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Predictions About Bisymmetry and Cross-Modal Matches From Global Theories of Subjective Intensities

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The article first summarizes the assumptions of Luce (2004, 2008) for inherently binary (2-D) stimuli (e.g., the ears and eyes) that lead to a "p-additive," order-preserving psychophysical representation. Next, a somewhat parallel theory for unary (1-D) signals is developed for intensity attributes such as linear extent, vibration to finger, and money. The 3rd section studies the property of bisymmetry in these 2 cases. For the 2-D case and the nontrivial p-additive forms, Proposition 3 shows that bisymmetry implies commutativity of the presentations. Bisymmetry has been empirically well sustained, whereas commutativity has been rejected for loudness, brightness, and perceived contrast, thus implying that pure additivity must obtain in the 2-D context. By contrast, bisymmetry and commutativity are automatically satisfied by the p-additive 1-D theory. The 4th section explores the resulting complex of cross-modal predictions. For the additive 1-D case and the 2-D case, the predictions are power functions. For the nonadditive 1-D cases, other relations are predicted (see Table 2). Some parameter estimation issues are taken up in Appendices B and C.

Keywords: bisymmetry, cross-modal matching, magnitude production, p-additive psychophysical scale, representations of unary and binary stimuli

This article is concerned with sensory attributes of subjective intensities, what Stevens (1975) called prothetic continua, and predictions about their cross-modal matches. The theories differ substantially according to whether or not nature has evolved pairs of sensors that normally work together in some interactive way. These paired sense organs are called binary or two-dimensional (2-D) attributes. Two very clear examples are the eyes and ears and probably the arms for lifting weights. Considerably less clear are the nostrils and smell. Sense organs that are not so paired lead to what I call unary or one-dimensional (1-D) sensations. Some examples that have been studied in the literature are vibration, shock, taste of saltiness, and preferences over money. Of course, binary attributes can be restricted to monocular or monaural stimulation that transforms them to certain unary cases (see Equations 52 and 62) that, like the 2-D case, exhibit power function cross-modal matches.[1]

The first section, A General Representation of Binary Sensory Intensities, summarizes the behavioral axioms and resulting representation that I arrived at in Luce (2002, 2004) and that has been evaluated for loudness, brightness, and contrast. Anyone familiar with those results should skip to the next section, A General Representation of Unary Sensory Intensities, which develops from known results a new theory for unary attributes with quite different properties from those in the binary cases. Considerable new experimentation is called for. The third section, Bisymmetry, defines the concept, points out that it seems to hold empirically, and works out the rather different properties it implies for the binary and unary cases. The final section, Cross-Modal Matching, explores a complex of predictions arising from these theories. True, in many cases involving binary and the simplest unary cases, Stevens' power law (see Stevens, 1975, for an extensive overview) is predicted, but in several other cases other predictions follow. I look at what Stevens (1975) reported for loudness, vibration, and shock. There seems to be evidence that matching shock and, indeed, several other 1-D continua to loudness and vibration may not satisfy a power law as was widely assumed in that early literature. Appendix A is proofs, Appendix B is estimation issues in the binary case, and C is estimation issues in the unary case.

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[1] The work pattern that Ragnar Steingrimsson and I have established over the 6 years of our collaboration is that I would work out, often at his instigation, a theoretical prediction; together we would then design one or more relevant experiments; then he would prepare the stimuli and run the experiments; and we would jointly write up the resulting article. The pace has been such that the predictions reported here will take at least 2 years to test experimentally and put in publishable form. At my age, I cannot be completely confident that I will live to see it done, so I concluded that it would be best to publish these theoretical results before the data are collected.

A General Representation of Binary Sensory Intensities

Luce (2004) presented several behavioral properties relating two structures that model the psychophysics of subjective intensity. The first structure has the form $\langle X \times X, \succsim \rangle$ for which X is interpreted to be the set of all physical intensities each minus its threshold intensity[2] (not a transformed function of physical intensity such as dB units) when other features of the signal, such as frequency, are fixed. So X is the nonnegative real numbers with the unit of intensity measurement. Thus, for this measure of intensity, the measure of each threshold is 0. Pairs (x, u) are interpreted in psychophysics as presenting physical signals x and u to, say, the left and right ears (or eyes) simultaneously.

Suppose that the experimenter presents two pairs of signals (x, u) and (y, v) and asks the respondent to report which pair seems (subjectively) more intense (e.g., louder in audition or brighter in vision). If, for example, (x, u) seems at least as intense as (y, v), we write $(x, u) \succsim (y, v)$. The ordering relation $\succsim$ is assumed to satisfy two properties:

Assumption 1: $\succsim$ is a *weak order* in the sense that every pair of signals can be ordered and that the ordering is transitive, meaning that for all signals

$$(x, u) \succsim (y, v) \;\&\; (y, v) \succsim (z, w) \Rightarrow (x, u) \succsim (z, w). \tag{1}$$

Assumption 2: $\succsim$ is *strictly monotonic*[3] in the sense that for all signals

$$(x, u) \succsim (y, u) \Leftrightarrow x \geq y, \tag{2}$$

$$(x, u) \succsim (x, v) \Leftrightarrow u \geq v. \tag{3}$$

We say that (*unrestricted*) *solvability* is satisfied in the sense that given any three of the signals x, y, u, v, the respondent can always select the fourth so that $(x, u) \sim (y, v)$, where $\sim$ means indifference in the sense that both $(x, u) \succsim (y, v)$ and $(y, v) \succsim (x, u)$ hold. Solvability is essential, for example, in Equation 4 below, which underlies the entire development.

Assumption 3: Solvability is assumed to be satisfied.

In the second part of Luce (2004) and using solvability (Assumption 3), I worked with the symmetric matches $(x, u) \sim (z, z)$. What is involved here is for the respondent to adjust an intensity z until the pair (z, z) matches (x, u) in the sense that they exhibit the same subjective intensity. Of course, in practice z really is a random variable when a fixed (x, u) is presented and empirically matched by (z, z). Some form of central tendency is reported as the estimate of z. The estimation error plays a very significant role in statistical evaluation of whether certain indifferences ($\sim$) are satisfied. See, for example, the articles of Steingrimsson (2009, 2011, 2012a, 2012b, 2012c) and Steingrimsson and Luce (2005a, 2005b) for detailed discussions of how we have dealt with it in practice. The randomness of data is not explicitly discussed in this article.

Because of both Assumption 1 and Assumption 2, the function that maps (x, u) to z can be thought of as an operation $x \oplus u$ from $X \times X \overset{\text{onto}}{\Longrightarrow} X$ that is strictly increasing in each variable.[4] Thus,

$$(x, u) \sim (x \oplus u, x \oplus u). \tag{4}$$

Because by Equation 4

$$(x, x) \sim (x \oplus x, x \oplus x),$$

which property is called *idempotence*, monotonicity, Assumption 2 implies that

$$x \oplus x = x. \tag{5}$$

In particular, $0 \oplus 0 = 0$.

The second structure involves formalizing the method of magnitude production first introduced into psychophysics in mid-20th century by S. S. Stevens (summarized in Stevens, 1975). This is described below in the section The Magnitude Production Operator $\circ_p$, where properties Assumption 4 and Assumption 5 are described.

