American Journal of Mathematics, 74, 1952, pp. 701-722.

TWO DECOMPOSITION THEOREMS FOR A CLASS OF FINITE ORIENTED GRAPHS.*

By R. DUNCAN LUCE.

1. Introduction.[1] The object of study in this paper is the class of finite oriented graphs which are subject to the conditions:

i. at most two branches exist between any pair of nodes (vertices), and

ii. whenever two branches do exist between a pair of nodes they shall have the opposite orientation.

Such a system will be called a network. The justification for introducing this word is its wide use in those applied sciences where oriented graphs of this type are playing an important role; for example: electrical networks, sociometric networks or diagrams, abstract programs for digital computers, and the neural networks of mathematical biology.

It is convenient to give a self-contained definition: A *network* N of order m is a system composed of two sets M and P, M being a finite non-empty set of m elements called the *nodes* of N and P a prescribed subset of the set of all ordered pairs of nodes. The members of P (i. e. the oriented branches) are called the *links* of N. The number of links of a network N will be denoted by $p(N)$, or simply by p when no ambiguity can arise. To indicate that N is of order m and has p links we shall write $N = N(m, p)$. Lower case Latin letters such as $a, b, c, \cdots$ will be used for nodes, and bracketed ordered pairs $(ab), (ca), \cdots$ to denote links. If (ab) is a link, the first node, a, will be called the *initial node* and the second, b, the *end node* of the link.

* Received October 2, 1951.

[1] Several of the concepts defined in the introduction have been assigned terms by D. König, *Theorie der Endlichen und Unendlichen Graphen*, New York, Chelsea Publishing Co., 1950. A brief glossary with page references to König is presented:

node	= Knotenpunkt, p. 1
link	= Gerichtete Kante, p. 4
disjoint	= Fremd, p. 3
arc of a network	= Zweifache Kante, p. 93
link of the form (aa)	= Schlinge, p. 3
non-reflexive graph	= Graph im engeren Sinn, p. 4
circuit	= Zyklus, p. 29
chain which is not a circuit	= Bahn, p. 30.

A *subnetwork* N' of a network N is a subset M' of the nodes, M, of N, with P' taken to be some subset (not necessarily proper) of those links of N which are definable on M'. If $M' = M$, we shall say the subnetwork is *complete*. Two subnetworks of a given network are *disjoint* if they have no nodes, and therefore no links, in common.

Each network is obviously a binary relation over a finite set, its nodes, and conversely every binary relation over a finite set can be interpreted as a network. This allows us to present all examples as relation matrices with entries 0 and 1 from the two element Boolean algebra. Furthermore, this suggests that if N and N' are two networks over the same (or isomorphic) set of nodes M, then by $N — N'$ we shall mean the complete subnetwork of N having those links of N which are not links of N'. If N' is a subnetwork of N, and if N' has the set of links P', then by the network formed from N by the *removal* of the links P', we mean $N — N'$. If N' has but one link, (ab), of N, then we shall write $N — N' = N — (ab)$.

We shall call a network *non-reflexive* if there are no links of the form (aa).

In case both the links (ab) and (ba) are present in a network, we shall say that an *arc* ab exists between a and b, the arc consisting of this pair of links, each of which will be said to be a member of the arc. This terminology is justified by the fact that when every link is a member of an arc the network is isomorphic (in the obvious sense of the word) to a graph without 2-circuits, to use a term of Whitney [2]; this is what we shall mean by saying that *a network is a graph*. Observe that the arcs of a network N are not the same as the branches (or arcs) of the graph which is oriented to form N. A link of the form (aa) is always the arc aa.

A (connected and oriented) *q-chain* from a to b is a set of q links of the form $(ac_1), (c_1c_2), \cdots, (c_{q-2}c_{q-1}), (c_{q-1}b)$, such that no node appears more than once, except in the case $a = b$ where a appears twice. Any q-chain from a to b will be denoted by (ab, q). Observe that $(ab, 1) = (ab)$. If c is a node included in a q-chain from a to b, then we may subdivide the chain into the "product" of two chains, one from a to c, and the other from c to b, i. e., $(ab, q) = (ac, q')(cb, q - q')$, $q' < q$.

An (oriented) *circuit* is a chain of the form (aa, q). A circuit of two links is an arc and conversely.

A network is *connected* if there exists a chain from each node to every other node. A network which is not connected is *disconnected*. When N is

[2] Whitney, H., "Non-separable and planar graphs," *Transactions of the American Mathematical Society*, vol. 34 (1932), p. 339.

treated as an oriented graph, connectedness is defined topologically; our definition implies topological connectedness but is not implied by it. However, as applied to networks which are graphs, the two definitions are equivalent.

2. A decomposition theorem for arbitrary networks. In this section we shall give four related definitions for networks of order m which will depend on the integers k between 0 and m. These definitions will be used in their full generality in Theorem 2.4, which shows that, in a certain sense, we need only consider the definitions in the case $k = 1$. Consequently the rest of the paper will be, for the most part, devoted to that special case.

First we need a measure of how easily a connected network is disconnected by the removal of links. We shall say a network is of degree 0 if it is not connected. A network is of *degree* k, $1 \leq k \leq m$, if there exists a set of k distinct links whose removal from the network will result in a complete subnetwork of degree 0, while the removal of any set of $q < k$ links results in a complete connected subnetwork.[3] The degree of a network is unique.

LEMMA 2.1. *If $N(m, p)$ is a network of degree k, then $p \geq km$.*

Proof. It will obviously suffice to show that each node is the initial node of at least k links. This is true, for if not, then the removal of the links for which such a node is the initial node will disconnect N. This contradicts the assumption that the degree is k.

In addition to the concept of degree, we need a condition implying that there is an even distribution of connectedness throughout the network; roughly, that the degree of any connected subnetwork is not greater than that of the network itself. That this is not always the case is evidenced by any graph formed of an $m - 2$ simplex, $m \geq 4$, and a single node joined by a single arc to one of the nodes of the simplex. The network is of degree 1, and the simplex, which is a connected subnetwork, is of degree $m - 2 \geq 2$. A definition which will suffice is the following. A network is said to be *k-minimal,* $1 \leq k \leq m$, if the removal of any link results in a complete subnetwork of degree $k - 1$. The existence of such networks is proved in Lemma 2.3. A network is *k-uniform* if every connected subnetwork is of degree $\leq k$. If a network is 1-uniform and connected we say it is *uniform.*

LEMMA 2.2. *If N is a k-minimal network with $k \geq 2$, then N is k-uniform and of degree k.*

[3] This definition of degree has no relation to the *Grad* defined by König, *op. cit.*, p. 3.

