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NETWORKS SATISFYING MINIMALITY CONDITIONS.*[1,2]

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1. Introduction. A *network* N is a system of two finite sets M and $P \subset M \times M$, in which the elements $a, b, \cdots \varepsilon M$ are called the *nodes* and the elements $(ab), (ca), \cdots \varepsilon P$ are called the *links* of N. The number of nodes is denoted by m and the number of links by $p(N)$. If (ab) is a link of N, a is called the *initial node* and b the *end node* of the link (ab). Thus, a network is a binary relation over a finite set and is also a finite oriented graph in which there is at most one oriented arc from one node to another. Our viewpoint is primarily that of graph theory rather than algebra.

Let I be the set of all links of the form $(aa), a \varepsilon M$. If $P \cap I = 0$, N is called *non-reflexive.* We shall, without further mention, take the word network to mean non-reflexive network.

A *subnetwork* N' of a network N, denoted $N' \subset N$, is any network with $M' \subset M$ and $P' \subset P$. If $M' = M$, we say N' is a complete subnetwork of N. If $N' \subset N$, $N - N'$ is the network with nodes M and links $P - P'$ and it is said to be formed from N by the removal of the links P'. If N' has but one link (ab) we write $N - N' = N - (ab)$. Similarly, the network with nodes $M - M'$ and links $P \cap [(M - M') \times (M - M')]$ is said to be formed from N by the removal of the nodes M' (and the incident links). N' is a *supernetwork* of N if N is a complete subnetwork of N'. We shall write in this case $N' = N + (N' - N)$ and say that N' is formed from N by adding the links $P' - P$ to N. If $N' - N$ contains but one link (ab), we write $N' = N + (ab)$.

A pair of links (ab) and (ba) is called an *arc* ab, and any network composed entirely of arcs is isomorphic to a graph and so is called a graph.

A *q-chain* from a to b, denoted (ab, q), is an ordered sequence of q links $(ac_1), (c_1c_2), \cdots, (c_qb)$ in which no node is repeated, except possibly $a = b$. In the latter case the chain is called an (oriented) *circuit.* A network is

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connected if there is a chain from every node to every other node; otherwise it is *disconnected.* Maximal connected subnetworks are called *components.*

In a previous paper [4], which will be referred to as (A), the following definitions were introduced:

A network has *degree* 0 if it is not connected; it has *degree* k, $k > 0$, if there exists $N' \subset N$ such that $p(N') = k$ and $N - N'$ is disconnected, but $N - N''$ is connected for all $N'' \subset N$ such that $p(N'') < k$.

A network N is *k-minimal* if the degree of $N - (ab)$ is $k - 1$ for every $(ab) \varepsilon N$. If N is 1-minimal and connected, it is called *minimal.*

In this paper we are concerned with three independent results which are each related to k-minimality. The definition is extended in a natural way to disconnected networks in Section 2 and these networks are completely characterized by Theorem 1. It is worth mentioning that the characterization problem for connected networks appears to be far more difficult. (The principal result of (A) is the solution to that problem for $k = 1$). In Section 3, the principal result is Theorem 4 which states that in a network of degree k, there is a set of at least k chains from any node to any other node, no two of which have a common link. This result is a generalization of a close analogue to the well known theorem of Menger that between any two nodes of a graph without a cut-node there are at least two chains that have no intermediate nodes in common. In the final section we turn to a generalization of transitivity. Connectedness and transitivity are each such strong requirements that combined they single out but one network—the case $P = M \times M$—so, in the presence of connectedness, transitivity must be weakened to be of interest. We require that every chain exceeding h links is "short-circuited" by a link, and that no chain of h or fewer links is short-circuited. It is shown that these connected networks fall into three classes: one having but one member which is of degree 2, the set of minimal networks, and a set non-minimal networks of degree 1 whose connected subnetworks also have degree 1.

2. $(-k)$-minimal networks.[3] To extend the above definitions of degree and minimality to disconnected networks, we simply interchange the roles of connected and disconnected as follows:

A network N has *degree* $(-k)$, $k \geqq 0$, if there exists a connected supernetwork N' of N such that $p(N' - N) = k + 1$, but every supernetwork N'' such that $p(N'' - N) < k + 1$ is disconnected.

[3] The author is indebted to Anatol Holt who suggested this problem to him.

A network N is (-0)-*minimal* if N is disconnected and for every $(ab) \notin N, a, b \varepsilon N$, $N + (ab)$ is connected; it is $(-k)$-minimal, $k \geqq 1$, if for every $(ab) \notin N, a, b \varepsilon N$, $N + (ab)$ has degree $(-k+1)$.