The p-Additive Representation

In the psychophysical context, Luce (2004, 2008) showed that there is a real, order-preserving representation Ψ onto the domain of physical intensities—that is, numbers with a fixed unit. We assume that Ψ is decomposable in the sense of Assumption 6.

Assumption 6: For some function F that is strictly increasing in each of two real variables it is the case that

$$\Psi(x, u) = F[\Psi(x, 0), \Psi(0, u)].$$

Then under our assumptions the representation simplifies to

$$\Psi(x, y) = \Psi(x, 0) + \Psi(0, y) + \delta\Psi(x, 0)\Psi(0, y), \quad \delta = -1, 0, 1, \tag{6}$$

which for $\delta = 0$ and 1 was Equation 18 of Theorem 1 of Luce (2004, p. 448).[5] The form with $\delta = -1$ implies that Ψ is bounded. Moreover, in the context of utility theory, the range of Ψ is an interval of both positive and negative real numbers, and Luce (2000, 2010) has made much of the $\delta = -1$ case.

The representation (Equation 6) is called a *p-additive representation* because it is the unique polynomial that can be transformed into an additive representation, as is shown below. Proving that it is the only polynomial form that can be transformed into an additive representation is quite a bit trickier but can be done (see Aczél, 1966, p. 61).

[2] Using intensities is not unusual, but subtracting the threshold intensity is a bit more unusual. It serves to establish a natural zero point for the subjective intensity measure.

[3] Because of the brightness phenomenon called Fechner's paradox, monotonicity breaks down when one intensity is substantially smaller than the other. This seems to be more an artifact of the laboratory than a real-world actuality. Moreover, with perceived contrast, there seems to be no evidence of an analog to Fechner's paradox.

[4] In Luce (2004), the operation $\oplus$ was called $\oplus_s$ because I also defined the asymmetric operations $\oplus_l$ and $\oplus_r$ by $(x, u) = (x \oplus_l u, 0) = (0, x \oplus_r u)$. Because of the very noticeable perceptual changes encountered with $\oplus_l$ and $\oplus_r$, here I work only with the symmetric operation, writing it simply as $\oplus$.

[5] The decomposition of Equation 6, as well as property in Equation 8 below, is problematic for brightness for two reasons: first, because binocular rivalry makes it difficult to actualize these signals directly, but they may be estimated; and second, the nonmonotonicity called the Fechner Paradox occurs when x or y approaches 0, as of course is the case for $\Psi(x, 0)$ and $\Psi(0, y)$.

Writing the function ψ as $\psi_\oplus$, we may rewrite $\Psi(x, y)$ as

$$\Psi(x, y) = \Psi(x \oplus y, x \oplus y) = [\psi_\oplus(x \oplus y), \psi_\oplus(x \oplus y)],$$

and so Equation 6 becomes equivalent to

$$\psi_\oplus(x \oplus y) = \psi_\oplus(x \oplus 0) + \psi_\oplus(0 \oplus y) + \delta\psi_\oplus(x \oplus 0)\psi_\oplus(0 \oplus y),$$
$$\delta = -1, 0, 1. \quad (7)$$

It was also shown that Ψ and $\psi_\oplus$ must satisfy that for some constant $\gamma > 0$

$$\frac{\Psi(x, 0)}{\Psi(0, x)} = \frac{\psi_\oplus(x \oplus 0)}{\psi_\oplus(0 \oplus x)} = \gamma. \quad (8)$$

Transforming the p-additive representation to an additive one. Because in Luce (2004) the important fact that the p-additive form can be transformed into an additive one was not mentioned, I show that here. For $\delta = 0$, Ψ is already purely additive, and for $\delta = -1, 1$, it can be transformed into an additive form by rewriting Equation 6 as

$$1 + \delta\Psi(x, y) = [1 + \delta\Psi(x, 0)]\,[1 + \delta\Psi(0, y)]. \quad (9)$$

If we define

$$\Phi(x, y) := \ln[1 + \delta\Psi(x, y)], \quad (10)$$

then Equation 6 transforms to

$$\Phi(x, y) = \Phi(x, 0) + \Phi(0, y). \quad (11)$$

Note that Equation 10 implies that $1 + \delta\Psi(x, y)$, and so $\delta\Psi(x, y)$, must be dimensionless. This is because the integer 1 is dimensionless, and because $\delta\Psi(x, u)$ is added to 1, it cannot have any unit. Thus, there is no gain of generality by setting δ different from either -1 or 1 in Equation 6. C. T. Ng (personal communication, April 4, 2008) made this observation to A. A. J. Marley and me in discussion about entropy-modified utility measures (Ng, Luce, & Marley, 2009).

Define

$$\Theta = \begin{cases} \Psi & \text{if } \delta = 0 \text{ in Equation 6} \\ \Phi & \text{if } \delta \neq 0 \text{ in Equation 6 and } \Phi \text{ is defined by Equation 10.} \end{cases} \quad (12)$$

Note that Θ is order preserving and additive,

$$\Theta(x, y) = \Theta(x, 0) + \Theta(0, y), \quad (13)$$

by Equation 6 when $\delta = 0$ and by Equation 11 when $\delta = 1$. Note also that from Equation 13 with $x = y = 0$, which are threshold values, yields

$$\Theta(0, 0) = 2\Theta(0, 0) \Rightarrow \Theta(0, 0) = 0. \quad (14)$$

The errata (Luce, 2008) simply said that $\delta = 0$ did not follow from the assumptions of Luce (2004), but it did not show why it might imply it. The purpose of the section Bisymmetry is to fill in some of the details about the relations among the concepts of bisymmetry, commutativity, and associativity. Specifically, a simple representation that is equivalent to bisymmetry is developed in Proposition 2 below, which is key to the next three propositions, which are somewhat surprising.

The Magnitude Production Operator $\circ_p$

In *magnitude production*, the respondent is presented with signal (x, x) and a number $p > 0$ and is asked to give the signal (y, y) that seems, in a sense to be discussed, to be "p times as intense as (x, x)." But we assume that the respondent interprets this to mean that there is a *reference signal* $\rho = \rho(p) < \min(x, y)$ such that the subjective interval from (ρ, ρ) to (y, y) is p times the subjective interval from (ρ, ρ) to (x, x). Either ρ is provided by the experimenter or generated by the respondent. We denote this operation

$$(x, x)\circ_p(\rho, \rho) := (y, y). \quad (15)$$

Two assumptions are made about the operation $\circ_p$:

Assumption 4: The operation $\circ_p$ is strictly increasing, nonconstant on a nontrivial interval, and continuous in p.

Assumption 5: The operation $\circ_p$ is idempotent in the sense that Equation 15 satisfies $(x, x) \circ_p (x, x) \sim (x, x)$.