Proof. Let S be any connected subnetwork of N, and suppose it has degree d. Let $(ab) \varepsilon S$. $N - (ab) = N'$ is of degree $k-1$, so in N' there exists a set U of $k-1 \geq 1$ links, whose removal from N' results in a complete disconnected subnetwork, N''. If in N'' there is a chain from a to b, we may replace (ab) and still have a disconnected network N^* which is formed from N by removing the links of U. In that case, the removal of any $(cd) \varepsilon U$ from N results in a complete subnetwork of degree $k-2 \geq 0$, since $k \geq 2$. This is contrary to the assumption that N is k-minimal, so there is no chain from a to b. Thus, the removal of no more than k links, those of U which are in S and (ab), from S, implies a is not connected to b by any chain. It follows that $d \leq k$.

Specifically, N has degree $d \leq k$. If $d \leq k-1$, then, since the removal of any link results in a complete subnetwork of degree $k-1$, it follows that $d = k-1$. Let U be a set of $k-1$ links whose removal from N results in a complete disconnected subnetwork. U is non-empty since $k \geq 2$. Remove $(ab) \varepsilon U$ *from* N. The resulting network is, by definition, of degree $k-1$; however, the remaining $k-2$ links of U disconnect $N - (ab)$. Hence $d = k$.

We note that the above argument does not apply for $k = 1$; in fact, any disconnected network is 1-minimal, since the removal of any link results in a complete subnetwork of degree 0. But some networks are both connected and 1-minimal; these we shall call *minimal.* A minimal network is clearly non-reflexive and uniform.

LEMMA 2. 3. *If N is a network of degree $k \geq 1$, then for every integer q, $1 \leq q \leq k$, there exists a complete connected subnetwork of N which is q-minimal.*

Proof. Let C_q be the set of all complete connected subnetworks of N having degree q. Since N is finite and $q \leq k$, it is obvious that C_q is non-empty. Let

$$p_q = \max_{N' \varepsilon C_q} p(N - N').$$

Since N is finite, there exists some $N_q \varepsilon C_q$ such that $p(N - N_q) = p_q$. N_q is, by choice, connected. Hence it will suffice to show that N_q is q-minimal. Suppose the removal of some link does not result in a complete subnetwork of degree $q-1$. Then, since the removal of one link cannot lower the degree by more than 1, the resulting network N' has degree q. Thus $N' \varepsilon C_q$ and $p(N - N_q) < p(N - N') \leq p_q$, which is contrary to choice.

A complete connected subnetwork N' of N such that, in the terms of the above proof, $p(N - N') = p_q$, is called a *q-descendant* of N. If N_q and N'_q are two q-descendants of a network N, then $p(N_q) = p(N'_q)$. It is clear that every connected network has at least one 1-descendant, but this is not generally true for $q > 1$. Because of their importance the 1-descendants will be called simply *descendants*. It is clear that a descendant is minimal.

A network N will be called the sum of complete subnetworks N_i, $i = 1, 2, \cdots, t$, and written $N = \sum_{i=1}^{t} N_i$, if each link of N is contained in exactly one of the N_i.

THEOREM 2.4 (*first decomposition theorem*). *To every network N there exists a unique number k, its degree, and at least one set of $k+1$ complete 1-minimal subnetworks, N_i, such that*

i. $N = \sum_{i=1}^{k+1} N_i$,

ii. N_{k+1} *is disconnected,*

iii. *if* $k \geq 1$, *then* N_1 *is minimal,*

iv. $\sum_{i=1}^{j} N_i$ *is a j-descendant of* $\sum_{i=1}^{j+1} N_i$, $1 \leq j \leq k$,

and

v. *the connected subnetworks of the* N_i, $1 \leq i \leq k$, *are minimal, and so these networks* N_i *are* 1-*uniform.*

Proof. By definition there is a unique degree k assigned to every network. If $k = 0$, then N is not connected and we are done. If $k > 0$, select, according to Lemma 2.3, a k-descendant N'_k of N, and define $N_{k+1} = N - N'_k$. N_{k+1} is not connected; for if so, N is the sum of two complete subnetworks having, respectively, degree k (Lemma 2.2) and degree ≥ 1. This, we will show, implies that N has degree $\geq k+1$, which is contrary to assumption.

To show this we prove the slightly more general statement: If $N = N_1 + N_2$, and these networks have degrees k, k_1, and k_2 respectively, then $k \geq k_1 + k_2$. For, by definition, there exists a set U having k links, such that their removal from N results in a complete disconnected subnetwork N', and this is not true for any smaller set. Of these k links, let u_1 be in N_1, and u_2 in N_2. By the definition of a sum, $k = u_1 + u_2$. Furthermore, $u_1 \geq k_1$, since we may remove from N first the links of U and then the remaining links of N_2. This complete subnetwork, which obviously is N_1 minus u_1 links, is disconnected, so $u_1 \geq k_1$. Similarly, $u_2 \geq k_2$, whence the result.

In the network N'_k select a $(k-1)$-descendant, N'_{k-1}, and let $N_k = N'_k$

$-N'_{k-1}$. N_k is 1-minimal, for if $(ab) \varepsilon N_k$, let $N^*_k = N'_k - (ab)$. Then, by the definition of k-minimal, N^*_k is of degree $k-1$. But since N^*_k contains N'_{k-1}, the latter is a $(k-1)$-descendant of the former. Then the argument given above shows that $N^*_k - N'_{k-1} = N_k - (ab)$ is not connected.

The argument proceeds inductively without difficulty, since the last argument is independent of k. When we get to the case N'_2, $N'_1 = N_1$ is a descendant of N'_2 and thus is minimal rather than simply 1-minimal.

Condition (iv) is satisfied by our choice of the N_i.

