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A DEFINITION OF STABILITY FOR n-PERSON GAMES[1,2]

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1. Introduction

In this paper we shall take an *n-person game* (3, 9), denoted (I_n, v), to be a set I_n of $n > 2$ elements, the *players*, and a real-valued set function v, called the *characteristic function*, having the properties

i. $$v(0) = 0,$$

and

ii. if S and T are disjoint subsets of I_n,

$$v(S \cup T) \geqq v(S) + v(T).$$

If v also satisfies

iii. $$v(T) + v(-T) = v(I_n), \qquad \text{for every } T \subset I_n,$$

the game is called *constant-sum*.

A game is called *inessential* if $\sum_{i \in I_n} v(\{i\}) = v(I_n)$, and its theory is trivial; otherwise, it is called *essential*.

A real n-tuple $X = \| x_1, x_2, \cdots x_n \|$ such that

i. $$\sum_{i \in I_n} x_i = v(I_n),$$

and

ii. $$x_i \geqq v(\{i\}) \qquad \text{for all } i \in I_n$$

is called an *imputation* of the game; it is interpreted as a possible set of payments to the players.

The term *coalition* is used for any non-empty set of players (who are thought to cooperate in the playing of the game); any proper partition of I_n will be called a *coalition structure*.

The symbols I_n, v, X, x_i with and without primes will be used throughout with the above meanings. S and T will denoted subsets of I_n and τ a coalition structure. The number of elements in a set T is denoted by $|T|$.

One major problem of the theory of general games is the introduction of

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[2] I am indebted to the referee for a number of valuable suggestions which led to a significant revision of Section 3. I also wish to thank Mr. Josiah Macy, Jr. for his critical discussions of the work.

suitable requirements, in terms of v, on the class of all imputations in order to isolate a subclass of "acceptable" ones, where the criteria on acceptable are, at best, vague. Several studies in this vein have been published, among them von Neumann and Morgenstern's "solutions" (9, p. 246), Milnor's "reasonable outcomes" (5), Shapley's "value" (6), and his "G-stable sets" (8). We shall begin yet another which is an attempt "To study n-person games with restrictions imposed on the forming of coalitions, . . . thus recognizing that the cost of communication among the players during the pregame coalition-forming period is not negligible but rather, in the typical economic model with large n, is likely to be the dominating consideration."[3]

DEFINITION 1. A coalition T is *X-admissible* if $v(T) \leqq \sum_{i \epsilon T} x_i$.

COMMENT. It is plausible to suppose that an imputation, i.e., a particular set of payments, will occur in the presence of a coalition only if that coalition receives a total payment at least as large as the smallest payment it need ever accept, hence the definition.

DEFINITION 2. If τ is a coalition structure and k a non-negative integer, then any set $(T \cup K) - H$, where $T \epsilon \tau$, $H \subset T$, $K \subset -T$, and $| H \cup K | \leqq k$, is called a *k-critical* coalition of τ.

COMMENT. In this definition we are considering the possibility of revising a coalition structure when there is a limitation on the degree of change that can occur. If we take this number to be k, and T is a coalition under consideration, then the total number of players added to T, $| K |$, plus the total number expelled from T, $| H |$, must not exceed k. It should be noted that the 0-critical coalitions of τ are the elements of τ itself.

DEFINITION 3. Let k be an integer such that $1 \leqq k \leqq n - 2$. A pair (X, τ) is *k-stable* if every k-critical coalition of τ is X-admissible and if $v(\{i\}) < x_i$ for all $i \epsilon T$ such that $T \epsilon \tau$ and $| T | > 1$. A game is *k-stable* if there is at least one k-stable pair for the game, otherwise it is called *k-unstable.*

COMMENT. For a pair to be k-stable we require that there be no positive gain assured any of the conceivable coalitions which could be formed from τ within the maximum allowable changes, in the sense of Definition 2. Further, we suppose that an individual will not trouble to participate in a non-trivial coalition unless he is assured more than he could obtain alone. The latter requirement is used only in Theorem 5, where it serves to restrict the number of 1-stable pairs.