Following Dirac's terminology ([1], p. 347), we shall call a network with every possible link present a *complete graph,* and any component which is a complete graph is simply called complete. A node which is neither an end node nor an initial node of any link is called an *isolated node.*

LEMMA 1. *If N is a $(-k)$-minimal network with $m \geqq 3$ and $k \geqq 2$ and a is an isolated node of N, then $N' = N - a$ is either $(-k+1)$-minimal or a complete graph.*

Proof. If N' is not a complete graph, then since $m \geqq 3$, there exist $b, c \varepsilon N'$ such that $(bc) \notin N'$. For any such b and c, consider $N^* = N' + (bc)$. Let $-q$ be the degree of N^*, then the lemma is proved if we show $q = k - 2$. Let U be any set of k links which connects $N + (bc)$, and observe, since a is isolated, there exist $e, f \varepsilon N'$ such that $(ea), (af) \varepsilon U$. Now, $U + (ef) - (ea) - (af)$ connects N^*, so $q \leqq k - 2$. Suppose $q < k - 2$ and let U' be a set of $q + 1$ links which connects N^*. U' is non-empty, for otherwise N^* is connected, whence $N + (ba)$ is connected by adding (ac), and this implies N *is* (-1)-minimal, which contradicts $k \geqq 2$. Let $(e'f') \varepsilon U'$, then $U' - (e'f') + (e'a) + (af')$ connects $N + (bc)$ using only $(q+1) + 1 < k$ links, which is a contradiction.

THEOREM 1. *A network N is $(-k)$-minimal if and only if either*

(i) *N is a graph which consists of $k+1$ complete components having no link between any pair, or*

(ii) *N consists of a set X of nodes which form $k+1$ complete components having no link between any pair and a complete component Y such that either*

1. $(xy) \varepsilon N$ *and* $(yx) \notin N$ *for all* $x \varepsilon X$ *and* $y \varepsilon Y$,

or

2. $(yx) \varepsilon N$ *and* $(xy) \notin N$ *for all* $x \varepsilon X$ *and* $y \varepsilon Y$.

Proof. The sufficiency is obvious.

The condition is clearly necessary for $k = 0$, so we restrict the proof to $k \geqq 1$. Let N' be any component of N. If N' is an isolated node, it is complete. If N' has more than one node, we show it is complete: If there exist $a, b \varepsilon N'$ such that $(ab) \not\varepsilon N'$, and if U is any set of k links which connects $N + (ab)$, then for any $(cd) \varepsilon U$, $U - (cd)$ connects $N + (cd)$, since N' is already connected. This contradicts the assumption that N is $(-k)$-minimal.

If N', N'' are two components of N, we show that if $a \varepsilon N'$, $b \varepsilon N''$, and $(ab) \varepsilon N$, then $(a'b') \varepsilon N$ for any $a' \varepsilon N'$, $b' \varepsilon N''$: Suppose $(a'b') \not\varepsilon N$, and let U be any set of k links connecting $N + (a'b')$. U connects N since N' and N'' are complete and $(ab) \varepsilon N$, which contradicts the assumption that N is $(-k)$-minimal.

Since the components of N are complete and since if there is one link from N' to N'' there are all possible links, it is sufficient to prove the theorem for networks having no components with more than one node.

First, $m \geqq k + 1$, for if not N can be connected as a circuit on all nodes with fewer than $k + 1$ links. If $m = k + 1$, no links are present, for if there were N could again be connected as a circuit using no more than $m - 1 = k$ links. In this case, N satisfies part (i) of the statement.

For networks with $m \geqq k + 2$, an induction on m will be used to show part (ii) holds. For $m = 3$, it is clear this is the case. Suppose $m > 3$ and part (ii) holds for $m' = m - 1$. If N has an isolated node a, then by Lemma 1, $N - a$ is $(-k+1)$-minimal, so by the induction hypothesis (ii) holds for $N - a$, since (i) cannot. Thus, there exists a node $d \varepsilon N - a$ such that for any other node $c \varepsilon N - a$, exactly one of (cd) and $(dc) \varepsilon N$. Suppose, without loss of generality, $(cd) \varepsilon N$. Then, $N + (ad)$ has degree $(-k)$ and $N + (da)$ has degree $(-k+1)$, which is a contradiction, so N has no isolated nodes.

Divide the nodes of N into three classes: $X =$ set of initial nodes, $Y =$ set of end nodes, and $Z =$ set of nodes which are both initial and end nodes. Let these sets have q, p, and $m - q - p$ members respectively. It is simple to see that if $q = 0$ or $p = 0$ there is a connected subnetwork of N, which is impossible. Suppose $q \geqq p$.