Finally, the connected subnetworks of N_j, $2 \leq j \leq k$ are minimal. For suppose S is a connected subnetwork of N_j such that $S - (ab)$ is a connected subnetwork of S. Let $N^*_j = N'_j - (ab)$. Since N'_j is j-minimal, N^*_j is of degree $j-1$, so there exists a set of $j-1$ links whose removal from N^*_j will result in a complete disconnected subnetwork. At least one of these links is in S, since there exists, in S, a chain $(ab, q) \neq (ab)$. Thus there are at most $j-2 \geq 0$ of these links not in N_j, so that the removal of at most $j-2$ links from the descendant N'_{j-1} results in a complete disconnected subnetwork. This is in contradiction to Lemma 2. 3 which shows that N'_{j-1} is $(j-1)$-minimal. Thus S is minimal. If $k \geq 1$, N_1 is minimal, and therefore the connected subnetworks are minimal. It follows immediately that these N_i are 1-uniform.

In the sense of this theorem, the study of an arbitrary network has been reduced to the study of a collection of 1-minimal networks. These 1-minimal networks are either connected, and so minimal, or disconnected. But a disconnected network consists of isolated nodes, isolated chains, and connected pieces. For k of the subnetworks, part (v) shows that the connected pieces are minimal. If the theorem is applied repeatedly to the connected pieces of N_{k+1}, it may, in the same sense, be reduced to isolated nodes, isolated chains, and minimal subnetworks. Thus we may say that, in a sense, the study of any network may be reduced to the study of minimal networks. This exaggerates the present state of the art, since we do not know whether this decomposition is sufficiently strong to allow general conclusions about networks, or even k-minimal networks, from a knowledge of minimal networks. In fact, an important unsolved problem is the relationship between two distinct decompositions of this type for a given network. That two distinct decompositions may exist is shown by:

$$\begin{pmatrix} 01000 \\ 00101 \\ 01010 \\ 01100 \\ 10010 \end{pmatrix} = \begin{pmatrix} 00000 \\ 00000 \\ 01000 \\ 00100 \\ 00010 \end{pmatrix} + \begin{pmatrix} 01000 \\ 00101 \\ 00010 \\ 01000 \\ 10000 \end{pmatrix} = \begin{pmatrix} 00000 \\ 00100 \\ 00010 \\ 01000 \\ 00000 \end{pmatrix} + \begin{pmatrix} 01000 \\ 00001 \\ 01000 \\ 00100 \\ 10010 \end{pmatrix} \cdot$$

On the basis of the preceding remarks we are led to devote the rest of this paper to beginning a study of minimal networks. Section 3 includes a decomposition of any minimal network and the deduction of several properties of minimal networks. These properties are used in section 4 to draw some conclusions about arbitrary connected networks. In section 5 we discuss the relationships between several of our concepts and that of a tree in graph theory. Finally, in the last section, we present an interesting inequality, and, from this, define a subclass of minimal networks, the members of which are shown to have a particularly simple form.

3. A decomposition theorem for minimal networks. This section presents a decomposition theorem for any minimal network, which may be used to show that there exists a close connection between the concept of a minimal network and the concept of a tree in graph theory. We note first that a network which is a tree is minimal. However, the class of minimal networks is much wider than that, for we know that every connected network has a descendant, which is minimal, and we have

LEMMA 3.1. *If N is a connected network and T a descendant of N, then T is a tree only if $T = N$.*

Proof. If T is a tree and $T \neq N$, then T must have been formed from N by the removal of at least one link. Reintroduce one of these, say (ab), into T. This must introduce an oriented circuit on at least three nodes, since T is connected; let it be $(ab)(bc_1)\cdots(c_q a)$. Since this circuit is on three or more nodes, and the links of T are members of arcs, it follows that $(ab, q) = (ac_q)(c_q c_{q-1})\cdots(c_1 b)$, $q \geq 2$, exists. Because of the existence of the circuit, this chain may be removed to result in the complete connected subnetwork N'. Now since $q \geq 2$, $p(N - N') > p(N - T)$, so T is not a descendant of N, which is contrary to assumption.

To carry further the work of this section we need two more definitions. First, a network is a *compound circuit of order* 1 if it is simply a non-reflexive oriented circuit on its nodes; assuming a compound circuit of order $s - 1$ defined, a *compound circuit of order* s is formed from one of order $s - 1$ by replacing some node c of that network by a non-reflexive circuit C, each link of the form (ac) by one and only one link of the form (ac') where $c' \varepsilon C$, and each link of the form (ca) by one and only one link of the form $(c''a)$, $c'' \varepsilon C$. We shall refer to this as an *inductive composition* of a compound circuit. Obviously any compound circuit is connected; furthermore, we have

LEMMA 3.2. *If $N(m,p)$ is a compound circuit of order s, then $s = p - m + 1$; i.e., if N is formed by orienting the graph G, s is the first Betti number of G.*[4]

Proof. Let N be formed by the inductive composition of the circuits C_i, $i = 1, 2, \cdots, s$, in their natural order, and suppose each C_i has m_i nodes and hence the same number of links. It follows by a simple induction that

$$m = m_1 - 1 + m_2 - 1 + \cdots + m_s = \Sigma m_i - (s - 1)$$

and $p = \Sigma p_i = \Sigma m_i$, so that $s = p - m + 1$. It is well known that $p - m + 1$ is the first Betti number of any connected graph having p branches and m nodes.

A connected network N will be said to be *reducible* into subnetworks N_1 and N_2 if:

i. N_1 and N_2 are disjoint,

ii. N_1 and N_2 are each either connected or consist of a single node,

iii. there exists a network N', formed of N_1 and N_2 joined by exactly one link from N_1 to N_2 and exactly one from N_2 to N_1, such that $N' = N$.

If a connected network is not reducible it is called *irreducible.*

THEOREM 3.3. *A connected network which is a graph is reducible if and only if it is of degree* 1.

Proof. It is clear that any reducible network is of degree 1, since we may disconnect it by removing either of the links joining the disjoint subnetworks.