DEFINITION 4. Two games (I_n, v) and (I_n, v') are *S-equivalent* if there exist a positive constant c and constants a_i, $i \epsilon I_n$, such that for every $T \subset I_n$

$$v'(T) = cv(T) + \sum_{i \epsilon T} a_i .$$

COMMENT. For zero-sum games McKinsey (4) has shown S-equivalence is a suitable formulation of the intuitive concept "strategic equivalence."

[3] Reference 2, p. xi. The editors of this book have listed 14 topics which they feel should be considered in the future development of game theory; I have quoted, in part, the ninth listing.

The constants c and a_i in Definition 4 are unique if the games are essential, for if c' and a'_i are another set, then for any coalition T and $i \in -T$,

$$(c - c')v(T \cup \{i\}) = \sum_{j \in T} (a_j - a'_j) + a_i - a'_i$$
$$(c - c')v(T) = \sum_{j \in T} (a_j - a'_j)$$
$$(c - c')v(\{i\}) = a_i - a'_i .$$

Subtracting the last two equations from the first, we have

$$(c - c')[v(T \cup \{i\}) - v(T) - v(\{i\})] = 0.$$

If $c \neq c'$, $v(T \cup \{i\}) = v(T) + v(\{i\})$, and so by a simple induction $v(I_n) = \sum_{i \in I_n} v(\{i\})$. Thus, (I_n, v) is inessential or $c = c'$. If the latter, it follows from the third equation that $a_i = a'_i$, and so the constants are unique if the game is essential. In footnote 5 we shall give explicit formulas for the constants.

THEOREM 1. *Two S-equivalent games are either both k-stable or both k-unstable.*

PROOF. Suppose (X, τ) is a k-stable pair of (I_n, v) and that (I_n, v') is S-equivalent to (I_n, v), with the constants c and a_i, then we show that $(\| cx_i + a_i \|, \tau)$ is a k-stable pair of (I_n, v'). If T is a k-critical coalition of τ, then

$$v'(T) = cv(T) + \sum_{i \in T} a_i \leqq c \sum_{i \in T} x_i + \sum_{i \in T} a_i = \sum_{i \in T} (cx_i + a_i).$$

If $T \in \tau$ and $| T | > 1$ and $i \in T$,

$$v'(\{i\}) = cv(\{i\}) + a_i < cx_i + a_i .$$

THEOREM 2. *Let k and k' be positive integers with $k < k'$. If a game is k'-stable it is k-stable and if it is k-unstable it is k'-unstable.*

PROOF. Trivial.

The paper consists of three parts. In the next section we show that all $(n - 2)$-stable constant-sum games are inessential and this is used to determine all k-stable 3- and 4-person constant-sum games. In Section 3 we generalize the concept of a simple game and characterize the k-stable simple games. Using this it is shown that the theory of 1-unstable simple games is identical to the theory of 1-unstable 3-person simple games, and it is an immediate consequence that there exist k-unstable games for all n and all $k \leqq n - 2$. The 1-stable pairs of a simple game are displayed, which is of some interest since it presents some of the phenomena encompassed by the definition. In the final section we present a class of games which are k-stable for all k, $1 \leqq k \leqq n - 2$.

2. k-Stable 3- and 4-Person Constant-Sum Games

THEOREM 3. *Any $(n - 2)$-stable constant-sum game is inessential.*

PROOF. Suppose (X, τ) is an $(n - 2)$-stable pair of (I_n, v). For any $T \in \tau$ and $j \in -T$, $| -T - \{j\} | \leqq n - 2$, so, using the $(n - 2)$-stability and constant-sum assumptions,

$v(T \cup [-T - \{j\}]) = v(-\{j\}) = v(I_n) - v(\{j\}) \leqq \sum_{i \epsilon -\{j\}} x_i = v(I_n) - x_j$.

Thus, $v(\{j\}) \geqq x_j$, but for any imputation $v(\{j\}) \leqq x_j$, so for all $j \epsilon I_n$, $v(\{j\}) = x_j$. Summing,

$$\sum_{j \epsilon I_n} v(\{j\}) = \sum_{j \epsilon I_n} x_j = v(I_n),$$

and the game is inessential.