Since the nodes of X terminate no links, at least q links will have to be added to N to produce a connected supernetwork. We shall now show that q links suffice. There are maximal subsets X_1 and Y_1 such that there is a $1:1$ correspondence $x_i \varepsilon X_1$, $y_i \varepsilon Y_1$, $i = 1, 2, \cdots, s$, and $(x_i y_i) \varepsilon N$. This follows from the fact that neither X nor Y are empty and from any $x \varepsilon X$ there is either a link to a $y \varepsilon Y$ or a chain via Z to a $y \varepsilon Y$. But in the latter case, $(xy) \varepsilon N$ for if not then N can be connected by the same set of links which connect $N + (xy)$.

The addition of the s links $(y_1 x_2), (y_2 x_3), \cdots, (x_s y_1)$ to N creates a connected network on the nodes $X_1 + Y_1$. Let $\xi \varepsilon X - X_1$ and $\eta \varepsilon Y - Y_1$; then $(\xi\eta) \not\varepsilon N$, since X_1, Y_1 are maximal. Thus, if $\xi \varepsilon X - X_1$, there exists $y \varepsilon Y_1$ such that $(\xi y) \varepsilon N$, and if $\eta \varepsilon Y - Y_1$, there exists $x \varepsilon X_1$ such that $(x\eta) \varepsilon N$. Now, from each of the $p - s$ nodes of $Y - Y_1$ introduce links to the nodes of $X - X_1$ such that no two terminate on the same node; this is possible since $q \geqq p$. To each of the remaining nodes of $X - X_1$, if any, introduce a link from a node of Y_1. It is easy to see the resulting network is connected and that $s + (q - s) = q$ links have been added.

If $z \varepsilon Z$ and $y \varepsilon Y$, then $N + (yz)$ still requires the addition of q links to connect, so $Z = 0$ and $p = m - q$. If p were > 1, then for $y_1, y_2 \varepsilon Y$, $N + (y_1 y_2)$ would also require the addition of q links to connect; hence $p = 1$ and $q = k + 1$. Thus, $m = q + 1 = k + 2$.

If $p \geqq q$, a similar argument applies.

3. Analogue to Menger's theorem. In graph theory, a node of a connected graph is called a *cut-node* if its removal, along with the incident arcs, results in a graph having two or more components. We generalize this notion: a set of nodes of a connected network is called a *cut-set* if it is one of the smallest sets of nodes whose removal, along with the incident links, results in a disconnected network. If the cut-sets of a network each have κ members, we say the network has *index* κ. It is clear that every connected network has a unique index κ, that $1 \leqq \kappa \leqq m - 1$, and that a connected graph has a cut-node if and only if the index is 1.

The notions of index and degree are parallel with respect to the removal of nodes and links, and so presumedly their values cannot be completely independent. Our first result establishes a relation between them.

THEOREM 2. *Let a connected network on m nodes have degree k and index κ, then $\kappa \leqq k \leqq (m - 1 + \kappa)/2$.*

Proof. To show the left side of the inequality we prove: If a connected network N has degree $k < m - 1$ and $(a_j b_j)$, $j = 1, 2, \cdots, k$, are a set of links whose removal disconnects N, then there exists a set of nodes c_j, $j = 1, 2, \cdots, k$, with $c_j = a_j$ or b_j, such that their removal results in a disconnected network. The c_j are not necessarily distinct.

For $m = 2$ this is obvious.

Consider $m \geqq 3$ and $k = 1$. Let c and d be two nodes such that there is no chain from c to d in $N' = N - (a_1 b_1)$. Let M_d consist of d and any node i such that there is a chain from i to d in N', and let $M_c = M - M_d$.

Since $m \geqq 3$ one of these sets has more than one member and neither is empty since $c \varepsilon M_c$ and $d \varepsilon M_d$. If either set, say M_c, has but one member, then the other contains either a_1 or b_1, say b_1. But there is no chain from c to $M_d - b_1$. If both sets have two or more members, remove either a_1 or b_1 and there is no chain between the resulting sets.

For $m \geqq 3$ and $k > 1$ we use an inductive argument. Remove the link $(a_k b_k)$ to obtain N' having degree $k - 1$. By the induction hypothesis there exists a set of no more than $k - 1$ nodes c_j, with $c_j = a_j$ or b_j, $j = 1, 2, \cdots, k - 1$, whose removal from N' results in a disconnected network N''. If either a_k or $b_k \not\varepsilon N''$ we are done; otherwise, call $N'' + (a_k b_k) = N^*$. Since $k < m - 1$, $m^* \geqq m - (k - 1) \geqq 3$, so if N^* is connected we may apply the $k = 1$ case to show that either the removal of a_k or b_k disconnects N^*. Thus, for $k < m - 1$, $\kappa \leqq k$; however, $\kappa \leqq k$ trivially when $k = m - 1$, so we are done.