Let N be a graph of degree 1 and (ab) a link such that $N - (ab) = N'$ is disconnected. Evidently, in N' there is no chain from a to b. Define M_b to consist of b and any nodes b' such that there is a chain from b' to b in N'. Let $M_a = M - M_b$. Clearly, $a \varepsilon M_a$. For any $a' \varepsilon M_a$, $a' \neq a$, there exists in N' a chain from a to a'. If not, then, since N is connected, any chain in N from a to a' must contain the link (ab), and at least one such chain exists. Since the node a can appear only one, this chain must be of the form $(aa', q) = (ab)(ba', q-1)$, and $(ba', q-1)$ does not contain a. But N is a graph, so $(ba', q-1)$ implies the existence of a chain $(a'b, q-1)$ which does not contain (ab). This, then, is a chain in N', and so $a' \varepsilon M_b$, which is

[4] König, *ibid.*, first Betti number = Zusammenhangszahl, p. 53; Whitney, *op. cit.*, first Betti number = nullity, p. 340.

contrary to choice. Thus we know a chain exists in N' from a to a'. Since N is a graph, this implies that the largest subnetworks of N defined on M_a and M_b are each either connected or consist of a single node. Now, if $a' \varepsilon M_a$ and $b' \varepsilon M_b$, where $a' \neq a$ or $b' \neq b$, then there exists no link of the form $(a'b')$, for otherwise there would be a chain from a to b in N'. Since N is a graph, it follows that there are no links of the form $(b'a')$. Thus N is reducible.

The following is our principal result:

THEOREM 3.4 (*second decomposition theorem*). *To any minimal network N, which is not a tree, there exist integers $t \geq 1$ and $y \geq 0$, such that N consists of t disjoint irreducible compound circuits C_i, $i = 1, 2, \cdots, t$, and y nodes C_{i+t}, $i = 1, 2, \cdots, y$, not included in the C_i, $1 \leq i \leq t$, subject to the conditions:*

- i. *there exists at most one link from any C_i to any C_j, $i \neq j$, $1 \leq i, j \leq t + y$;*
- ii. *no arc is contained in any of the C_i, $1 \leq i \leq t$;*
- iii. *the network formed by treating the C_i, $1 \leq i \leq t$, as nodes, all other nodes and links remaining unchanged, is a tree, or, if $t = 1$ and $y = 0$, a single node.*

Proof. This proof will be carried out in two stages. First we shall show that if N is a minimal network in which there exists an arc ab, then N is reducible into two subnetworks joined only by ab. Since N is minimal, it is non-reflexive, so that $a \neq b$. Define the set of nodes M_a to consist of a and any other nodes, a', of N such that there exists a chain from a to a' which does not include the link (ab). Let $M_b = M - M_a$. $b \varepsilon M_b$, for if not, then $b \varepsilon M_a$, and so there exists (ab, q) not including (ab). Then $N - (ab)$ is a connected subnetwork of N; this violates the condition that N is minimal.

We shall now show some properties N must satisfy which will lead ultimately to a proof of the statement:

1. If $a' \varepsilon M_a$, $a' \neq a$, there exists a chain from a' to a not including the link (ba). Clearly some chain exists from a' to a, since N is connected. If all such chains include (ba), then, since the node a may only appear once, each of them may be written in the form $(a'a, q) = (a'b, q-1)(ba)$. Moreover, $(a'b, q-1)$ does not include (ab) since $a' \neq a$. Now, by the definition of M_a, there exists a chain (aa', u) which does not include (ab), so that $(aa', u)(a'b, q-1)$ does not include (ab). This is contrary to the assumption that N is minimal.

2. Let $b' \neq b$. $b' \varepsilon M_b$ if and only if there exists a chain from b to b' which does not include the link (ba). Suppose first that $b' \varepsilon M_b$. Since N is connected, there exists at least one chain from b to b'. Suppose each (bb', q) contains (ba). Then, since each node may appear only once, $(bb', q) = (ba)(ab', q-1)$. If $(ab', q-1)$ does not contain (ab), then, by definition, $b' \varepsilon M_a$, which is impossible. But $(ab', q-1)$ cannot contain (ab), for if it did, then (bb', q) would not be a chain. Hence (bb', q) does not contain (ba).

Conversely, suppose there exists a chain from b to b' not including (ba). If $b' \varepsilon M_a$, there exists, by 1, a chain from b' to a not including (ba); these two combine into a chain from b to a not including (ba), which is contrary to N being minimal.

3. If $b' \varepsilon M_b$, $b' \neq b$, there exists $(b'b, q)$ not including (ab). We may parallel the proof of property 1 by replacing the words "definition of M_a" by "property 2."

4. If $a' \varepsilon M_a$, $b' \varepsilon M_b$, and either $a \neq a'$ or $b \neq b'$, then no link of the form $(a'b')$ exists. Suppose such a link does indeed exist. Then, by the definition of M_a, there exists a chain (aa', q) which does not include (ab), and, by property 3, a chain $(b'b, r)$ which does not include (ab). Thus the chain $(aa', q)(a'b')(b'b, r)$ does not include (ab), since $(a'b') \neq (ab)$, which is impossible.

5. Under the same conditions as in 4, there is no link of the form $(b'a')$. The argument is exactly parallel to that of 4, using properties 1 and 2.

It thus follows that the maximal subnetworks of N on the sets M_a and M_b are each either connected and minimal, or consist of a single node. From 4 and 5, one concludes that the subnetworks are joined only by the arc ab. This exhausts N, and the result is proved.

Since an arbitrary network has a finite number of arcs, it follows from a finite number of applications of the above result that a minimal network which is not a tree consists of a set of $t' \geq 1$ disjoint arc-free minimal subnetworks C_i, $i = 1, 2, \cdots, t'$, and $y' \geq 0$ nodes $C_{t'+i}$, $i = 1, 2, \cdots, y'$, not included in the C_i, $1 \leq i \leq t'$, such that:

i. any link not in a C_i, $1 \leq i \leq t'$, is a member of an arc;

ii. there exists at most one arc between any C_i and C_j, $i \neq j$, $1 \leq i$, $j \leq t' + y'$;

iii. the network formed by treating the C_i, $1 \leq i \leq t'$, as nodes, is a tree;

iv. the decomposition is unique.

By virtue of this decomposition, the problem is reduced to examining the case of an arc-free minimal network. We show: An arc-free minimal network consists of $t'' \geq 1$ disjoint irreducible compound circuits C_i, $i = 1, 2, \cdots, t''$, and $y'' \geq 0$ nodes $C_{i+t''}$, $i = 1, 2, \cdots, y''$, not included in the C_i, $1 \leq i \leq t''$, such that:

i. there exists at most one link from any C_i to any C_j, $i \neq j$, $1 \leq i$, $j \leq t'' + y''$;
ii. the network formed by treating the C_i, $1 \leq i \leq t''$, as nodes is a tree, or, if $t'' = 1$ and $y'' = 0$, a single node.