As with the psychophysical function, the 2-D structure can be put in 1-D form. And Luce (2004) showed that the resulting general representation is

$$\psi_{\circ_p}(x\circ_p\rho) = \psi_{\circ_p}(x)W(p) + \psi_{\circ_p}(\rho)[1 - W(p)], \quad (16)$$

which for those familiar with utility theory is the subjective expected utility representation if W is a probability measure. The expression (Equation 16) is equivalent to the magnitude production representation:

$$W(p) = \frac{\psi_{\circ_p}(x\circ_p\rho) - \psi_{\circ_p}(\rho)}{\psi_{\circ_p}(x) - \psi_{\circ_p}(\rho)}. \quad (17)$$

The special case $x \circ_p 0$ is separable in the sense that it has a representation

$$\psi_{\circ_p}(x, p) := \psi_{\circ_p}(x\circ_p 0) = \psi_{\circ_p}(x)W(p), \quad (18)$$

where $\psi_{\circ p}$ is the psychophysical scale for the structure $\langle X, \succsim, \circ p\rangle$ and $W(p)$ is a cognitive distortion of perceived positive numbers. This is a multiplicative form of "additive" conjoint representation: The several measurement studies of that representation have rested on three necessary properties of increasing advantage in carrying out empirical evaluation. The first was *double cancellation* (Krantz, Luce, Suppes, & Tversky, 1971, p. 250). Next came the *Thomsen condition* (Holman, 1971; Krantz et al., 1971, p. 250), which is the special case of double cancellation by restricting $\succsim$ to $\sim$, that is, with boldface signifying estimated signals:

$$x\circ_p 0 \sim y\circ_q 0 \;\&\; y\circ_r 0 \sim z\circ_p 0, \;\&\; x\circ_r 0 \sim \mathbf{z}'\circ_q 0$$
$$\Leftrightarrow (x, p) \sim (\mathbf{y}, q) \;\&\; (\mathbf{y}, r) \sim (\mathbf{z}, p) \;\&\; (x, r) \sim (\mathbf{z}', q)$$
$$\Rightarrow \mathbf{z} = \mathbf{z}'. \quad (19)$$

And finally, Luce and Steingrimsson (2011) have recently noted that Falmagne's (1976) *conjoint commutativity rule* (originally stated for random conjoint measurement) is, in the context of the other axioms, equivalent to the Thomsen condition (Equation 19). It may be stated as follows. Define

$$m_{p,q}(a) = b \Leftrightarrow (a, p) \sim (b, q), \quad (20)$$

which by Equation 18 is equivalent to

$$a\circ_p 0 \sim b\circ_q 0.$$

Then conjoint commutativity is said to hold if and only if

$$m_{r,s}[m_{p,q}(a)] = m_{p,q}[m_{r,s}(a)]. \tag{21}$$

Recast in the operator notation, this is equivalent to the four indifferences

$$a\circ_p 0 \sim \mathbf{b}\circ_q 0 \Leftrightarrow (a, p) \sim (\mathbf{b}, q)$$

$$\mathbf{b}\circ_r 0 \sim \mathbf{c}\circ_s 0 \Leftrightarrow (\mathbf{b}, r) \sim (\mathbf{c}, s)$$

$$\mathbf{d}\circ_s 0 \sim a\circ_r 0 \Leftrightarrow (\mathbf{d}, s) \sim (a, r)$$

$$\mathbf{e}\circ_q 0 \sim \mathbf{d}\circ_p 0 \Leftrightarrow (e, q) \sim (\mathbf{d}, p)$$

implying that

$$\mathbf{c} = \mathbf{e}.$$

Notice that these four conditions and the conclusion form a quadruple cancellation condition.

From early on (e.g., Gigerenzer & Strube, 1983), it was recognized that double cancellation is a very inefficient property to test empirically because of numerous inherent redundancies. And the Thomsen condition, which bypasses the redundancies of double cancellation, is also difficult to test because of response variability, especially the fact that the two hypotheses entail estimates, $\mathbf{y}$ and $\mathbf{z}$, where $\mathbf{z}$ depends upon $\mathbf{y}$; whereas the conclusion entails only one estimate, $\mathbf{z}'$ and the question is whether or not $\mathbf{z} = \mathbf{z}'$ holds. The conjoint commutativity condition (Equation 21) has the great advantage of being statistically symmetric, but also the decided disadvantage that both sides of the indifference rest upon double compound estimates and so are quite variable at least when the compounds are not trivial.

Linking the Two Structures $\oplus$ and $\circ_p$

At this point we have two psychophysical measures of subjective intensity $\psi_\oplus$ and $\Psi_{\circ_p}$. Clearly, we need to understand when it is possible to prove that there is a single psychophysical measure Ψ that is p-additive and that satisfied a conjoint condition analogous to Equation 18.

I introduced testable behavioral invariances for each of the two structures separately and invariances that link the two structures in a fashion somewhat like the classical theory for the measurement of mass: Mass orderings can be generated either by the concatenation of (homogeneous) masses on a pan balance or by the trade-offs between volume of homogeneous substances and the various substances themselves. And a linkage in the form of a testable property is assumed between the two structures that permits one to prove that the two resulting measures of mass are, in fact, identical (see, e.g., Luce, 2009; Luce et al., 1990, pp. 312–318). The appropriate linking properties, not explicitly called that, were formulated in Luce (2004) as segregation and joint presentation decomposition. These were tested for auditory intensities by Steingrimsson and Luce (2005b) and for brightness by Steingrimsson (2011). Appendix B outlines some estimation issues in the binary case.

A General Representation of Unary Sensory Intensities

Of course, a substantial number of prothetic attributes, such as force, linear extent, vibration, money, and, perhaps, odor, are unary as discussed in the introduction. That means that the theory we have used above (Luce, 2004, 2008) has to be altered to take account of the inherent dimensional difference.

The primitives in the 1-D case are formally highly parallel to $\oplus$ and $\circ_p$ of the 2-D case, but the former has such a different interpretation, as is discussed in the next section, that I use a different symbol, $\odot$, for it, whereas magnitude production operation $\circ_p$ is unchanged.

The Primitives and Their Representations

The two binary operations need to be explicitly defined for unary (1-D) signals.

Concatenation $\odot$. In the 2-D cases, $\oplus$ was interpreted as the symmetric match of the joint presentation of, say, $x \in X$ to the left and $y \in X$ to the right ear (or eye), respectively. Several testable properties were assumed and sustained in the first article in each series for loudness, brightness, and perceived contrast—each being a 2-D modality. For the 1-D cases, the interpretation is altered to mean simple physical concatenation of x and y, which, of course, means that the physical measure of $x \odot y$ is simply the sum of the two intensities: $x + u$. Thus, there is no experimental issue about finding $\odot$. This case was modeled in physics by Hölder's (1901) axioms, heavily involving commutativity and associativity that for all signals x, y, and z,

$$x \odot y \sim y \odot x, \tag{22}$$

$$(x \odot y) \odot z \sim x \odot (y \odot z). \tag{23}$$

This led, via Hölder's theorem or Falmagne's (1975) weakened version, to a simple additive representation.