We now show the right half of the inequality. If $\kappa = m - 1$, it is trivially true, so we suppose $\kappa < m - 1$. Let S be a cut-set of N and call the resulting disconnected network N'. Let $M' = M - S$. M' consists of more than one node since $m - \kappa > 1$, hence there exist $c, d \varepsilon M'$ such that there is no chain from c to d in N'. Let M_c and M_d be defined as in the first half of the proof. One or the other contains no more than half of the nodes of M', i. e., no more than $(m - \kappa + 1)/2$ nodes. Without loss of generality, we may suppose M_c is the smaller set. In N consider the removal of the links (ci) where $i \varepsilon M_c + S$, which are no more than $(m - \kappa + 1)/2 + \kappa - 1 = (m - 1 + \kappa)/2$ in number. Clearly the resulting network is not connected because there is no link for which c is the initial node, which concludes the proof.

The right inequality is weak and may be improved by relating the degree to the diameter of a network. Let δ_{ab} be the shortest chain from a to b in a connected network, then $\delta = \max_{a,b} \delta_{ab}$ is called the *diameter* of the network.

THEOREM 3. *For a connected network of diameter $\delta > 2$ and degree k, $k \leqq (m - \delta)/2 + 1$.*

Proof. If $\delta = m$, then there is a circuit on the nodes of N such that at least one of the nodes is the initial node of only one link, thus $k = 1 = (m - m)/2 + 1$.

Consider $2 < \delta < m$. Let a and b be two nodes having no chain with fewer than δ links from a to b. If $a \neq b$, there are $\delta + 1$ nodes S in the shortest chain from a to b and $m - \delta - 1$ nodes in $M - S$. If $i \varepsilon M - S$

then not both (ai) and $(ib) \varepsilon N$ since $\delta > 2$. Thus, either a is the initial node of no more than $(m - \delta - 1)/2$ links to $M - S$ or b is the end node of no more than $(m - \delta - 1)/2$ links from $M - S$. Furthermore, a is the initial node of only one link to the nodes of S and b is the end node of only one from S, else there is a chain with fewer than δ links from a to b. Consequently, the removal of at most $(m - \delta - 1)/2 + 1 < (m - \delta)/2 + 1$ links disconnects N.

If $a = b$, S has δ nodes and $M - S$ has $m - \delta$, and by a similar argument $k \leqq (m - \delta)/2 + 1$.

Observe that for $\delta > 2$, Theorem 3 implies the right side of Theorem 2, for $k \leqq (m - \delta + 2)/2 \leqq (m - 1)/2 < (m - 1 + \kappa)/2$.

We turn now to Menger's theorem [3]. It is proved for graphs; however, substantially the same proof holds for networks and so we state it in that form: If a network is connected and has no cut node, i. e., index $\kappa \geqq 2$, then from any node a to any node b there are at least two chains which have no intermediate nodes in common. Because of the parallel definitions of degree and index, one is led to inquire if the following analogue to Menger's theorem is true: If a network has degree $k \geqq 2$, then from any node a to any node b there are at least two chains which have no links in common. It is indeed true; one proof parallels very closely the demonstration given by Dirac for a strengthened form of Menger's theorem; cf. [2], p. 72. We shall not include this proof, for the result is included in the following considerably stronger result.

THEOREM 4. *If a network has degree k, then from any node a to any other node b there is a set of at least k chains such that no two have a common link.*

Proof. We proceed by induction on k; for $k = 1$ the theorem is trivial.

If N has degree $k > 1$, select a k-descendant N' of N (i. e., one of the smallest complete k-minimal subnetworks of N, see p. 705 of (A)). It suffices to show the theorem for N'. Let n be the length of the shortest chain from a to b. If $n = 1$, remove the link (ab) yielding a network of degree $k - 1$, which, by the induction hypothesis, has $k - 1$ chains from a to b with no link in more than one of them. But (ab) is not common to any of them, so there are k chains from a to b in N such that no pair has a common link.

The remainder of the argument is an induction on n with k fixed. Let λ be a chain from a to b of length n and let c be the node of λ immediately preceding b. The shortest chain from a to c has $n - 1$ links, so by the induction hypothesis there exists a set A_1 of k chains from a to c having no link common to any pair. Similarly, there is a set B of k chains from c to b having no link common to any pair. We may suppose that at least one chain of B has a link in common with a chain of A_1, else we are done.

Notation. If g and h are two nodes of a chain λ, let $\lambda(g, h)$ denote the part of λ from g to h.

Suppose $\beta \varepsilon B$ has a link in common with a chain of A_1. Proceed along β opposite to its orientation, i. e., from b toward c, until the first link which is common to a chain, say α, of A_1. Continue further along β until either there is a link common to some $\alpha' \varepsilon A_1$, $\alpha' \neq \alpha$, or until c is reached. Let g be the end node of the common link or c, whichever is appropriate. Observe that α and $\beta(g, b)$ may have several common links. Let h be the first node of α, measured along α from a, such that the links of α and $\beta(g, b)$ for which h is the initial node are different. We call $\beta(h, b)$ the *tail* of β.