The first arc-free minimal network occurs for $m = 3$, and this obviously satisfies the conditions since it is a circuit on three nodes. Suppose now that the statement, except for the condition that the compound circuits are irreducible, has been proved for all networks through $m - 1$ nodes, and let $N(m, p)$ be an arc-free minimal network. In N there exists a circuit consisting of at least three links, since N is connected, arc-free, and non-reflexive; let C be one such circuit on the nodes c_i, $i = 1, 2, \cdots, q \geq 3$. The maximum subnetwork of N on these nodes is only C, for if there exists any other link $(c_i c_k)$, $k \neq i + 1$, then this link can be removed without disconnecting N, since the chain $(c_i c_{i+1})(c_{i+1} c_{i+2}) \cdots (c_{k-1} c_k)$ exists. This is impossible since N is minimal. Now, if C exhausts all the nodes of N we are done. If not, let M' be the set of nodes remaining. If $a \varepsilon M'$, and $(a c_k)$, $1 \leq k \leq q$, exists, then no link of the form $(a c_i)$, $1 \leq i \leq q$, $i \neq k$, exists. For if so, then the chain from c_k to c_i, which is a part of C, and so does not include $(a c_i)$, shows that $(a c_i)$ may be removed without disconnecting N. This is impossible. Similarly, if $(c_k a)$, $1 \leq k \leq q$, exists, then $(c_i a)$, $1 \leq i \leq q$, $i \neq k$, does not exist.

Now consider the network N' formed by letting the nodes of C coalesce into a single node which we shall call c. Evidently, since N is minimal, so is N', and N' has at least two nodes fewer than N, since $q \geq 3$. Several possibilities exist for N'. First, it may be a graph, and hence a tree, in which case the statement is proved. Second, it is not a tree, but there exists at least one arc. By the first part of this theorem, N' may be decomposed into several arc-free minimal subnetworks connected in such a fashion that if they are treated as nodes, the resulting network is a tree. By the induction hypothesis, these arc-free minimal subnetworks satisfy the conditions of the statement we are proving. But replacing the node c by C, reconstructing N, only increases the order of one of these compound circuits, or introduces a new compound circuit, so the result is true for N. Third, N' is arc-free, in which case the induction hypothesis may be applied directly, and the introduction of C for c only increases the order of the compound circuit.

Thus, we may decompose N into several compound circuits and nodes not in these compound circuits and connecting links satisfying the conditions i and ii of the second intermediate statement. Carry this decomposition as far as possible; the process will terminate in a finite number of steps, since N is finite. We will show that the resulting compound circuits are irreducible. For suppose that C_k is reducible into the disjoint subnetworks A and B connected by the links (ab), $(b'a')$, $a, a' \varepsilon A$, $b, b' \varepsilon B$. By condition ii it follows that any C_i, $i \neq k$, $1 \leq i \leq t' + y'$ is linked "symmetrically" to C_k if at all. In fact, it is either linked symmetrically to A or to B; for if not, then there exists a link from A to C_i and a link from C_i to B, in which case (ab) may be removed, or, in the other case, $(b'a')$ may be removed without disconnecting N. This is impossible. A and B are either compound circuits or, by the result proved for arc-free minimal networks, may be reduced to several compound circuits and nodes not in them such that i and ii hold. By an argument similar to the one just made, the conditions i and ii hold for N with this finer decomposition. This is contrary to choice, so C_k must be irreducible.

The proof of the theorem follows almost immediately from the two intermediate results, if we note that the last argument may be applied to show condition iii.

This decomposition of a minimal network is not unique, for

$$\begin{Bmatrix} 010000 \\ 000001 \\ 010100 \\ 000010 \\ 001000 \\ 100010 \end{Bmatrix}$$

may be decomposed into either a tree consisting of one arc, or one of two arcs.

The next result gives a little more information about the components into which we have decomposed a minimal network, the irreducible compound circuits. This result is unsatisfactory in the sense that it does not give a complete characterization of these networks. For this proof and succeeding results we need the following definition. A node is *simple* if it is the initial node of exactly one link and the end node of exactly one link.

THEOREM 3. 5. *Let N be a minimal network. N is irreducible if and* *only if it is a compound circuit such that in any inductive composition of N, none of the circuits introduced are arcs.*

Proof. Suppose that at some stage of the composition of N, an arc ab is introduced into a compound circuit C to form a compound circuit C'. If ab does not have a simple node, then in C' there exists either a chain from a to b not containing (ab), or one from b to a not containing (ba), since such a chain exists in C. The introduction of further circuits can only lengthen this chain, so $N - (ab)$ is a complete connected subnetwork, which is impossible. Hence a node of ab is simple. The introduction of further circuits merely adds to C to form a larger compound circuit, and hence a connected subnetwork or a single node of N, or it may replace the simple node of ab by a compound circuit. Between these two connected subnetworks, or single nodes, are only the links arising from ab, now no longer an arc in general. Thus N is reducible, which is contrary to assumption, proving that no arc can be introduced.

Conversely, if we suppose N is reducible, then Theorem 3. 4 implies N may be decomposed into one or more irreducible compound circuits and nodes not included in these compound circuits. The circuits of any compound circuit C may be coalesced into nodes in the inverse order of an inductive composition of C. This clearly leads to the tree of Theorem 3. 4. But any tree is a compound circuit formed only of arcs. Thus we have an inductive composition of N involving arcs, the arcs of the tree. As this is contrary to assumption, N must be irreducible.

The principal theorem will be utilized sometimes through two properties of minimal networks derivable from it. They are presented as

THEOREM 3. 6. *A minimal network is a compound circuit which contains at least two simple nodes.*

Proof. The last part of the above proof suffices to show that a minimal network is a compound circuit.