The p-additive representation of $\odot$. As was true of Equation 17 above, the mapping of the operator $\circ_p$, in Equation 29 below, entails both addition and multiplication, that is, the mapping is to $\langle \mathbb{R}^+, \geq, +, \times \rangle$. So there seems no good reason to limit the representation of Hölder's concatenation axioms to a purely additive representation but rather to the full positive real numbers. Luce (2000) showed[6] that doing this yields the p(olynomial)-additivity representation

$$\psi_\odot(x \odot y) = \psi_\odot(x) + \psi_\odot(y) + \delta\psi_\odot(x)\psi_\odot(u),\ \ \delta = -1, 0, 1. \tag{24}$$

The production operator $\circ_p$. Let x and $\rho(p)$, with $x > \rho(p)$, be signals with x given by the experimenter, with a reference signal $\rho(p)$ either given by the experimenter or generated by the respondent and that we estimate from the observed behavior in the light of the theory, and $p > 0$ is a number given by the experimenter. The respondent is asked to provide the signal z that is perceived as yielding the subjective interval from $\rho(p)$ to z that is p times the

[6] I failed to make clear there that δ really is no more general than either -1, 0, or 1. That simple but important fact was pointed out in Ng et al. (2009).

subjective interval from $\rho(p)$ to x. This is Stevens's method of magnitude production for the unary case. Note that z is a function of x, $\rho(p)$, and p that can be treated as an operation: $x \circ_p \rho(p) := z$. The operation $\circ_p$ is interpreted just as it was in the 2-D case. I have used the same symbol W for both the 1-D and 2-D cases on the grounds that it represents a cognitive function having nothing to do with the type of domain. It, therefore, leads to the equivalent representation in Equations 16 and 17. Of course, it is definitely an empirical question whether this assumption is justified.

Linking the two structures $\odot$ and $\circ_p$. As in physical measurement, if there are two ways of manipulating an attribute—often a concatenation operation and a trade-off, as in mass concatenation and density–volume trade off—there has to be some linkage between the two structures to ensure that there is but a single measure of mass. In such a case, the link is a distributive law discussed by Luce et al. (1990, p. 125) and most succinctly by Luce (2009). The key linkage between our two psychophysical structures in the 1-D cases is simply the property of segregation:[7] For all x, $y \in X$ and $p \in \mathbb{R}^+ = [0, \infty]$,

$$(x \circ_p 0) \odot y \sim (x \odot y) \circ_p y. \tag{25}$$

This link is appreciably simpler than the linkage in the 2-D cases (see section Linking the two structures $\oplus$ and $\circ_p$; Luce, 2004).

Given the assumptions in Luce (2010) including Equation 25, then Theorem 4.4.6 of Luce (2000) showed the existence of a strictly monotonic real function, which, to distinguish from the 2-D case, is denoted φ with the following two properties: It is p-additive (Equation 24),

$$\varphi(x \odot y) = \varphi(x) + \varphi(y) + \delta\varphi(x)\varphi(u),\ \ \delta = -1,0,1, \tag{26}$$

and it satisfies Equation 16,

$$\varphi(x \circ_p \rho) = \varphi(x)W(p) + \varphi(\rho)[1 - W(p)]. \tag{27}$$

In Luce (2010), Equation 27 was numbered Equation 12.

1-D Theory and the Form of φ

The p-additive utility case was worked out in detail in Luce (2010), who derived Equations 26 and 27 (but using the utility notation U rather than the more generic φ that I use here), and Proposition 4 of that article derived the following representations: There exists a strictly increasing function g: $X \xrightarrow{\text{into}} \mathbb{R}^+$ that is additive over $\odot$, which, because money is itself additive, means that $g(x)$ is proportional to x, and therefore

1. If $\delta = 0$, then φ_0 is strictly increasing and onto $\mathbb{R}$ and

$$\varphi_0(x) = \eta x \ \ (\eta > 0). \tag{28}$$

2. If $\delta = 1$, then $\theta = 1 + \varphi_+$ is strictly increasing, onto $\mathbb{R}_+$, multiplicative, and

$$\varphi_+(x) = e^{\lambda x} - 1\ (\lambda > 0). \tag{29}$$

3. If $\delta = -1$, then $\theta = 1 + \varphi_-$ is strictly decreasing, onto $\mathbb{R}_+$, multiplicative, and

$$\varphi_-(x) = 1 - e^{-\kappa x}\ (\kappa > 0). \tag{30}$$

Although clearly Cases 2 and 3 are different from power functions

$$\alpha x^{\beta},\ \alpha > 0,\ \beta > 0, \tag{31}$$

that does not preclude the possibility that in the region for which data can be collected—say 0–100 kg in the case of one-arm-lifted weights where the upper asymptote for younger, athletic men is probably less than 80 kg—the approximation is reasonably good. As was true in most of the Stevens-based literature, the reported data are based on geometric averaging over respondents, which is surely misleading in the nonlinear cases $\delta = -1, 1$. For example, Figure 1A[8] plots Equation 30 so that at $I = 100$, $\varphi_-(100) = 2/3$ of the asymptote,[9] which yields $\kappa = .011$, and the dotted one is the exponential (Equation 29) so that $\varphi_+(100) = \varphi_-(100)$, which implies $\lambda = .051$. The two "straight lines" are the mean and geometric mean. Figure 1B shows two more exponentials with $\lambda = .051 \pm .010$. Figure 1C shows both the mean and the geometric mean of φ_- for each of these three φ_+. Table 1 gives the linear regressions and least squares goodness of fit normalized by the range of the estimated means. In no case are linear fits rejected, and the geometric mean is slightly better than the mean.

It appears to be a challenge to devise experiments that can distinguish these average functions from Equation 31. Galanter (1962) did some empirical work on power function utility measures, which Stevens (1975) describes, but he did not try Equation 29 or 30 on individual respondents.

A good deal of further empirical work is needed to try to understand the relation of data to theory. One taxonomy we need is just how many of the 1-D cases have an additive physical measure of the attribute in question and so can be encompassed by Hölder's theorem. For those that do not, the current approach is not viable.

General criterion for δ. Unlike the 2-D case, I do not so far know of any reason to rule out the $\delta \neq 0$ cases. So for each respondent, one must estimate δ from data. That can be done by directly adapting the criterion that was established for utility theory (Proposition 2 of Luce, 2010). Define $p_{1/2}$ to be the p value such that

$$W(p_{1/2}) = \frac{1}{2}. \tag{32}$$

The number $p_{1/2}$ exhibits the following simple behavioral property: For any x, y with $x > y$

$$x \circ_{p_{1/2}} y \sim y \circ_{p_{1/2}} x, \tag{33}$$

which follows immediately from Equations 27 and 32. Of course, we cannot be certain in advance that respondents' data will prove to be so consistent. Note that Equations 27 and 32 together imply

[7] Segregation was first stated in a form suitable for utility in Luce (2000, Section 4.4). But that formulation is not quite suitable here because many physical signals must be nonnegative, whereas for money that need not be the case—it can be lost as well as gained.

[8] I thank Ragnar Steingrimsson for carrying out these calculations and developing Figure 1.

[9] I am following the convention that φ_+ and φ_- are absolute scales, whereas the empirical literature tends to use for $\varphi(I)$ the same range of numbers as for I. Thus, to be in that realm, the numbers below for slope and intercept should be multiplied by 100.