The remainder of the proof is concerned with the construction of k chains from a to b which satisfy the conditions of the theorem. Parts of chains in A_1 and B will be used. The construction is expedited by dividing A_1 into a number of classes.

A_1 is given. Suppose A_{j-1}, C_{j-1}, D_{j-1}, E_{j-1}, F_{j-1}, and G_{j-1} to be defined. Then define $A_j = D_{j-1} + E_{j-1}$.

Now, for any $\alpha \varepsilon A_j$, let $\beta^j{}_\alpha$ be the j-th distinct chain of B as measured along α from a, which has a link in common with α. Let $g^j{}_\alpha$ be the first node in α which is initial to a link of $\beta^j{}_\alpha$ which is not also a link of α. Then we define

$$C_j = [\alpha \varepsilon A_j \mid \beta^j{}_\alpha(g^j{}_\alpha, b) \text{ is the tail of } \beta^j{}_\alpha].$$

$$D_j = [\alpha \varepsilon A_j \mid \alpha \varepsilon A_j - C_j, \beta^j{}_\alpha(g^j{}_\alpha, b) \text{ has a link in common with some } \alpha' \varepsilon A_j - C_j, \alpha' \neq \alpha].$$

$$E_j = [\alpha \varepsilon A_j \mid \alpha \varepsilon A_j - C_j, \beta^j{}_\alpha(g^j{}_\alpha, b) \text{ has links in common only with members of } \sum_{\sigma=1}^{j} C_\sigma + \sum_{\sigma=1}^{j-1} F_\sigma \text{ and } \beta^j{}_\alpha \text{ is associated with some } \alpha' \varepsilon \sum_{\sigma=1}^{j} C_\sigma + \sum_{\sigma=1}^{j-1} F_\sigma, \text{ by its defining property}].$$

$$F_j = [\alpha \varepsilon A_j \mid \alpha \varepsilon A_j - C_j, \beta^j{}_\alpha(g^j{}_\alpha, b) \text{ has links in common only with members of } \sum_{\sigma=1}^{j} C_\sigma + \sum_{\sigma=1}^{j-1} F_\sigma, \text{ and } \beta^j{}_\alpha \text{ is not associated with any } \alpha' \varepsilon \sum_{\sigma=1}^{j} C_\sigma + \sum_{\sigma=1}^{j-1} F_\sigma \text{ by its defining property}].$$

$$G_j = [\alpha \varepsilon A_j \mid \text{no } g^j{}_\alpha \text{ exists}].$$

Continue this inductive subdivision of A_1 until $A_\eta \neq 0$ and $A_{\eta+1} = 0$.

Let $W_j = [\alpha(a, g^j{}_\alpha)\beta^j{}_\alpha(g^j{}_\alpha, b) \mid \alpha \in C_j + F_j]$. As above, we shall speak of the β's as being associated with the corresponding α's according to the definition of W_j. Now suppose $\omega \in W_i$ and $\omega' \in W_j$ have a link in common. For simplicity we write $\omega = \alpha\beta$, $\omega' = \alpha'\beta'$, and suppose $i \leqq j$. Either β' has a link in common with α or β with α'. Consider the former case. Certainly $\alpha' \notin C_j$ since β' is not a tail, so $\alpha' \in F_j$. But since β' has a link in common with α, then for some $\rho < i$, $\alpha \in D_\rho + E_\rho$, which implies that β' has a link in common with some $\alpha^* \in A_\rho - C_\rho$ or that β' has already been associated with some $\alpha'' \in \sum_{\sigma=1}^{\rho} C_\sigma + \sum_{\sigma=1}^{\rho-1} F_\sigma$. The latter is impossible since $\alpha' \in F_j$. In the former, continue along β' toward b; there is a last chain $\lambda \in A_\rho - C_\rho$ which has a link in common with β'. Either $\lambda \in C_\rho + F_\rho$ or β' has already been associated with a member of A_1, both of which are impossible. So β' and α do not have a common link. For the second case, in which β has a link in common with α', we may suppose $i < j$ since the case $i = j$ has already been covered. Thus, $\alpha \in F_i$, but since $\alpha' \in A_j \subset A_i - C_i$ and β has a link in common with α', then $\alpha \in D_i$ by definition. This is a contradiction.

Let $W' = \sum_{j=1}^{\eta} W_j$ have r members; we have shown there are r chains from a to b such that no two have a link in common.