To show that a minimal network has two simple nodes, we shall perform an induction on the order s of the compound circuit. It is certainly true for $s = 1$, since the compound circuit is then a non-reflexive circuit. Assume the result true up through circuits of order $s - 1$. Suppose N is a minimal compound circuit of order s, and let C be the last circuit introduced in some inductive composition of N. Let C coalesce into a single node c, and call the resulting network N'. N' is readily seen to be minimal and of order $s - 1$, so by the induction hypothesis it contains at least two simple nodes. If two of these are different from c, then we are done. If not, c is simple. Consider N; if C is not an arc then it must introduce a simple node, for C has at least three nodes, and there exists only one link to C from the rest of the nodes, and only one from C, since c is simple. If, on the other hand, C is an arc, then the first argument in the proof of Theorem 3. 5 shows that one of its nodes is simple, and the result follows.

That not every compound circuit is minimal or has a simple node is shown by:

$$\begin{pmatrix} 0101 \\ 0010 \\ 0101 \\ 1000 \end{pmatrix}$$

4. Applications to connected networks. Two applications to connected networks are given of the results on minimal networks; the first examines limits on the number of links a connected network may have, and the second discusses the maximum number of "independent" circuits a connected network may have.

THEOREM 4. 1. *Let $N(m, p)$ be a connected network, not a tree. Let N have a descendant N' which is decomposable in the terms of Theorem 3. 4 into t irreducible minimal subnetworks and y nodes not in these subnetworks. Then*

$$p \leq (3m + t + y - 4)/2 + p(N - N') < 2(m - 1) + p(N - N').$$

If N is a tree, $p = 2(m - 1)$.

Proof. If N is a tree, the result is well known from graph theory.[5]

Suppose N is not a tree. Then it is sufficient to show the result for the class of minimal networks which are not trees, since, in the general case, the network N has $p(N - N')$ more links than any descendant N'. By Lemma 2. 3 a descendant is minimal, and, by Lemma 3. 1, it is not a tree. So we consider N minimal. Decompose N as in Theorem 3. 4, and let the irreducible compound circuits C_i have m_i nodes, p_i links, and order s_i. Let there be p' links not in any irreducible compound circuit. Then, by the result on trees, $p' = 2(t + y - 1)$. For each of the irreducible minimal subnetworks, Theorem

[5] Whitney, *ibid.*, pp. 340-341.

3.5 implies that each of the s_i circuits used in forming C_i has at least three nodes, so that $m_i \geq 3 + 2 + 2 + \cdots + 2 = 2s_i + 1$. By Lemma 3.2,

$$p_i = s_i + m_i - 1 \leq 3(m_i - 1)/2.$$

Thus,

$$p = \Sigma p_i + p' \leq \Sigma 3(m_i - 1)/2 + 2(t + y - 1)$$
$$= (3/2)(\Sigma m_i + y) + (t + y - 4)/2 = (3m + t + y - 4)/2.$$

This may be simplified a little by noting that each of the irreducible minimal subnetworks must have at least three nodes, so $m \geq 3t + y$; hence,

$$p \leq (3m + t + y - 4)/2 = (4m - 4 - 2t + 3t + y - m)/2$$
$$\leq 2(m - 1) - t < 2(m - 1).$$

This concludes the proof.

It is clear that in a given network we may define the *addition of chains* (*mod* 2). Thus we may also define *linear independence* (*mod* 2). We shall be concerned with sets of linearly independent (mod 2) circuits such that no other linearly independent set contains a greater number of circuits. These sets will be called *maximal.* The result proved in the next theorem is, in statement, formally the same as a result of graph theory: [6]

THEOREM 4.2. *In any connected network $N(m, p)$ there exists a maximal set of $p - m + 1$ linearly independent (mod 2) circuits.*

Proof. First, it is sufficient to show this for minimal networks. For, if N is not minimal, then it has a descendant N' which is. N may be considered to be formed from N' by the addition of links one at a time. Each such link adds at least one new circuit which is independent (mod 2) of the circuits of the network to which it was added, since in a connected network every link is contained in at least one circuit. Thus, if there exists a set U' of $p - K - m + 1$, $K = p(N - N')$, linearly independent (mod 2) circuits in N', there will exists a set U of at least $p - m + 1$ linearly independent circuit in N.

Furthermore, if U' is maximal in the descendant, U will be in N also. If not, then there is a first subnetwork, N^*, for which any set of $p^* - m + 1$ linearly independent (mod 2) circuits is maximal, and to which the addition of a link (ab) produces a linearly independent set U'' having more than

[6] Lefschetz, Solomon, *Introduction to Topology*, Princeton, Princeton University Press (1949), p. 71.

$p^* - m$ circuits. It is clear that this set U'' must contain at least two circuits which include the link (ab), for otherwise the subset of U'' in N^* would contain more than $p^* - m + 1$ linearly independent circuits. Let two of the circuits be denoted by $(ab)(ba, q)$ and $(ab)(ba, q')$. Since N^* is connected and does not contain (ab), there exists a chain from a to b not including (ab); select a shortest: (ab, q''), $q'' > 1$. In general, (ab, q'') will coincide with (ba, q) over a certain number of links, i. e., over a set of several chains of the form (cd, t), each a part of (ba, q). The argument does not change in principal, and a great saving in notation is gained, if we assume that at most one such chain occurs. Similarly, (ab, q'') will be assumed to coincide with (ba, q') over the chain $(c'd', t')$. Furthermore, we shall assume that (cd, t) and $(c'd', t')$ have no links in common; if they do, a slight modification of the following argument will suffice. So we may write:

$$(ab, q'') = (ac, u)(cd, t)(dc', z)(c'd', t')(d'b, v),$$

the order of (cd, t) and $(c'd', t')$ being immaterial.

$$(ba, q) = (bc, x)(cd, t)(da, y), \qquad (ba, q') = (bc', x')(c'd', t')(d'a, y').$$

Observe that the following formal products are in fact circuits of N^*:

$$\text{A:} \quad (ac, u)(cd, t)(da, y) \qquad \text{B:} \quad (d'b, v)(bc', x')(c'd', t')$$

$$\text{C:} \quad (ac, u)(cd, t)(dc', z)(c'd', t')(d'a, y')$$

$$\text{D:} \quad (bc, x)(cd, t)(dc', z)(c'd', t')(d'b, v).$$

Since these are circuits of N^*, they are expressible (mod 2) in terms of the circuits in U''. But observe that $(ab)(ba, q) + A + B + C + D = (ab)(ba, q')$ (mod 2) so that $(ab)(ba, q)$ and $(ab)(ba, q')$ are not linearly independent. Thus only one of them can be in U'', and so we have shown that if the theorem is true for minimal networks it is true in general.