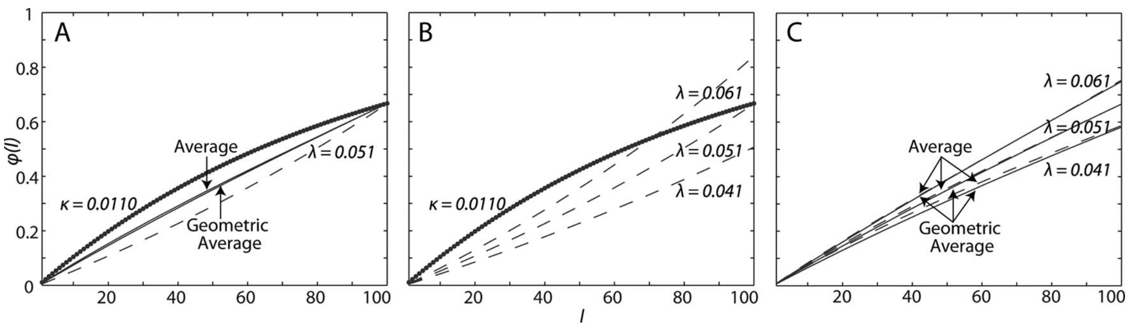

Figure 1. Figure 1A shows the negative exponential (Equation 29) function chosen so that $\varphi_{-}(100) = 2/3$ and the exponential (Equation 28) chosen to agree at $I = 100$. Figure 1B shows this exponential and two others, one above and one below it, with the parameter λ at three values. Figure 1C shows the mean and geometric mean of the negative exponential function with each of these exponential functions.

$$\varphi(x \circ_{p_{1/2}} y) = \frac{\varphi(x) + \varphi(y)}{2}. \tag{34}$$

Proposition 1. Under the above assumptions about unary stimuli, then for signals $x > x' > y > y'$

$$\delta = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \Leftrightarrow (x \oplus x')\circ_{p_{1/2}} (y \oplus y') \begin{Bmatrix} \succ \\ \sim \\ \prec \end{Bmatrix} (x \oplus y)\circ_{p_{1/2}} (x' \oplus y'). \tag{35}$$

All proofs are given in Appendix A.

Thus, the data collection involves, first, the experimenter using Equation 33 to establish $p_{1/2}$ for the respondent. Then the experimenter chooses several sets of signals $x > x' > y > y'$. From these the experimenter generates the pairs of signals $(x \oplus x')\circ_{p_{1/2}} (y \oplus y')$ and $(x \oplus y)\circ_{p_{1/2}} (x' \oplus y')$ and presents them to the respondent, who is asked to state which is more subjectively intense. These data used with Equation 35 should decide the value of δ for each respondent unless the data are not consistent.

For example, suppose that we are studying perception of weight to one arm and the measure of the attribute is grams, which is additive. For the values

$$x = 100 \text{ g}, x' = 80 \text{ g}, y = 60 \text{ g}, y' = 50 \text{ g},$$

then the comparison is which alternative is subjectively more intense,

$$180 \text{ g} \circ_{p_{1/2}} 110 \text{ g versus } 160 \text{ g} \circ_{p_{1/2}} 130 \text{ g?}$$

Some pilot work may be required to get realistic values.

It is not a priori obvious whether a particular 1-D attribute involves just a single value of δ or whether the three values correspond to individual differences of people. I think that the latter may be true for the utility of money (Luce, 2010), but I am not at all sure what is true for other attributes such as electric shock.

Summary

So, in summary, we need to verify empirically in the 1-D cases that the following three properties are satisfied:

1. The key properties of Hölder, namely, commutativity (Equation 22) and associativity (Equation 23) of ⊙ (as was noted, there is no good reason to exclude representations involving multiplication as well as addition);

2. The key axiom of magnitude productions is the Falmagne conjoint commutativity rule (Equation 21) in terms of the defined function $m_{p,q}(a)$ (Equation 20);

3. The linking segregation law (Equation 25), which is far simpler in the 1-D cases than the 2-D ones.

Table 1
Linear Regressions and Least Squares Goodness of Fit

Measure	.61	.51	.41
Mean			
Regression	$.0256 + .0058 \times I$	$.0204 + .0066 \times I$	$.0136 + .0074 \times I$
Goodness of fit (%)	1.47	0.74	0.21
Geometric mean			
Regression	$.0146 + .0058 \times I$	$.0132 + .0066 \times I$	$.0110 + .0075 \times I$
Goodness of fit (%)	0.57	0.34	0.22

Note. For the mean and geometric mean of the negative exponential $\varphi_{-}(I) = 1 - e^{-\kappa I}$, $\kappa = .0110$ with the three for the exponentials $\varphi_{+}(I) = e^{\lambda I} - 1$ with the values of λ shown, the linear regression, and a goodness-of-fit measure. The latter is the least squares measure normalized by the range of mean or geometric mean of φ_{-} and φ_{+} at $I = 100$ and expressed as a percentage.

A method was outlined for estimating the value of δ for a respondent. Appendix C explores some of the other estimation issues that arise in the unary cases.

Bisymmetry

Binary Bisymmetric Representation

An operation $\oplus$ over binary stimuli is said to be *bisymmetric* if for all signals

$$(x \oplus y) \oplus (u \oplus v) \sim (x \oplus u) \oplus (y \oplus v). \tag{36}$$

Note that bisymmetry (Equation 36) simply says that switching the two interior signals y and u does not alter the overall subjective intensity. To evaluate this empirically entails replacing each $\oplus$ with its empirical matching definition:

$$(x \oplus y, x \oplus y) \sim (x, y), \ \ (u \oplus v, u \oplus v) \sim (u, v),$$

$$((x \oplus y) \oplus (u \oplus v), (x \oplus y) \oplus (u \oplus v)) \sim (x \oplus y, u \oplus v),$$

and similarly for the right side of Equation 36.

Four Propositions Concerning Binary Bisymmetry

The representation of binary bisymmetry.

Proposition 2. Consider a structure $\langle X \times X, \succsim \rangle$ satisfying the Assumptions 1–6, listed above, of Luce (2004). Then Parts 1 and 2 are equivalent:

1. The operation $\oplus$ is bisymmetric.
2. There exist a strictly increasing function $\psi{:}X \xrightarrow{\text{onto}} X$ and a constant $1 \geq \mu \geq 0$ such that

$$\psi(x \oplus y) = \mu\psi(x) + (1 - \mu)\psi(y). \tag{37}$$

And Parts 1 $\Leftrightarrow$ 2 imply

3. For $\rho = u/(1 - u)$, φ satisfies

$$\frac{\psi(x \oplus 0)}{\psi(0 \oplus x)} = \rho. \tag{38}$$

Because in the 2-D cases commutativity generally seems to fail (Steingrimsson, 2009, 2012b; Steingrimsson & Luce, 2005a), we conclude $\rho \neq 1$ and $\mu \neq 1/2$.

The form of φ in Equation 37 is discussed in Proposition 6.

Binary bisymmetry and commutativity of $\oplus$. The next result draws upon the paragraph following the corollary of Theorem 2 of Luce (2004), which asserts that for some γ Equation 8 is satisfied. Proposition 3 replaces Luce's incorrect inference that bisymmetry alone forces $\delta = 0$.

Proposition 3. Under the assumptions of Proposition 2 and assuming that Equations 6 and 8 hold, then

1. For $\delta = 0$, bisymmetry (Equation 36) is satisfied.
2. For $\delta \neq 0$, bisymmetry is satisfied if commutativity[10] is satisfied in the sense that, for $x, y \in X$,

$$(x, y) \sim (y, x) \Leftrightarrow x \oplus y \sim y \oplus x. \tag{39}$$

For loudness, and brightness, and perceived contrast, the data strongly support bisymmetry of the operation $\oplus$ (Equation 36; Steingrimsson, 2011, 2012b; Steingrimsson & Luce, 2005b) and equally strongly reject its commutativity (Equation 39; Steingrimsson, 2009, 2011; Steingrimsson & Luce, 2005a, 2005b). So, we conclude from these data that $\delta = 0$, that is, pure additivity, is satisfied and that it is impossible for $\delta \neq 0$. This fact simplifies considerably the section Cross-Modal Matching.