Let $G = \sum_{j=1}^{\eta} G_j$ have s members, then we show $s + r = k$. If S is a finite set we denote by $N(S)$ the number of elements in S. By definition

$$A_j = C_j + F_j + G_j + A_{j+1},$$

and since the sets on the right are mutually exclusive,

$$\sum_{j=1}^{\eta} N(A_j) = \sum_{j=1}^{\eta} [N(C_j) + N(F_j)] + \sum_{j=1}^{\eta} N(G_j) + \sum_{j=2}^{\eta+1} N(A_j).$$

By choice, $N(A_{\eta+1}) = 0$ and $N(W_j) = N(C_j) + N(F_j)$, so

$$N(A_1) = k = \sum_{j=1}^{\eta} N(W_j) + \sum_{j=1}^{\eta} N(G_j).$$

But, $W_i \cap W_j = 0$, $G_i \cap G_j = 0$, so $N(W') = \sum_{j=1}^{\eta} N(W_j)$ and $N(G) = \sum_{j=1}^{\eta} N(G_j)$, so $k = r + s$.

Let B' be the set of $\beta \in B$ which have no link in common with any $\omega \in W'$. Clearly, $N(B') = N(A_1) - N(W') = k - r = s$. Thus, we may set up an arbitrary $1:1$ relation between the s elements of G and the s elements of B'.

Denote it by a subscript q and call the set of chains $\alpha_q\beta_q$, $q = 1, 2, \cdots, s$, $\alpha_q \varepsilon G$, $\beta_q \varepsilon B'$, W''. $W = W' + W''$ is a set of k chains from a to b, which we now show concludes the proof.

First, let $\omega' = \alpha'\beta' \varepsilon W'$ and $\omega'' = \alpha''\beta'' \varepsilon W''$. By definition of W'', β'' has no link in common with α'. If β' has a link in common with α'', then since $\alpha'' \varepsilon G$, β' must have been associated with some $\alpha^* \neq \alpha'$, whence $\alpha'\beta' \not\varepsilon W'$. Finally, if $\omega = \alpha\beta$ and $\omega' = \alpha'\beta' \varepsilon W''$ and, say, β has a link in comon with α', then since $\alpha' \not\varepsilon \sum_{\sigma=1}^{\eta} C_\sigma + \sum_{\sigma=1}^{\eta} F_\sigma$, β must have been associated with some α^*, whence $\beta \not\varepsilon B'$. The proof is concluded.

From the analogue to Menger's theorem one may deduce the structure of 2-minimal graphs.

THEOREM 5. *If G is a 2-minimal graph, there exist minimal subnetworks N_1 and N_2 such that $G = N_1 + N_2$ and N_1 and N_2 are (opposite) orientations of G.*

Proof. Let N be a descendant of G and suppose there is an arc $ab \varepsilon G - N$. Let α be a chain of N from a to b and β from b to a. There is a link (cd) of α such that (dc) is in β, for otherwise there are two chains of arcs between a and b in $G - ab$ having only nodes in common. Thus $G - (ab)$ has degree $\geqq 2$, which is contrary to assumption. Let $N' = N + (ab) - (cd)$. If N' is connected it is also minimal since it has the same number of links as N. To show it connected it is sufficient to show a chain from b to a and one from c to d. The chain β from b to a remains and $\beta(c, a)(ab)\beta(b, d)$ exists.

If $G - N'$ has an arc, continue the process until N_1 is obtained such that $G - N_1 = N_2$ is arc-free. N_1 is also arc-free, for if $ab \varepsilon N_1$ then by Theorem 3. 4 of (A) N_1 consists of two disjoint connected subnetworks joined only by ab. But since G is 2-minimal there is another chain of arcs from a to b not including ab, so N_2 has an arc, a contradiction. Since N_1 is arc-free it is an orientation of G, hence N_2 is a connected orientation of G, and so is minimal.

Finally, it should be observed that Theorem 4, a generalization of a result suggested by Menger's theorem, in turn suggests a generalization to his theorem, to wit: *If a network has index κ, then from any node a to any other node b there exists a set of at least κ chains such that no two have a common intermediate node.* Since the proof of Theorem 4 is based on two sets of chains with a common node c, it is evident that no minor modification of that proof will suffice to demonstrate the above statement, and I have been unable to develop a proof of it.

Some interest attaches in either proving it or giving a counter example, for if it is true there are theorems in graph theory (cf. [2], Theorems 1, 4, 5) of the form "If a graph has no cut-node, then . . ." which presumedly can be strengthened to a form "If a graph has index κ, then . . ."

4. *h*-transitive networks. As was pointed out in the introduction, the conditions of transitivity and connectedness result in the single class of networks, the complete graphs, so it is desirable to weaken the transitivity condition. We shall call a network N *h-transitive* if there is at least one chain $(ab, h) \varepsilon N$ such that $a \neq b$, and if for every chain (cd, q) such that $c \neq d$, then $(cd) \varepsilon N$ if $q \geqq h + 1$ and $(cd) \notin N$ if $1 < q \leqq h$. Clearly, $1 \leqq h \leqq m - 2$, and for connected networks, 1-transitivity implies transitivity.