The minimal case will be proved by induction on m. For $m = 2$ it is trivially true. Suppose it is true for all minimal networks having $m - 1$ or fewer nodes, and let $N(m, p)$ be minimal. By Theorem 3. 6, N has a simple node a which is the initial node of only one link, (ab), and the end node of only one, (ca). We may distinguish three cases:

i. $b = c$. Remove a and the arc ab, leaving the subnetwork $N'(m - 1, p - 2)$. N' is obviously minimal, and so it has a maximal set of $p - m$ linearly independent (mod 2) circuits. The arc ab adds exactly one circuit to this set.

ii. $b \neq c$, and there does not exist $(cb, q) \neq (ca)(ab)$. Remove a and the adjoining links and introduce the link (cb) to form $N'(m-1, p-1)$, which is minimal. Thus, by the induction hypothesis, has a maximal set of $p-m+1$ linearly independent circuits. But forming N' from N cannot essentially change any set of linearly independent circuits, since the chain $(ca)(ab)$ is, in this case, formally the same as (cb).

iii. $b \neq c$, and there does exist $(cb, q) \neq (ca)(ab)$. Again remove a and the adjoining links to form $N'(m-1, p-2)$, which is minimal. By the induction hypothesis, N' has a maximal set of $p-m$ linearly independent circuits. Replacing a and the two links (ab) and (ca) to form N adds one or more new circuits, depending on the number of chains from b to c. This situation is not essentially different from the one discussed in the first part of this proof, except that we are adding a 2-chain and a new node, rather than a single link. Since this node is simple, the argument is formally the same, and it shows that there is a maximal set of $p-m+1$ linearly independent (mod 2) circuits in N. This, then, concludes the proof.

We note the trivial corollary: A connected network $N(m, p)$ has exactly $p-m+1$ circuits if and only if the set of all circuits of N is linearly independent (mod 2).

5. Generalizations of a tree. We shall show in this section that several of our definitions, when applied to networks which are graphs, are identical with the concept of a tree. This can be shown directly and easily in each case; however we shall first prove two results which are true in general, and then we shall use them to prove Theorem 5. 3. Thus, that result is not as deep as it first appears to be.

We shall call a connected network $N(m, p)$ having exactly $p-m+1$ circuits *circuit minimal.* This definition makes sense because of Theorem 4. 2. By the corollary to that theorem, N is circuit minimal if and only if N is connected and the set of all circuits is linearly independent (mod 2).

LEMMA 5. 1. *A circuit minimal network is uniform.*

Proof. Let $N(m, p)$ be circuit minimal. Let (ab) be any link, and $N-(ab)=N'$. If N' is not connected, then N has degree 1. If N' is connected it is circuit minimal, for at least one circuit of N was destroyed by the removal of (ab), and according to Theorem 4. 2, no more than one.

Since N' is connected, there exists at least one chain from b to a, but only one, for if there were more then the addition of the single link (ab) would introduce more than one circuit, and N would have more than $p - m + 1$ circuits. In N, interrupt the chain from b to a by removing a single link from it, thus disconnecting N. This proves N is of degree 1.

To show N is uniform it will thus suffice to show that every connected subnetwork is circuit minimal. If S is a connected network which is not circuit minimal, then there exists a circuit in S which is linearly dependent (mod 2) on the other circuits of S. This remains true in N, so N is not circuit minimal, a contradiction.

LEMMA 5. 2. *A compound circuit is uniform.*

Proof. This may be demonstrated by an induction on the order of compound circuits. It is trivially true for compound circuits of order 1. Let N be a compound circuit of order $s > 1$, let C be the last circuit introduced in some inductive composition of N, and let S be any subnetwork of N. Coalesce C into a single node c, and under this operation let S become S'. If S' is a single node, then $S = C$, and the degree of S is 1. Otherwise, S' is connected, and therefore, by the induction hypothesis, it is of degree 1. Select $(ab) \in S, \notin C$, such that $S' - (ab)$ is not connected. This is possible since S′ is of degree 1. Now the introduction in S' of that part of C in S, subject to the conditions of N, can only replace the node c by a chain or a circuit, but cannot introduce a link or a chain from a to b; thus S is of degree 1. So N is uniform.

The next result is the justification for the title of this section.

THEOREM 5. 3. *For a connected network N which is a graph, the following are equivalent*: i. *N is a tree,* ii. *N is minimal,* iii. *N is a compound circuit,* iv. *N is uniform,* v. *N is circuit minimal.*

Proof. i. implies ii trivially. ii. implies iii by Theorem 3. 6. iii. implies iv by Lemma 5. 2. iv. implies i. For if N is not a tree, then there exists a circuit in the sense of graph theory. But this is clearly a connected subnetwork of degree 2, so that N is not uniform. v. implies iv by Lemma 5. 1. i. implies v. If $N(m, p)$ is a tree, it follows, from theorem 3. 5, that $p - (m - 1) = m - 1$. Furthermore, the only circuits in a tree are 2-circuits (arcs) of which there are exactly $m - 1$, so the number of circuits is $p - m + 1$.

The several results of this section and Theorem 3. 6 suggest the following class of unsolved problems: Conditions on a uniform network that it be a compound circuit. Conditions on a uniform network that it be circuit minimal. Conditions on a compound circuit that it be minimal. These four concepts are indeed all distinct. The network.

$$\begin{pmatrix} 010000 \\ 001001 \\ 000100 \\ 100010 \\ 010000 \\ 000100 \end{pmatrix}$$

is minimal, and hence a compound circuit, but not circuit minimal. The network

$$\begin{pmatrix} 0101 \\ 1000 \\ 1101 \\ 1010 \end{pmatrix}$$

is both uniform and circuit minimal, but not a compound circuit. The network

$$\begin{pmatrix} 0111 \\ 1010 \\ 1001 \\ 1000 \end{pmatrix}$$

is uniform, but neither a compound circuit nor circuit minimal. The example following Theorem 3. 6 shows that not every compound circuit is minimal.

