The Incompleteness Of Hölder's Theorem During Most of The 20^{th} Century

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Abstract

In 1901, O. Hölder formulated a set of axioms for extensive measurement from which he showed there is a ratio scale, additive measure into the additive real numbers. Neither he nor anyone else apparently noticed until 1991 that had he considered mappings into the additive and multiplicative real numbers, there are two other types of representation. It turns out that for the physical cases that he had in mind, they are ruled out. This, however, is not true for many attributes of subjective intensity in economics and psychology. This note summarizes the currently known results including a complex of cross-modal predictions.

Dedication This report is based on my presentation at the Stanford University celebration of Patrick Suppes 90^{th} birthday – actually a week before St. Patrick's Day, 2012, his actual 90^{th} birthday. Pat and I have known each other for 57 years and have collaborated extensively on the representational theory of measurement, both on original research and on an exposition that resulted in the 3 volume *Foundations of Measurement*. What is reported here was only noticed after Volume III was published in 1990 and so can be thought of as an addendum to it.

1 Hölder's (1901) Theorem

1.1 The Classic Formulation

One very important measurement result involved structures $\langle X, \succsim, \odot \rangle$ where X is a set of objects; $\succsim$ is a weak ordering according to some attribute of the objects; and $\odot$ is a binary concatenation of X that is closed under $\odot$, i.e., if $x, y \in X$, then $x \odot y \in X$. A concrete example: X is a set homogeneous masses, which are ordered by using a equal-arm pan balance to determine which of two masses is heavier, and concatenation simply means placing two masses on the same pan of the balance.

Hölder's (1901) theorem, formalizing some of Helmholtz's earlier ideas, stated axioms about $\langle X, \succsim, \odot \rangle$ such that when mapped via an order preserving function φ into $\langle \mathbb{R}^+, \geq, + \rangle$, then

$$\varphi(x \odot y) = \varphi(x) + \varphi(y). \tag{1}$$

This class of models has been called *extensive* measurement. For a detailed critique of Hölder's formulation and a discussion of related but more satisfactory axiomatizations, see pp. 53-55 of Krantz, Luce, Suppes, & Tversky (1971).

1.2 What Hölder and the Rest of Us Missed for Most of 20^{th} Century

Typically, one does not invoke Hölder's theorem in isolation. For example, in mass measurement one also considers the multiplicative conjoint representation of mass as volume times density. And in order to force a common measure φ of mass for both the concatenation and conjoint scales, a linking property must be discovered between the two structures. In this case, two distribution conditions serve as that link.

But, given that this involved both addition and multiplication, why did Hölder map $\langle X, \succsim, \odot \rangle$ just into $\langle \mathbb{R}^+, \geq, + \rangle$ rather than into the full real numbers $\langle \mathbb{R}^+, \geq, +, \times \rangle$?

In fact, if we do map into $\langle \mathbb{R}^+, \geq, +, \times \rangle$, then his axioms about $\langle X, \succsim, \odot \rangle$ lead to three possible, very distinct, representations, namely,

$$\varphi(x \odot y) = \varphi(x) + \varphi(y) + \delta\varphi(x)\varphi(y), \qquad \delta = -1, 0, 1. \tag{2}$$

(Luce, 1991, 2000). This representation is called *polynomial-additive* (for short *p-additive*) because it is the only polynomial that can be transformed into an additive representation (Aczél, 1966). That (2) can be so transformed is easily shown (see Section 2.2.2); that it is the only such polynomial is more subtle.

Clearly, p-additivity (2) with for $\delta = 0$ is, in fact, additive. The distribution laws of the physical applications force $\delta = 0$. So the cases $\delta = -1$ and 1 do not matter for physics.

But they very much do matter for the behavioral sciences, as I outline next.

2 Testable Predictions From Luce's (2004) Global Psychophysical Theory:

2.1 Binary Senses

2.1.1 Primitives for Binary Receptors

The ears, eyes, and arms are binary receptors that function as cooperative pairs. Of course, as was long ago remarked, had there had been a grand designer, rather than evolution, we would undoubtedly have 3 arms and hands – 2 to hold and the middle one to manipulate. But, in reality, we need only consider binary and unary senses (Section 2.2).

For any intensity attribute, let X denote the set of physical intensities less their respective threshold intensities. (Not dB!) Thus, for any aspect of the signal, such as frequency, then intensity $x = 0$ is the threshold level of that signal. If $x, u \in X$, then the stimulus presented is $(x, u) \in X \times X$. The subjective intensity ordering $\succsim$ over $X \times X$ is assumed to be a weak order.