This theorem is not generally true as the following non-constant-sum game shows.

$$v(\{i\}) = -1, \quad i = 1, 2, 3, \qquad v(\{1, 2, 3\}) = 0,$$
$$v(\{1, 2\}) = v(\{1, 3\}) = 1, \qquad v(\{2, 3\}) = -2.$$

The pair $(\| 2, -1, -1 \|, [\{1\}, \{2\}, \{3\}])$ is 1-stable in this game. Furthermore, even if the constant-sum assumption is retained, the theorem cannot be extended to $k < n - 2$, as the following example shows.

$$v(T) = \begin{cases} -1 \\ 1 \quad \text{when } |T| = \\ 0 \end{cases} \begin{cases} 1 \\ n-1 \\ \text{all other values.} \end{cases}$$

The addition of $n - 3$ or fewer elements to any coalition of the pair $(\| 0, 0, \cdots 0 \|, [\{1\}, \{2\}, \cdots \{n\}])$, i.e. to any element, results in a set T having at most $n - 2$ members, so $v(T) \leqq 0 = \sum_{i \epsilon T} x_i$. Thus, the pair is $(n - 3)$-stable.

It is immediate from Theorem 3 that there are no essential 1-stable 3-person constant-sum games, nor any 2-stable constant-sum 4-person games. The possibility of essential 1-stable constant-sum 4-person games remains; we shall give the 1-stable pairs for the reduced $(v(I_n) = 0$ and $v(\{i\}) = -1)$ cases. The essential reduced constant-sum 4-person games may be shown to be of the following form (9, p. 291):

$$v(T) = \begin{cases} 0 \\ -1 \\ 1 \\ 0 \end{cases} \text{when } |T| = \begin{cases} 0 \\ 1 \\ 3 \\ 4 \end{cases} \text{and } \begin{aligned} v(\{1, 4\}) &= 2y_1 = -v(\{2, 3\}) \\ v(\{2, 4\}) &= 2y_2 = -v(\{1, 3\}) \\ v(\{3, 4\}) &= 2y_3 = -v(\{1, 2\}) \end{aligned}$$

where $-1 \leqq y_i \leqq 1, \quad i = 1, 2, 3.$

Let[4]

$$Q = \| y_1 - y_2 - y_3,\ y_2 - y_1 - y_3,\ y_3 - y_1 - y_2,\ y_1 + y_2 + y_3 \|.$$

By a simple exhaustive examination of cases, the following (X, τ) pairs may be shown to be the only 1-stable ones.

i. $(Q, [\{1\}, \{2\}, \{3\}, \{4\}])$ if the components of Q are all $\geqq -1$.

[4] The referee has pointed out that Q is the "quota" defined by Shapley (7) and that it is easy to show that in any quota game without a weak player $(Q, [\{1\}, \{2\}, \cdots \{n\}])$ is a 1-stable pair if Q is the quota.

ii. $(Q, [\{1, 2\}, \{3\}, \{4\}])$ if $y_1 = -y_2$, $-1 < y_1 < 1$, and $y_3 = -1$.
In addition, all cases that arise from this by permutations on I_4 are 1-stable.

iii. $(Q, [\{1, 2, 3\}, \{4\}])$ if $y_1 + y_2 + y_3 = -1$ and the other components of Q are > -1. As in ii, all cases arising from permutations of I_4 are 1-stable.