Binary bisymmetry and associativity of $\oplus$. The associativity notion used next is related to that defined by Aczél (1966, p. 253) for functions. For every $x, y, u \in X$,

$$((x, y), u) \sim (x, (y, u)). \tag{40}$$

Proposition 4. Given the definition of $\oplus$, associativity is equivalent to

$$(x \oplus y) \oplus u \sim x \oplus (y \oplus u). \tag{41}$$

Proposition 5. Under the assumptions of Proposition 2 and assuming that bisymmetry is also satisfied, then the structure is not associative.

Recall that commutativity and associativity are both satisfied in Hölder's (1901) representation. In this connection Aczél (1966, p. 278) made the insightful comment that "[bisymmetry is] most used in structures without the property associativity—in a certain respect, it has been used as a substitute for associativity and also for commutativity (symmetry)."

Estimating φ and u. Recall that Proposition 2 showed that bisymmetry is equivalent to $x \oplus y$, satisfying

$$\psi(x \oplus y) = \mu\psi(x) + (1 - \mu)\psi(y). \tag{42}$$

In principle, were one able to collect sufficient data, one could estimate the unknowns ψ and μ. But in practice, doing that entails too much data collection to be really feasible.

Matters are greatly simplified when, for any $\kappa > 0$, the following testable *multiplicative invariance* property is satisfied: There exists a constant $\beta > 0$ such that for every x, y with $x \neq y$

$$\kappa x \oplus \kappa y = \kappa^{\beta}(x \oplus y). \tag{43}$$

Proposition 6. Under the assumptions of Proposition 2 and assuming that bisymmetry is also satisfied, then multiplicative invariance (Equation 43) is equivalent to

$$\psi(x) = \alpha x^{\beta}. \tag{44}$$

So β is estimated by converting Equation 43 into decibel form, leading to

$$(kx \oplus ky)_{\mathrm{dB}} = \beta k_{\mathrm{dB}} + (x \oplus y)_{\mathrm{dB}}$$

$$\Leftrightarrow \beta = \frac{(kx \oplus ky)_{\mathrm{dB}} - (x \oplus y)_{\mathrm{dB}}}{k_{\mathrm{dB}}}. \tag{45}$$

By using a number of (x, y) pairs, one can see the degree to which the right ratio in Equation 45 is constant. The degree to which it is constant yields an estimate of β. Then the estimation problem (Equation 42) reduces to finding the one parameter μ in the expression

$$(x \oplus y)^{\beta} = \mu x^{\beta} + (1 - \mu)y^{\beta}. \tag{46}$$

[10] In Luce (2004) I called this property joint presentation symmetry, but in mathematics it is usually called commutativity.

Unary Bisymmetry

There is a close 1-D analog to bisymmetry, which in the 2-D cases forced $\delta = 0$, because of the empirical failure of commutativity. The following proposition shows that the 1-D case simply predicts bisymmetry:

Proposition 7. If the representation Equation 24 holds, then bisymmetry

$$(x \odot y) \odot (u \odot v) \sim (x \odot u) \odot (y \odot v) \tag{47}$$

holds.

So far, I have not discovered a property for the 1-D case that forces $\delta = 0$.

This means that there is ample room for three kinds of individual differences in the 1-D cases. This was discussed in Luce (2010) for utility of money, where it appears to correspond to risk-seeking, risk-neutral, and rise-averse types of people. It admits interpersonal comparisons of utility involving the first and third types. Of course, that fact considerably complicates the discussion of cross-modal matching below.

Summary

The goal of this section was to work out some important implications of the behaviorally supported 2-D invariance property of bisymmetry and to study it in the 1-D case. There are five theoretical findings:

- A simple weighted averaging representation is equivalent to bisymmetry (Proposition 2).
- A proof that when the 2-D p-additive forms $\delta \neq 0$ hold, then bisymmetry implies that the binary operation $\oplus$ must be commutative (Proposition 3). Because the loudness, brightness, and contrast data strongly support bisymmetry and strongly reject commutativity, we conclude that the binary representation must be additive, that is, $\delta = 0$.
- Bisymmetry precludes the commonly assumed property of associativity (Proposition 5).
- A fairly simple estimation scheme is given for the weighted averaging representation of bisymmetry mentioned above (Proposition 6).
- In the unary (1-D) case, the p-additive representation simply implies bisymmetry (Proposition 7).

Cross-Modal Matching

During the 1960s and 1970s, a substantial empirical literature developed concerning cross-modal matching; much of it was summarized by Stevens (1975). The major theoretical contribution was Krantz (1972), and some of it is related to what was discussed under the 2-D case. He did not make the 1-D and 2-D distinction and so none of the distinctions I make below.

In this section I use the subscript b to identify functions, such as ψ_b, and parameters of the attribute with intensity z, which is being matched to an attribute a with intensity x and scale ψ_a. And the notation differences—$\odot$ and $\oplus$, φ and ψ—identify whether an attribute is, respectively, 1-D or 2-D.

Because empirical data for loudness and brightness favor Equation 43 (Steingrimsson, 2009; Steingrimsson & Luce, 2006), which implies Equation 44, the 2-D functions ψ are power functions of intensity y:

$$\psi(y) = \alpha y^{\beta}. \tag{48}$$

The 1-D scales, according to the section 1-D theory and the form of φ, have the following representations for intensity y:

$$\varphi(y) = \begin{cases} e^{\lambda y} - 1 & \delta = 1,\ \lambda > 0 \\ \eta y & \delta = 0,\ \eta > 0 \\ 1 - e^{-\kappa y} & \delta = -1,\ \kappa > 0 \end{cases}. \tag{49}$$

Keep in mind that, unlike the 2-D cases, the $\delta \neq 0$ cases are absolute scales. So λ and κ must have the unit of $1/x$. Furthermore, we see that for $\delta = -1$,

$$\lim_{x \to \delta} \varphi(x) = \lim_{x \to \delta} (1 - e^{-\kappa x}) = 1.$$

So, unlike either $\delta = 0$ or $\delta = 1$, which are unbounded, the $\delta = -1$ cases are bounded. This has implications below.

Because this section is nothing more than the various combinations of matching Equations 48 and 49, formal proofs are hardly needed.

2-D Matched to 2-D

Suppose that a signal x from a 2-D modality a is presented and that the respondent matches it by a signal z from a 2-D modality b. Of course, z is a function of x. Then because each psychophysical function is a power function

$$\psi_a(x) = \alpha_a x^{\beta_a},\ \ \psi_b(z) = \alpha_b z^{\beta_b}, \tag{50}$$

we have Proposition 8.

Proposition 8. Assuming Equation 48, the 2-D representations for $\oplus_a$ and $\oplus_b$, then signal x on modality a and signal z on modality b match if and only if

$$z = \left(\frac{\alpha_a}{\alpha_b}\right)^{1/\beta_b} x^{\beta_a/\beta_b}, \tag{51}$$

where the parameters are given by Equation 50.

This predicted power function accords well with the empirical findings for loudness and brightness (see Stevens, 1975, for a summary). A very few articles cite some individual data rather than group averages.