Suppose that a respondent matches (z, z) to (x, u), then define the operator $\oplus$ by

$$x \oplus u := z. \tag{3}$$

The nature of $\oplus$ characterizes the intensity trade-off between the binary receptors.

Suppose also that the experimenter presents signal x and a number p, and assume there is a reference signal $\rho < x$ either given by the experimenter or generated by the respondent. The respondent is asked to report the signal y such that the "interval from from the reference signal ρ to y" is perceived as p "times" as intense as the "interval from ρ to x". It is convenient to think of y as an operator: $y = x \circ_p \rho$.

When $\rho = 0$, this is nothing but S. S. Stevens' (1975) method of magnitude production.

2.1.2 Linking $\oplus$ and $\circ_p$

Luce (2002, 2004, 2008) formulated behavioral (i.e., testable) axioms that assert invariances among the primitives. Included were two linking properties between matching and production somewhat analogous to the distribution properties in physics. These allow us to use the same psychophysical function for both matching and production.

2.1.3 Representations of Binary Intensities

These assumed properties imply the following numerical representation: A *p–additive* order preserving psychophysical function ψ:

$$\psi(x \oplus y) = \psi(x \oplus 0) + \psi(0 \oplus y) + \delta\psi(x \oplus 0)\psi(0 \oplus y), \quad \delta = -1, 0, 1, \tag{4}$$

with

$$\psi(x \oplus 0) = \gamma\psi(0 \oplus x). \tag{5}$$

And a weighting function W over positive numbers such that

$$W(p) = \frac{\psi(x \circ_p \rho) - \psi(\rho)}{\psi(x) - \psi(\rho)}. \tag{6}$$

For loudness and focusing on the data from individual respondents, Steingrimsson & Luce (2005a, 2005b, 2006, 2007) strongly supported the behavioral axioms. Steingrimsson (2009, 2011, submitted, in preparation a,b) has and is running a parallel series for brightness and for "perceived" contrast, and the axioms are equally strongly supported.

2.1.4 Form of the Psychophysical Function ψ

A simple behavioral invariance implies ψ is a power function

$$\psi(x) = \alpha x^{\beta}. \tag{7}$$

The invariance has been empirically supported for loudness and brightness in the third article of above sequences.

In the 1960s and 1970s this was thought to be sustained for all intensity (prothetic) attributes (summarized in Stevens, 1975). Using geometric averaging, the log-log plots of binary attributes were "plausibly" linear. Caution: fitted lines do guide one's eyes.

2.1.5 Several Familiar Operator Properties

Commutativity:

$$x \oplus u \sim u \oplus x. \tag{8}$$

This property was rejected for loudness, brightness, perceived contrast with individuals analyzed separately (Steingrimsson & Luce, 2005a; Steingrimsson, 2011, in preparation a).

Associativity:

$$(x \oplus u) \oplus v \sim x \oplus (u \oplus v). \tag{9}$$

No data have been reported so far. However, we will shortly see why it is expected to fail.

Bisymmetry:

$$(x \oplus y) \oplus (u \oplus v) \sim (x \oplus u) \oplus (y \oplus v). \tag{10}$$

Accepted for loudness, brightness, and contrast with individuals separately evaluated (Steingrimsson & Luce, 2005b; Steingrimsson, submitted, in preparation b).

None of these have yet been tested for two-arm weight lifting.

2.1.6 Predictions of the Binary Theory

Suppose the p-additive representation holds. Then Luce (2012, in press) has proved:

- For $\delta = 0$, bisymmetry is satisfied.
- For $\delta \neq 0$, bisymmetry is satisfied iff commutativity is satisfied. Conclusion: The data − Yes to bisymmetry and No to commutativity − imply $\delta = 0$, i.e. simple additivity.

This argument corrects Luce's (2004) unconditional claim that $\delta = 0$. The error was pointed out by Dr. C. T. Ng (Luce 2008).

And If bisymmetry holds in this context, then associativity cannot hold.

2.2 Unary Theory

Many intensity senses are unary, not binary. Examples: taste, electric shock, vibration, force, linear extent, preference for money, etc.

Consider those unary attributes for which a physical signal concatenation $\odot$ exists that has an additive physical ratio scale representation. So $\odot$ must satisfy both commutativity and associativity.