3. Simple Games

For an essential game the transformation

$$m(T) = \frac{v(T) - \sum_{i \in T} v(\{i\})}{v(I_n) - \sum_{i \in I_n} v(\{i\})}$$

is a real-valued non-negative set function satisfying

i. $$m(I_n) = 1,$$

ii. if S and T are disjoint subsets of I_n,

$$m(S \cup T) \geqq m(S) + m(T),$$

iii. $$m(\{i\}) = 0 \qquad \text{for } i \in I_n,$$

iv. $$m(0) = 0.$$

If the game is also constant-sum we have

v. $$m(T) = 1 - m(-T) \qquad \text{for every } T \subset I_n.$$

It is easy to show that two n-person games have the same normalized superadditive m measure if and only if they are S-equivalent,[5] thus, by Theorem 1, it is sufficient to deal with (I_n, m) in the study of k-stability. Any measure m satisfying i through iv above is a 0, 1 reduced form of a game which can be transformed into the usual -1, 0 form by

$$v(T) = nm(T) - |T|.$$

The corresponding transformation on imputations,

$$p_i = \frac{x_i - v(\{i\})}{v(I_n) - \sum_{j \in I_n} v(\{j\})},$$

has the properties $\sum_{i \in I_n} p_i = 1$ and $p_i \geqq 0$ for $i \in I_n$, i.e. $P = \| p_i \|$ is a probability distribution over I_n.

The advantages of this reduced form are three: the relation of game theory to

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[5] This, incidentally, gives an explicit method to check whether two essential games (I_n, v) and (I_n, v') are S-equivalent, for the unique set of constants in Def. 4 are:

$$c = \frac{v'(I_n) - \sum_{i \in I_n} v'(\{i\})}{v(I_n) - \sum_{i \in I_n} v(\{i\})} \qquad \text{and} \qquad a_i = v'(\{i\}) - cv(\{i\}).$$

measure theory is clearer, certain definitions become transparently simple, and no distinction is made among S-equivalent games.

It should be noted that Definitions 1 and 3 remain absolutely unchanged if v is replaced by m and the X's by the corresponding P's.

DEFINITION 5. A coalition T is called *losing* whenever $m(T) = 0$ and *winning* whenever $m(T) = 1$. An essential game is *simple* if every coalition is either winning or losing.

COMMENT. Von Neumann and Morgenstern (9, pp. 423–428) used the term *flat* for what we have called losing, but in simple games they called flat sets losing and introduced the word winning for the complement of a losing coalition. For zero-sum games they introduced the concept of simple in a different manner; we have simply translated their definition into the m notation and dropped the zero-sum requirement. Among the games included in the more general definition are the (n, k) games of Bott (1).

It is clear that in any game a subset of a losing coalition is losing, that a superset of a winning coalition is winning, that any one element coalition is losing, and that if T is winning $-T$ is losing. It is not true, in general, that if T is losing $-T$ is winning. This property, which in effect was the defining property used by von Neumann and Morgenstern for simple zero-sum games, holds in a simple game, as we have defined it, if and only if it is constant-sum.

If T is a losing coalition, $m(T) = 0 \leqq \sum_{i \epsilon T} p_i$, so it is clear that the first condition of stability is met by a k-critical losing coalition. If T is winning, $m(T) = 1 \geqq \sum_{i \epsilon T} p_i$. Thus, if T is k-critical and winning and if the conditions of stability are met, the equality must hold, so $p_i = 0$ for all $i \epsilon -T$; and if $p_i > 0$ for $i \epsilon -T$, the conditions for k-stability are violated.

THEOREM 4. *A sufficient condition for a game to be k-unstable, which for simple games is also a necessary condition, is that there exists a $(k + 1)$-element winning coalition and the intersection of all $(k + 1)$-element winning coalitions is vacuous.*

PROOF. Let $S_1, S_2, \cdots S_s$ be the set of $(k + 1)$-element winning coalitions and suppose (P, τ) is a k-stable pair, where $\tau = (T_1, T_2, \cdots T_t)$. If S_i intersects two elements of τ, say T_q and T_r, then both $T_q \cup S_i$ and $T_r \cup S_i$ are winning and k-critical, so

$$m(T_q \cup S_i) = 1 = \sum_{j \epsilon T_q \cup S_i} p_j = m(T_r \cup S_i) = \sum_{j \epsilon T_r \cup S_i} p_j .$$