1-D Matched to 1-D

Here there are nine cases.

Proposition 9. Assume that modalities a and b have the representation of that shown in the section A General Representation of Unary Sensory Intensities, then matching $\varphi_b(z) = \varphi_a(x)$ occurs under the following conditions:

1. $\delta_a = 0$

$$\delta_b = 0 \Leftrightarrow \frac{z}{x} = \frac{\eta_a}{\eta_b} \tag{52}$$

$$\delta_b = 1 \Leftrightarrow z = \frac{\ln(1+\eta_a x)}{\lambda_b} \tag{53}$$

$\delta_b = -1 \Leftrightarrow$ matches are not always possible.

2. $\delta_a = 1$

$$\delta_b = 0 \Leftrightarrow z = \frac{e^{\lambda_a x} - 1}{\eta_b} \tag{54}$$

$$\delta_b = 1 \Leftrightarrow \frac{z}{x} = \frac{\lambda_a}{\lambda_b} \tag{55}$$

$\delta_b = -1 \Leftrightarrow$ matches are not always possible.

3. $\delta_a = -1$

$$\delta_b = 0 \Leftrightarrow z = \left[\frac{1 - e^{-\kappa_a x}}{\eta_b}\right]^{1/\beta_b} \tag{56}$$

$$\delta_b = 1 \Leftrightarrow z = \frac{\ln(2 - e^{-\kappa_a x})}{\lambda_b} \tag{57}$$

$$\delta_b = -1 \Leftrightarrow \frac{z}{x} = \frac{\kappa_a}{\kappa_b} \tag{58}$$

Note that the three cases where $\delta_a = \delta_b$ assert that a match is simply proportionality. These predictions seem to offer some guidance about what to look for in the literature on 1-D scales and their cross-modal matching.

2-D Matched to 1-D

We know that

$$\psi_b(z) = \alpha_b z^{\beta_b},$$

and there are the three 1-D cases of φ_a yielding Proposition 10.

Proposition 10. Assume that modality b has the representation of Equation 48 and modality a has the representation of Equation 49, then matching $\psi_b(z) = \varphi_a(x)$ occurs under the following conditions:

$$\delta_a = 0 \Leftrightarrow z = \left(\frac{\eta_a}{\alpha_b}x\right)^{1/\beta_b} \tag{59}$$

$$\delta_a = 1 \Leftrightarrow z = \left(\frac{e^{\lambda_a x} - 1}{\alpha_b}\right)^{1/\beta_b} \tag{60}$$

$$\delta_a = -1 \Leftrightarrow z = \left(\frac{1 - e^{-\kappa_a x}}{\alpha_b}\right)^{1/\beta_b}. \tag{61}$$

1-D Matched to 2-D

For the 2-D cases, the theory and data have led us to the representation

$$\psi_a(x) = \alpha_a x^{\beta_a}.$$

And for the 1-D cases, Equation 52 summarizes the three φ_b cases.

So establishing a match of modality b of 1-D to modality a of 2-D yields three cases (Proposition 11).

Proposition 11. Assume that modality a has the representation of Equation 33 and modality b has the representation of Equation 52, then matching $\varphi_b(z) = \psi_a(x)$ occurs under the following conditions:

$$\delta_b = 0 \Leftrightarrow z = \left(\frac{\alpha_a}{\eta_b}\right)x^{\beta_a} \tag{62}$$

$$\delta_b = 1 \Leftrightarrow z = \left[\frac{\ln(1 + \alpha_a x^{\beta_a})}{\lambda_b}\right]^{1/\beta_b} \tag{63}$$

$\delta_b = -1$ matches are not always possible.

Note that $\delta_b = 0$ is a power function but that $\delta_b = 1$ is not. Thus, it is clear that we need to collect data for several 1-D cases and to see whether Equation 62 or Equation 63 fits these data. The only case that I know of that involves $\delta = -1$ is the utility of money (Luce, 2010), which of course is important. But according to this theory, it is not an effective attribute for cross-modal matching.

Table 2
Predictions of the Two Theories for Cross-Modal Matches of Modality b to Modality a

Theory	2-D	1-D	
2-D			
$\delta_a = 0$	$\delta_b = 0$	$\delta_b = 0$	$\delta_b = 1$
x	power	power	$\left[\frac{1}{\lambda_b}\ln(1 + \eta_a x^{\beta_a})\right]^{\frac{1}{\beta_b}}$
1-D			
$\delta_a = 0$	power	proportion	$\left[\frac{1}{\lambda_b}\ln(1 + \eta_a x)\right]^{\frac{1}{\beta_b}}$
$\delta_a = 1$	$\left[\frac{1}{\alpha_b}(e^{\lambda_a x} - 1)\right]^{\frac{1}{\beta_b}}$	$\frac{1}{\eta_b}(e^{\lambda_a x} - 1)$	proportion
$\delta_a = -1$	$\left[\frac{1}{\alpha_b}(1 - e^{-\kappa_a x})\right]^{\frac{1}{\beta_b}}$	$\frac{1}{\eta_b}(1 - e^{-\kappa_a x})$	$\frac{1}{\lambda_b}\ln(2 - e^{-\kappa_a x})$

Note. Because there are three representations possible for one-dimensional (1-D) modalities, the predictions are more complex than normally recognized. 2-D = two-dimensional.

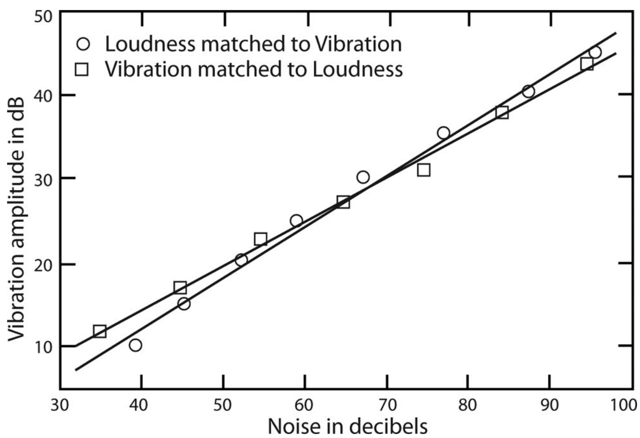

Figure 2. Matches of loudness and vibration to each other as reported by Stevens (1975, Figure 33). Both scales are given in decibel form and so should be straight lines if each is a power function. Adapted from *Psychophysics: Introduction to Its Perceptual, Neural, and Social Prospects* (p. 103), by S. S. Stevens, 1975, New York, NY: Wiley. Copyright 1975 by John Wiley & Sons, Inc. Adapted with permission.

Summary

A major difference between $\odot$ in the 1-D and $\oplus$ in the 2-D cases lies in the key property of bisymmetry, Equations 47 and 36, respectively. It is a necessary property in the 1-D case but not in the 2-D case. But it has been sustained empirically in the 2-D cases (see earlier reference citations). And the section Bisymmetry showed (Proposition 3) that for the cases of $\delta = -1, 1$ of p-additivity (Equation 6) that bisymmetry implies that $\oplus$ is commutative (Equation 22), which was rejected empirically for the ears and eyes. So the data force $\delta = 0$ in all 2-D cases. By contrast the 1-D case rests upon both commutativity (Equation 22) and associativity (Equation 23) being satisfied.