For connected networks, two cases can be distinguished: either there exists a chain of length $\geqq h + 1$, or there does not. In the latter case, it is easy to see that the network is minimal . This case has been discussed in (A), so we shall be interested only in the former case.

The following are a set of examples of non-minimal, h-transitive networks with $m \geqq h + 2 \geqq 5$. Let Q be a set of four nodes 1, 2, 3, and 4, R a set of $h - 2$ nodes distinct from Q labeled $5, 6, \cdots, h + 2$, and S a set of $m - h - 2$ nodes disjoint from $Q + R$. Let the following links be present on $Q + R + S$: (13), (14), (23), (24), (35), (45), (56), $\cdots$, $(h + 1, h + 2)$, $(h + 2, i)$, $(i3)$, $(i4)$, where $i \varepsilon S$. It is not difficult to show these networks satisfy the above requirements.

LEMMA 2. *If N is h-transitive, $h > 1$, and there exists $(ab, q) \varepsilon N$ with $q > h$, then $q = h + 1$.*

Proof. $(ab, q) = (ac)(cb, q - 1)$, and if $q > h + 1$, $q - 1 > h$, so (cb), $(ab) \varepsilon N$. But for $h \geqq 2$, $(ac)(cb) \varepsilon N$ implies $(ab) \notin N$, a contradiction.

In (A) a network was called *uniform* if every connected subnetwork has degree 1. A graph which consists of only a circuit of arcs encompassing all the nodes is called a *circle.*

THEOREM 6. *If a network is connected and h-transitive, $h > 1$, then it is uniform or a circle (which is 2-minimal and for which $h = m - 2$).*

Proof. If N is minimal, it is uniform (p. 704 of (A)).

If N is non-minimal, there is an $h + 1$ chain and, by Lemma 2, it is the longest chain in N. Let its nodes be ordered by the orientation of the chain and $M_1 = \{a, a + 1, \cdots a + h, a + h + 1 = b\}$ and $M_2 = M - M_1$.

If $h < m - 2$, then $M_2 \neq 0$. If $(ba) \varepsilon N$, then a simple induction shows there is a circle on M_1. Then, any link from a node of M_1 to one of M_2 results in an $h + 2$-chain, and at least one such link exists since N is connected. By Lemma 2, this is impossible, so $(ba) \not\varepsilon N$. Let $c \varepsilon M_2$, then $(bc) \not\varepsilon N$ or $(ab, h+1)(bc)$ would be an $h+2$ chain. But since N is connected, there exists at least one $a + i \varepsilon M_1$, $1 \leqq i \leqq h$, such that $(b, a+i) \varepsilon N$. However, for $j > i$, $(b, a+j) \not\varepsilon N$ since $(b, a+i)(a+i, a+i+1) \cdots (a+j-1, a+j)$ is a chain of length no greater than $1 + (h-1) = h$. Therefore, b is the initial node of exactly one link, so N has degree 1.

If $h = m - 2$, then $M_2 = 0$. The only possible links to the node $a + h$ are $(b, a+h)$ and $(a+h-1, a+h)$, since any others produce a chain (ab, q) with $q \leqq h$. Thus, the degree of N is, in this case, no greater than 2.

Suppose N is $(m-2)$-transitive and of degree 2, then we show N is a circle (the converse is trivial). Since $k = 2$, $(b, a+h) \varepsilon N$. Now node $a + h - 1$ must be the end node of at least two links, one being $(a+h-2, a+h-1)$. Of the other two possibilities, $(b, a+h-1)$ and $(a+h, a+h-1)$, the former is excluded because $(b, a+h-1)(a+h-1, a+h)$ implies $(b, a+h) \not\varepsilon N$, contrary to what we have just shown. Proceed inductively and a circle results.

Now consider the non-minimal h-transitive networks of degree 1. Let S be a connected subnetwork of N, and let h' be the length of the longest chain in S. Either $h' = h + 1$ or $h' \leqq h$. In either case, S is h'-transitive, and so the degree of S is 1 except, possibly, if $h' = 1$ or $h' = m' - 2$. If $h' = 1$, then since $h > 1$, $m' = 2$, and so the degree is 1. If $h' = m' - 2$, the only interesting case is degree 2, which, by what we have just seen, implies S is a circle. But, then, $h' = h$, and N is a circle, for $h = m - 2$, else there is an $h + 2$ chain. This is contrary to assumption, so S has degree 1, and N is uniform.

The second example on p. 719 of (A) shows there are uniform networks which are not h-transitive.