Now, suppose that there is $\sigma \epsilon T_q - S_i$ such that $p_\sigma > 0$. Since $T_q \cap T_r = 0$, $\sigma \notin T_r - S_i$, so $\sum_{j \epsilon T_r \cup S_i} p_j < 1$, which is impossible. Thus, we have $\sum_{j \epsilon S_i} p_j = 1$. If, however, $S_i \subset T_r$, then $\sum_{j \epsilon S_i} p_j = 1$ except possibly if $T_r - S_i \neq 0$. Suppose there is $\sigma \epsilon T_r - S_i$ such that $p_\sigma > 0$. Observe that $T_r - \{\sigma\} \supset S_i$ is winning and k-critical for every k, so we obtain the contradiction $m(T_r - \{\sigma\}) = 1 > 1 - p_\sigma = \sum_{j \epsilon T_r} p_j - p_\sigma = \sum_{j \epsilon T_r - \{\sigma\}} p_j$. But S_i was any of the winning coalitions, so we obtain the absurdity $0 = \sum_{j \epsilon \cap_i S_i = 0} p_j = 1$, and thus we must conclude that the game is k-unstable.

Suppose (I_n, m) is simple and k-unstable, then there is at least one $(k + 1)$-element winning coalition, otherwise $(\| 1/n \|, [\{1\}, \{2\}, \cdots, \{n\}])$ is a k-

stable pair. This follows from the fact that if every $(k + 1)$-element coalition is losing, then every k-critical coalition is also losing. Now, suppose there is an element common to all $(k + 1)$-element winning coalitions, say n, then $(\| 0, 0, \cdots, 0, 1 \|, [\{1\}, \{2\}, \cdots \{n\}])$ is k-stable since every coalition T such that $| T | \leqq k + 1$ is either losing and so $m(T) = 0 \leqq \sum_{i \epsilon T} p_i$ or it is winning and so contains n, in which case $m(T) = 1 = \sum_{i \epsilon T} p_i$.

COROLLARY 1. *If a game is simple, it is k-stable if and only if either there is no $(k + 1)$-element winning coalition or the intersection of all $(k + 1)$-element winning coalitions is non-empty.*

PROOF. This is a simple restatement of part of the theorem.

COROLLARY 2. *The 1-stable simple constant-sum 4-person games are those with a player i such that $\{i, j\}$ is winning for all $j \neq i$ (or, in the notation of Section 2, those for which either one or three of the $y_i = 1$ and the others $= -1$). In these cases, the only 1-stable pair is case i of Section 2.*

PROOF. A 4-person constant-sum game is simple if and only if $y_i = \pm 1$, and so there are always three 2-element winning coalitions. If the game is 1-stable it follows from corollary 1 that these coalitions have a common element, hence the first assertion. It is readily verified that this is equivalent to the assertion that one or three of the $y_i = 1$. Of the 1-stable pairs, case iii obviously cannot occur and case ii is eliminated by the condition $-1 < y_1 < 1$, which violates the assumption that the game is simple.

THEOREM 5. *Let (I_n, m) be 1-stable and simple and let R be the intersection of all 2-element winning coalitions, which by relabelling we may take to be $R = \{1\}$ or $\{1, 2\}$. The 1-stable pairs of the game are:*

i. *if* $R = \{1\}$, $(\| 1, 0, \cdots, 0 \|, [\{1\}, \{2\}, \cdots \{n\}])$;

ii. *if* $R = \{1, 2\}$, $(\| p, 1 - p, 0, \cdots, 0 \|, [\{1\}, \{2\}, \{3\}, \cdots, \{n\}])$, where $0 \leqq p \leqq 1$,
and $(\| p, 1 - p, 0, \cdots, 0 \|, [\{1, 2\}, \{3\}, \cdots, \{n\}])$, where $0 < p < 1$;

iii. *if there are no winning 2-element coalitions, any (P, τ) with $p_i > 0$ for $i \epsilon T \epsilon \tau$ where $| T | > 1$ and such that either*
  a. *every 1-critical coalition of τ is losing,*
  or b. $\tau = (\{1\}, \{2\}, \cdots, \{s\}, T)$ *and*
    1. *every 1-critical winning coalition is of the form $T \cup \{i\}, i \epsilon I_n$,*
    and 2. *If $T \cup \{i\}, i \epsilon I_n$, is winning, then for every $j \epsilon -(T \cup \{i\})$, $p_j = 0$.*

PROOF. First, consider cases i and ii. It is clear that these pairs are 1-stable and from the first part of the proof of Theorem 4 it follows that they are the only 1-stable pairs. The remainder of the proof is restricted to the case where there are no winning 2-element coalitions.

