including $\{r\}$ is winning. If T is any coalition not including $\{r\}$ and if $|T| \leqq n - 2$, then there exists $j \in -(T \cup \{r\})$. Since $\{r, j\}$ is winning and $-T \supset \{r, j\}$, $-T$ is winning and so T is losing. The only remaining coalition is $-\{r\}$ which if it is losing results in a non-constant-sum game and if it is winning results in a constant-sum game.

Corollary 1. *Every non-exceptional constant-sum simple quota game is k-stable for all $k < n - 2$ and is $(n - 2)$-unstable. Every non-constant-sum simple quota game is k-stable for all $k \leqq n - 2$. The exceptional game is 1-unstable.*

Proof. Theorem 5 and the conditions for the k-stability of simple games (Theorem 4, [1]).

Corollary 2. *The only k-stable pair of a non-exceptional simple quota game is (Q, Δ_n).*

Proof. Let (X, τ) be k-stable. By Theorem 3 we know that $X = Q$ except possibly when $k = 1$ and n is even. In the latter case the fact that n is even implies $n \geqq 4$, and so the intersection of the 2-element winning coalitions is $\{r\}$, where r is the element described in theorem 5. From this we may conclude by

Theorem 5 of [1] that $X = Q$. Since $X = Q$ in all cases and since $q_j = 0$ for $j \neq r$, the second condition of k-stability implies $\tau = \Delta_n$.

5. A remark

It follows from Theorem 4 of [1] and from Theorems 1 and 2 of this paper that any 1-stable simple, symmetric, or quota game has a 1-stable pair of the form (X, Δ_n). In the case of the non-exceptional simple quota games the only k-stable pair is of this form (corollary 2, Theorem 5). In other words, for these games there is at least one n-tuple X such that

i. $x(I_n) = 1$,

ii. $m(\{i, j\}) \leqq x_i + x_j$, $i, j \in I_n$, $i \neq j$,

and iii. $x_i \geqq 0$,

and so all of these 1-stable games lie in a class of games which is a generalization of Shapley's notion of a quota game without a weak player.

There are two grounds for thinking that this is a comparatively special case and that some effort should be expended to isolate classes of games which do not possess this property and to characterize their stability properties. First, there exist 1-stable games which do not have this property. Consider any (I_n, m) having a set T such that

i. $|T| = n/2 = t$,

ii. for $i, j \in T$ or $i, j \in -T$, $m(\{i, j\}) > 2/n$

and iii. for $i, j \in T$ *and* $k \in -T$, *or* $i \in T$ *and* $j, k \in -T$, $m(\{i, j, k\}) \leqq 3/n$.

It is easy to see that such games exist, e.g., a game decomposable along T with components which are 1-unstable symmetric games. Now suppose a pair (X, Δ_n) is 1-stable, then for $i, j \in T$, $x_i + x_j \geqq m(\{i, j\}) > 2/n$. If we sum over all possible pairs in T, $(t - 1)x(T) > t(t - 1)/2 \cdot 2/n$, and so $x(T) > t/n = 1/2$. Similarly, $x(-T) > 1/2$, and a contradiction results. On the other hand, if we relabel the players so that $T = \{1, 2, \cdots, t\}$ and $-T = \{t + 1, t + 2, \cdots, n\}$ then the pair $(\| 1/n \|, [\{1, t + 1\}, \cdots \{t, n\}])$ is 1-stable. The only 1-critical coalitions S which need be considered have three elements, two in either T or $-T$ and the third in the other. By assumption in this case $m(S) \leqq 3/n$ and so the pair is 1-stable.

A second aspect of games with k-stable pairs (X, Δ_n) is that one can give a plausible argument to show that such pairs are not very likely to occur in a trial and error hunting for a stable state. We shall say that a coalition structure τ is *k-inaccessible* if for every pair (X, τ') such that

i. $m(S) \leqq x(S)$, $S \in \tau'$,

and ii. the coalitions of τ are k-critical coalitions of τ',

then iii. $m(T) \leqq x(T)$ for $T \in \tau$.

In words, we say τ is k-inaccessible if for each pair (X, τ') which is "admissible" (Condition i) and for which the changes involved in going from τ' to τ are all "acceptable within the constraints of k" (Condition ii), then there is no "positive motive" for any of the coalitions in τ to make the move from τ' to τ (Condition iii). It is clear that a sufficient condition for τ to be k-inaccessible is that each of the coalitions of τ be losing, and so Δ_n is k-inaccessible for every k. Thus, within the framework of the implicit dynamic model underlying k-stability theory we must conclude that the players either happen on a 1-stable pair (X, Δ_n) at the start or it will not arise, for they cannot reach it by any sequence of "acceptable" coalition changes involving only "admissible" pairs.

These comments suggest that it would be interesting to characterize the k-inaccessible coalition structures of any game and to distinguish between those k-stable pairs which involve such structures and those which do not.

CENTER FOR ADVANCED STUDY IN THE BEHAVIORAL SCIENCES

REFERENCES

1. LUCE, R. D., *A definition of stability for n-person games*, Ann. of Math., 59 (1954), 357–366.
2. McKINSEY, J. C. C., Introduction to the theory of games, McGraw-Hill, New York, 1952.
3. NEUMANN, J. VON, and MORGENSTERN, O., Theory of games and economic behavior, Princeton University Press, Princeton, 1947.
4. SHAPLEY, L. S., *Quota solutions of n-person games*, Contributions to the theory of games, II, Annals of Mathematics Studies, no. 28, Princeton University Press, 1953, 343–359.