2.2.1 Unary Magnitude Production

Exactly as with the binary theory, we assume magnitude production and a linking axiom between the $\circ_p$ production structure and the $\odot$ structure. And that means we definitely need $\langle \mathbb{R}^+, \geq, +, \times \rangle$, so we have to allow the p-additive form. And, quite unlike the binary case, $\odot$ satisfies all of: bisymmetry, commutativity, and associativity.

I do not know of any principled argument that forces the $\delta = 0$ case for unary attributes.

2.2.2 p-Additive Scale Types

The following observations were first made in connection with utility theory (Luce, 2010, 2011)

For the additive case ($\delta = 0$), φ is well known to be a ratio scale.

For the non-additive cases ($\delta = -1, 1$), p-additivity is equivalent to

$$1 + \delta\varphi(x \odot y) = [1 + \delta\varphi(x)]\,[1 + \delta\varphi(y)] = 1 + \delta\varphi(y \odot x)$$

Clearly, ln of this is an additive representation. Because we have $1 + \delta\varphi(x)$ and $\delta = -1, 1$, φ has to be dimensionless and so is an **absolute**, not a ratio, scale.

2.3 Form of the Psychophysical Function φ

Unlike the binary case, there are 3 types corresponding to the value of δ, and Luce (2012, in press) shows

$$\begin{array}{ll} \varphi_0(x) = \eta x \quad (\eta > 0) & \text{if } \delta = 0 \\ \varphi_1(x) = e^{\lambda x} - 1 \ \ (\lambda > 0) & \text{if } \delta = 1 \\ \varphi_{-1}(x) = 1 - e^{-\kappa x} \ \ (\kappa > 0) & \text{if } \delta = -1 \end{array} .$$

For $\delta = 0$, this is a special case of a power function. For $\delta \neq 0$, these two exponential function clearly are not power functions.

2.4 But the Empirical Claim is Just Power Functions

The empirical literature seemed to defend power functions; what gives?

For example, Stevens (1959) reported averaged cross-modal matches between loudness of noise, vibration, and shock each measured in dB:

- loudness versus vibration seemed to be a power function – but fitted lines can deceive.
- shock versus loudness and versus vibration were equally not power functions, although Stevens tried – not very convincingly – to explain that fact away.

3 Predictions of Cross-Modal Matches

From the binary and unary theories, it is fairly apparent that predictions of cross-modal mappings should follow. They do. They are moderately complicated because of the unary case's 3 representations $\delta = -1, 0, 1$. Because we know of no $\delta = -1$ attribute other than utility of money for some people, that attribute is omitted from the columns in the following already complex table.

Insert Table 1 about here
Reprinted with permission from Luce (2012, in press).

- The unary predictions differ from a power function only when $\delta = \pm 1$.
- I suspect that further research will confirm that for shock, pain, and vibration $\delta \neq 0$.
- Utility of money, which is unusual because the domain includes losses and well as gains, appears to be a case where all 3 can occur: risk seeking, risk neutral, and risk averse types.

4 Closing Remarks

The overlooked solutions to Hölder's axiomatizations did not matter at all for physics, but they certainly appear to matter greatly for the behavioral and economic sciences. Much experimentation is needed to check these predictions. But it is most important to realize that such experiments *must be* analyzed for each individual respondent; averaging respondents is clearly inappropriate especially if they have different values of δ.

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Table 1

				Match $\varphi_b(z)$	to $\varphi_a(x)$
			2-D		1-D
			$\delta_b = 0$	$\delta_b = 0$	$\delta_b = 1$
	2-D	$\delta_a = 0$	power	power	$\frac{1}{\lambda_b} \ln\left(1 + \eta_a x^{\beta_a}\right)$
x					
		$\delta_a = 0$	power	proportion	$\frac{1}{\lambda_b} \ln\left(1 + \eta_a x\right)$
	1-D	$\delta_a = 1$	$\left[\left(e^{\lambda_a x} - 1\right)/\alpha_b\right]^{1/\beta_b}$	$\frac{1}{\eta_b}\left(e^{\lambda_a x} - 1\right)$	proportion
		$\delta_a = -1$	$\left[\left(1 - e^{-\kappa_a x}\right)/\alpha_b\right]^{1/\beta_b}$	$\frac{1}{\eta_b}\left(1 - e^{-\kappa_a x}\right)$	$x\lambda_a/\lambda_b$