From the remarks preceding Theorem 4 it is clear that any pair falling in case a is 1-stable. In case b, if $T \cup \{i\}, i \epsilon I_n$, is winning then condition b.2 implies $m(T \cup \{i\}) = \sum_{j \epsilon T \cup \{i\}} p_j$. By condition b.1 there are no other 1-critical winning coalitions so the pair is 1-stable.

Conversely, suppose (P, τ) is a 1-stable pair. First, if there is a $T \epsilon \tau$ which is winning, then, by the remarks preceding Theorem 4, $p_i = 0$ for all $i \in -T$. This and the second part of Definition 3 imply that, for some labelling of the players, $\tau = (\{1\}, \{2\}, \cdots, \{s\}, T)$, so condition b.2 is met. We turn to condition b.1. Since T is winning, $| T | > 1$, so for any $i \in T$, $p_i > 0$. Thus, $T - \{i\}$ must be a losing coalition. By assumption, $\{i, j\}$, $i \in -T$ and $j \in I_n$, is not winning, so condition b.1 is met.

We may, therefore, assume every $T \in \tau$ is a losing coalition. It is clear that if $T \cup \{i\}$ is a losing coalition for every $T \in \tau$ and $i \in -T$ then condition a is met, so we assume that for some T and some $i \in -T$, $T \cup \{i\}$ is a winning coalition. It then follows that $p_j = 0$ for $j \in -(T \cup \{i\})$ and so there are no coalitions in $-T$ aside from single elements, hence condition b.2 is met. It follows immediately that b.1 is also met.

DEFINITION 6. If there is a set T such that

$$m(S \cap T) + m(S - T) = m(S)$$

for all $S \subset I_n$, the game (I_n, m) is said to be *decomposable*, into games on the sets of players T and $-T$ (9, p. 342). If $m(T) = 0$, the game on T is inessential; if $m(T) > 0$, the game on T is $(T, m(S)/m(T))$. If T consists of a single player, he is called a *dummy*.

It is easily seen that a decomposable game is simple if and only if one component game is simple and the other is inessential.

THEOREM 6. *An n-person game, $n \geqq 3$, is 1-unstable and simple if and only if it is decomposable into an essential 3-person constant-sum game and an $(n - 3)$-person inessential game.*

PROOF. Suppose (I_n, m) is 1-unstable and simple, then by Theorem 4 we know that there exist at least two 2-element winning coalitions. Let L be the intersection of the complements of all 2-element winning coalitions. For $n > 3$, $L \neq 0$, for if it were empty then the set of all 2-element winning coalitions would span I_n, which coupled with the fact that their intersection is vacuous implies, for $n \geqq 4$, there are two, S and T, without a common element. Thus, $-S$ is a losing coalition since S is winning, but since $T \subset -S$, $-S$ is winning, a contradiction.

Let T be a losing coalition, we show $T \cup L$ is also losing. This is trivial if $T \subset L$, so we suppose $T - L \neq 0$. If there is a winning coalition $\{i, j\}$ such that $i, j \in -(T \cup L)$, then since $-\{i, j\}$ is losing, $T \cup L$ is losing. We may therefore suppose every winning 2-element coalition is of the form $\{i, j\}$ with $i \in -(T \cup L)$ and $j \in T - L$. Furthermore, for every such i there exists a $j \in T - L$, and for every such j there exists an $i \in -(T \cup L)$, such that $\{i, j\}$ is winning, otherwise L does not have the maximal property of its definition. In order that all 2-element winning coalitions do not have a common element, it is necessary that $| -(T \cup L) | \geqq 2$ and $| T - L | \geqq 2$. Thus, the 2-element winning coalitions span the 4 or more elements in $-L$ and, as above, it follows there are two without a common element, and the same contradiction results. Thus, $T \cup L$ is losing and so for any $i \in L$, $m(T \cup \{i\}) = m(T)$ if T is losing. The same equation holds if T is winning since $T \cup \{i\}$ is also winning. It therefore follows easily that for any T,

$$m(T) = m(T - \{i\}) + m(T \cap \{i\}),$$

and so i is a dummy.