COROLLARY. *For $m \geqq 5$, there are no 2-transitive, connected, non-minimal networks.*

Proof. Suppose N is 2-transitive, connected, and non-minimal. Let the nodes of one of the 3-chains be a, $a+1$, $a+2$, b. As in the first part of the above proof, if $m \geqq 5$, there is a link from b to $a+i$, $1 \leqq i \leqq 2$. If $(b, a+1) \varepsilon N$, then $(ab)(b, a+1)(a+1, a+2) \varepsilon N$ implies $(a, a+2) \varepsilon N$, which is impossible. Thus, for N to be connected, $(b, a+2) \varepsilon N$. If $(a+2, a) \not\varepsilon N$, then there is a 3-chain from b to a, which is impossible.

But, $(b, a+2)(a+2, a)(a, a+1) \varepsilon N$ implies $(b, a+1) \varepsilon N$, which we have just shown is impossible. Thus, N does not exist.

THEOREM 7. *Let N be connected, non-minimal, h-transitive, $h > 1$, and not a circle. If N contains an arc ab, then N consists of two connected subnetworks N_a and N_b joined only by the arc ab. Either N_a or N_b is h-transitive and non-minimal, and the other is either minimal or h-transitive and non-minimal.*

Proof. Let $M_a = [i \varepsilon M \mid \text{there exists } (ai, q) \text{ not including } b] + a,$

$$M_b = [i \varepsilon M \mid \text{there exists } (bi, q) \text{ not including } a] + b,$$

$$M_{ab} = M_a \cap M_b,$$

and

$$M'_a = [i \varepsilon M \mid \text{there exists } (ia, q) \text{ not including } b] + a,$$

$$M'_b = [i \varepsilon M \mid \text{there exists } (ib, q) \text{ not including } a] + b,$$

$$M'_{ab} = M'_a \cap M'_b.$$

Observe that $M = M_a + M_b = M'_a + M'_b$. If $i \varepsilon M_a \cap (M - M'_a)$, then there exist $(ai, r) \varepsilon N$ not containing b and all $(ia, s) \varepsilon N$ do contain b, so there exists a chain α from a to b not including (ab). Since $(ab) \varepsilon N$, which is h-transitive, α must be of length $h+1$. Since $(ba) \varepsilon N$, we may use the same induction as in the proof of Theorem 6 to show there is a circle of arcs on the nodes of α. This contradicts the fact that N is uniform. Thus, $M_a \subset M'_a$. Similarly, $M'_a \subset M_a$, so $M_a = M'_a$. In like manner, $M_b = M'_b$. Thus, $M_{ab} = M_a \cap M_b = M_a \cap M'_b$, so by the same argument $M_{ab} = 0$. Similarly, $M'_{ab} = 0$. Let N_a and N_b be the maximal subnetworks of N on M_a and M_b. By what we have just shown they are connected and they have no node in common. They are joined by ab, and no other link exists between them since $M_{ab} = 0$.

Not both N_a and N_b are minimal, for if they were then N would be minimal. Indeed, no $h+1$ chain traverses the arc ab, for if it did, there would exist another link between N_a and N_b. Thus, one of them is h-transitive and non-minimal, and the other is minimal or h-transitive, non-minimal.

The class of non-minimal, h-transitive, uniform networks on m nodes is smaller than the class of minimal networks on m nodes, and the former can be readily obtained from the latter. Observe, if N is h-transitive and non-minimal, it contains a minimal N' as a descendant. N' is $h+1$-transitive, and N is obtained inductively from N' by introducing a link (ab) every time a chain $(ab, h+1)$ appears. It is easy to find examples of minimal networks for which this operation does not result in an h-transitive network, so the class of minimal networks is the larger.

For example, if $m = 5$, it is easy to construct the 15 possible minimal networks using Theorem 3. 4 of (A). Of these, 10 have arcs and in each case the longest chain in the network passes through the arc, so by Theorem 7 they cannot be descendants of an h-transitive non-minimal network. Of the remaining five, one is the circuit which obviously becomes the circle, and one has $h = 3$ which by the corollary to Theorem 3 cannot yield a 2-transitive case. Performing the inductive operation described above on the other three gives the complete graph in two cases and a 3-transitive network in the third case (which is included in the example at the beginning of this section).

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REFERENCES.

[1] G. A. Dirac, "Note on the colouring of graphs," *Mathematische Zeitschrift*, vol. 54 (1951), pp. 347-353.

[2] ———, "Some theorems on abstract graphs," *Proceedings of the London Mathematical Society*, ser. 3, vol. 2 (1952), pp. 69-81.

[3] D. Konig, *Theorie der endlichen und unendlichen Graphen*, New York, Chelsea, 1950.

[4] R. D. Luce, "Two decomposition theorems for a class of finite oriented graphs," *American Journal of Mathematics*, vol. 74 (1952), pp. 701-722.