6. Rank minimal networks. In section 1 we noted the representation of networks by relation matrices with entries from the two-element Boolean algebra. Equally well, we may interpret this as a representation by real matrices with the numbers 0 and 1 as entries. Thus, since it is well known that matrix rank is a similarity invariant, to each network there is a uniquely defined number r, $1 \leq r \leq m$, called the *rank of the network*, which is the rank of any of the corresponding real matrix representations.

THEOREM 6. 1. *If $N(m, p)$ is a connected network*[7] *having rank r, then* $p + r \geq 2m$.

[7] Luce, R. D., "Connectivity and generalized cliques in sociometric group structures," *Psychometrika*, vol. 15 (1950), pp. 169-190. In this paper the diameter, n, of a connected network was defined as $n = \max_{a,b \in M} \min_{q} (ab, q)$, and it was conjectured that $p + n \geq 2m$. This is now known to be false; however, Theorem 6. 1 is a correct result which is closely related to the conjecture, for it may also be shown that $r \geq n$.

Proof. Suppose $p + r < 2m$. Select any set R of r linearly independent rows in a particular matrix representation. Since N is connected, there exists a non-zero entry in each column j; but since each row can be written as a linear combination of rows from R, it follows that for each column j there exists an $i \varepsilon R$, such that the ij entry is 1. Thus, in the rows of R there are at least m 1's. By our assumption, there remain $p' \leq p - m < m - r$ links (entries that are 1). Each of the $m - r$ rows not in R must contain a non-zero entry, since N is connected, and therefore $p' \geq m - r$, a contradiction.

We shall call a network $N(m, p)$ *rank minimal* if it is connected, and $p + r = 2m$.

THEOREM 6.2. *If a connected network is rank minimal, it is minimal.*

Proof. As in the proof of Theorem 6.1, we consider a matrix representation N of the rank minimal network N, and let R be a set of r linearly independent rows. Each column has a non-zero entry in some row of R, since N is connected. The set R' of the $m - r$ remaining rows must have a non-zero entry in each row for the same reason. However, since $p = 2m - r$, it is necessary that R have exactly one non-zero entry in each column, and R' exactly one in each row.

Let (ab) be any link of the network. We shall show that its removal results in a disconnected network, which will prove the theorem.

If $a \varepsilon R'$, then the removal of (ab) results in a network N' having no link for which a is the initial node, since the rows of R' have exactly one non-zero entry.

If $a \varepsilon R$, then either the row a has only one non-zero entry, and we use the above argument, or it has another non-zero entry, say in column c, $c \neq b$. We show that in the latter case column b has only the one non-zero entry, N_{ab}. For suppose another link (db), $d \neq a$, exists. Then $d \varepsilon R'$, for we showed above, essentially, that the rows of R have exactly one non-zero entry in each column, and we have assumed (ab) to exist and $a \varepsilon R$. Since the rows of R' have exactly one non-zero entry, it follows that N_{db} is the one for row d. But since the rows of R are a set of linearly independent ones for this matrix, row d must be a linear combination of rows of R. The row a must be in this combination, as it is the only one of R having an entry in the b column. However, we assumed that row a has a non-zero entry in column c. This must be subtracted, since row d cannot have an entry $N_{dc} = 1$. But this is impossible using only rows of R, since no other row of R has an entry in the

c column. This contradiction implies that column b has only $N_{ab} = 1$, and so $N - (ab)$ is a complete disconnected subnetwork of N. Thus N is minimal.

The converse statement is not true, as will be obvious from a comparison of Theorem 6. 4 and Theorem 3. 4.

The next lemma will be used in conjunction with Theorem 3. 4 to decompose any rank minimal network.

LEMMA 6. 3. *Let N be rank minimal. If N is reducible into the subnetworks N_1 and N_2, then either N_1 or N_2 is a single node.*

Proof. Let the N_i, $i = 1, 2$, have m_i nodes, p_i links, and rank r_i; and let m, p, and r denote the corresponding quantities in N. If neither of the N_i is a node, they are both connected subnetworks, so by Theorem 6. 1, $p_i \geq 2m_i - r_i$. It is evident from the definition of a reducible network that $p = p_1 + p_2 + 2$, and $m = m_1 + m_2$. Furthermore, if we let the matrix minor repersentation of N_i be denoted by the same symbols, we then have, for an appropirate labeling of the nodes, the following type of matrix representation for N:

$$\begin{bmatrix} N_1 & \begin{matrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{matrix} & N_2 \end{bmatrix},$$

whence one sees that $r \geq r_1 + r_2 - 1$. Thus, $p = p_1 + p_2 + 2 \geq 2m_1 - r_1 + 2m_2 - r_2 + 2 = 2(m_1 + m_2) - (r_1 + r_2 - 1) + 1 > 2m - r$, which is contrary to the assumption that N is rank minimal.

A tree such that the arcs all have one end node in common is called a star. It is not difficult to show that a star is rank minimal.

THEOREM 6. 4. *Let N be a rank minimal network which is not a star. If N has any arcs, there exists one arc-free rank minimal subnetwork C of N, such that the network formed from N by treating C as a node, all other links and arcs remaining unchanged, is a star. An arc-free rank minimal network C consists of exactly one irreducible rank minimal subnetwork C', not a single node, and, possibly, some simple nodes such that the network formed from C by treating C' as a node, all other links and nodes remaining unchanged, is a star. Furthermore, if C' is the irreducible rank minimal subnetwork of the arc-free rank minimal subnetwork C of N, then the network formed from N by treating C' as a node, all other links and nodes remaining unchanged, is a star.*

Proof. In the first case, apply Lemma 6. 3 to the first statement presented in the proof of Theorem 3. 4 to show that each arc which exists must have a simple node. The arc-free subnetwork C is rank minimal, for the removal of any arc from N results in a network N' for which $p' = p - 2$, $m' = m - 1$, and $r' \leq r$, implying $p' \leq 2m' - r'$. Thus, by Theorem 6. 1. N' is rank minimal. To C, first apply Theorem 3. 4 and then Lemma 6. 3 to show any nodes, not in an irreducible subnetwork, must be simple, and there is only one irreducible subnetwork C'. The same argument as applied above suffices to show that C' is rank minimal. The final statement is proved in the same manner.

In conclusion one may mention two more unsolved problems: Conditions on a minimal network that it be rank minimal, and a characterization of an irreducible rank minimal network.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.