The game on $-\{i\}$ is also 1-unstable for by the choice of i all 2-element winning coalitions remain, so by a repeated induction we may isolate and remove dummies until $n = 3$. Thus, (I_n, m) is decomposable into a 1-unstable simple 3-person game and an inessential game. It is clear from Theorem 4 that a 3-person game is 1-unstable and simple if and only if it is constant-sum and essential.

The converse is immediate since if (I_{n-1}, m) is 1-unstable and simple, (I_n, m'), where

$$m'(T) = m(T \cap I_{n-1}),$$

is a simple game, formed by adding a dummy to (I_{n-1}, m), which has exactly the same 2-element winning coalitions as (I_{n-1}, m); hence, it is 1-unstable by Theorem 4.

COROLLARY 1. *Every simple non-constant-sum game is* 1-*stable, and every simple constant-sum game is* 1-*stable except those which are decomposable into the* 3-*person constant-sum game and an inessential game.*

PROOF. This follows immediately from the theorem provided that a decomposable game is constant-sum if and only if its components are constant-sum. This is readily verified from Definition 6.

COROLLARY 2. *For every* $n \geqq 3$ *and* k, $1 \leqq k \leqq n - 2$, *there exists a* k-*unstable* (*simple and constant-sum*) *game.*

PROOF. By Theorem 2 it is sufficient to show this for $k = 1$, and this has been done in Theorem 6.

The result of Theorem 6—the decomposition of a 1-unstable game into a 3-person game and an inessential game—does not hold in general for non-simple games, as the following example shows. Consider the 4-person constant-sum game in which $y_1 = y_2 = 1$, $y_3 = 0$. Then $Q = \| 0, 0, -2, 2 \|$ is not an imputation, so the game is 1-unstable by the results of Section 2. Further, for each $i \in I_4$ the equation $v(T) = v(T - \{i\})$ is violated:

i	T	$v(T)$	$v(T - \{i\})$
1	$\{1, 2, 3\}$	1	-1
2	$\{1, 2, 3\}$	1	-1
3	$\{1, 2, 3\}$	1	0
4	$\{2, 3, 4\}$	1	-1

Thus, this 1-unstable 4-person game is not decomposable into a 3-person game and a dummy.

4. Negative Games

DEFINITION 7. An essential game (I_n, m) is called *negative* if $m(T) \leqq |T|/n$ for every $T \subset I_n$.

COMMENT. The term negative is motivated by the fact that the condition implies $v(T) = nm(T) - |T| \leqq 0$, where v is the usual reduced form.

THEOREM 7. *Every negative game is k-stable for* $1 \leqq k \leqq n - 2$.

PROOF. We shall prove the slightly stronger statement that there exist distributions P such that for every coalition structure τ and integer k, $1 \leqq k \leqq n - 2$, (P, τ) is a k-stable pair. Let $\sigma_i = \min_{i \in T \subsetneqq I_n} [1/n - m(T)/|T|]$. Let P satisfy $p_i \geqq 1/n - \sigma_i$ and $p_i > 0$. Such a distribution exists, e.g. $p_i = 1/n$, since the game is negative and so $\sigma_i \geqq 0$. Let $\delta(T) = \max_{i \in T} \sigma_i$. Then,

$$m(T) \leqq \frac{|T|}{n} - \delta(T)\,|T| \leqq \sum_{i \in T}\left(\frac{1}{n} - \sigma_i\right) \leqq \sum_{i \in T} p_i .$$

Thus, the pair is k-stable since T is arbitrary.

COROLLARY. *A negative game is not constant-sum*

PROOF. Theorem 7 and Theorem 3.

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