The matching predictions, except for the apparently uninteresting $\delta_b = -1$ cases, derived above are summarized in Table 2.

An Application to Stevens (1959) Cross-Modality Matching

Stevens (1959) reported cross-modal matches between loudness of noise, vibration to a finger, and electric shock (see Figures 2 and 3, which adapt Stevens, 1975, Figures 33–35).

Keep in mind that loudness is a binary modality and both vibration and shock are unary attributes. Figure 3 shows loudness matched to vibration, which means the first column of Table 2 with $\delta_a^V = 0$, $\delta_b^L = 0$ is relevant. That predicts a power function (Equation 59) with exponent $1/\beta_b$. Also shown in that figure is vibration matched to loudness with $\delta_a^L = 0$ and $\delta_b^V = 0$, which is a power function (Equation 62) with power β_a. These predictions may seem well confirmed by the straight lines of Figure 2 (Stevens's Figure 33), but if one can suppress the lines and look at the data directly, that may not be true. One can entertain the hypothesis that $\delta^V = 1$ when vibration is in the role of either attribute *a* or *b*. The so-called regression effect (Stevens, 1975, pp. 102–104, 271–272) may, in fact, reflect the difference between shallow exponential and logarithmic curves being separately fit to straight lines. The curvature in Figure 3 may be misleading by combining data that should be fit separately. There is, indeed, the possibility

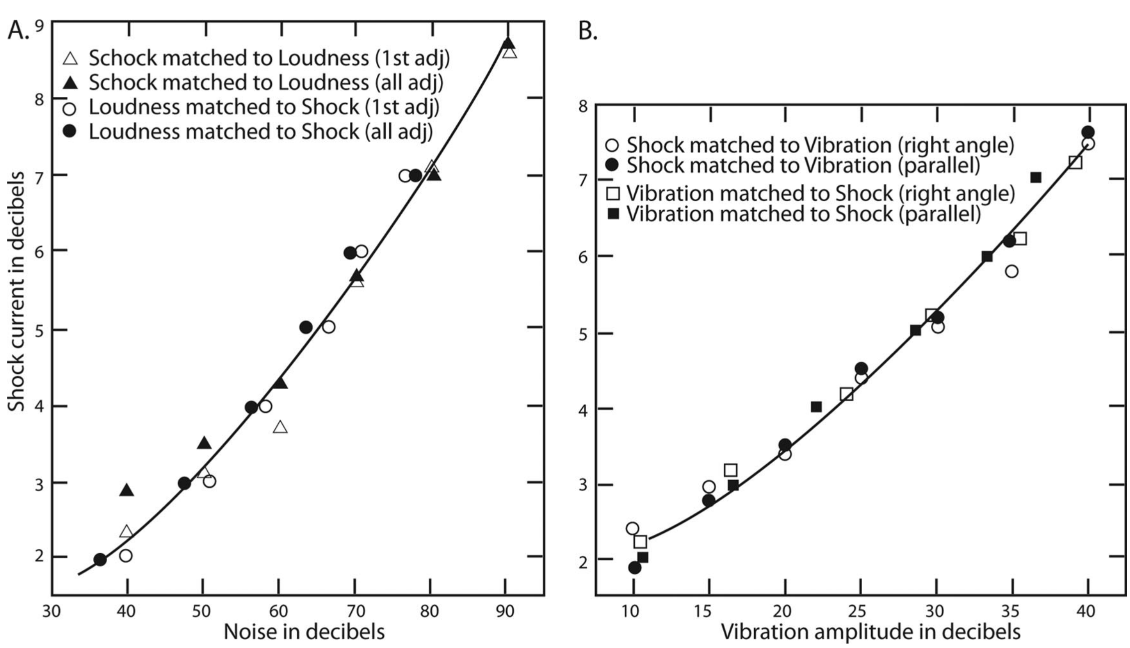

Figure 3. Matches of electric shock to loudness and vibration as adapted from Figures 34 (A) and 35 (B) of Stevens (1975). The scales are all given in decibel form. Quite clearly, these are not straight lines. adj = adjusted. Adapted from *Psychophysics: Introduction to Its Perceptual, Neural, and Social Prospects* (pp. 104, 106), by S. S. Stevens, 1975, New York, NY: Wiley. Copyright 1975 by John Wiley & Sons, Inc. Adapted with permission.

that also $\delta^S = 1$. A detailed evaluation of these predictions to Stevens's data has not been carried out. But such a potential explanation of the data, if sustained, seems less strained than Stevens's claim that they all are really power functions, with the latter two distorted by "adaptation" to the shock.

General Summary

The first section summarizes the theory and representation that Luce (2004, 2008) had earlier generated for subjective intensity judgment of inherently binary (2-D) stimuli such as for loudness and brightness. That theory has been favorably evaluated for individual respondents in articles involving various combinations of Luce and Steingrimsson. Except for my extensive work in utility, the second section is my first attempt at a general theory and representation for the inherently unary (1-D) signals of many other prothetic continua, such as utility of money, shock, and odor intensity. It is fundamentally dependent upon the commutativity and associativity properties underlying Hölder's (1901) theorem. The two theories have much in common, such as p-additive psychophysical functions and a common magnitude production function. However, the third section on the property of bisymmetry brings out a quite sharp difference between the two cases. The unary representation simply implies that bisymmetry must hold, and I do not know any argument to exclude the nonadditive cases. The binary representation implies several things, including that associativity of the joint presentations of signals cannot hold and that for the nonadditive psychophysical functions, commutativity of joint presentations must hold. Earlier data on loudness and brightness established strong support for bisymmetry and equally strong rejection of commutativity of joint presentations. So we conclude that for the 2-D case, the psychophysical function must be purely additive. The fourth section uses these findings to predict cross-modal matching. The predictions for the purely additive psychophysical functions are simply a power function. However, when the nontrivial p-additive representations hold in the unary case, other possibilities occur, which are summarized in Table 2. Of course, in practice, testing the predictions is quite difficult because of the randomness of the data. Unfortunately, no one has figured out how to formulate theories as complex as these in terms of random variables. The difficulty seems to center on linking properties.

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Appendix A

Proofs

Proposition 1

Proof. Applying order preserving φ to

$$(x \odot x')\circ_{1/2}(y \odot y')\left\{\begin{matrix} \succ \\ \sim \\ \prec \end{matrix}\right\}(x \odot y)\circ_{1/2}(x' \odot y')$$

yields

$$\varphi((x \odot x')\circ_{1/2}(y \odot y'))\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\varphi((x \odot y)\circ_{1/2}(x' \odot y'))$$

$$\Leftrightarrow \frac{\varphi(x \odot x') + \varphi(y \odot y')}{2}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\frac{\varphi(x \odot y) + \varphi(x' \odot y')}{2}$$

by Equations 27 and 32 and

$$\Leftrightarrow \delta[\varphi(x)\,\varphi(x') + \varphi(y)\varphi(y')]\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\delta[\varphi(x)\,\varphi(y) + \varphi(x')\,\varphi(y')]$$

by Equation 26. This is clearly satisfied for $\delta = 0$. For $\delta \neq 0$, it reduces to

$$\delta\varphi(x)[\varphi(x') - \varphi(y)]\left\{\begin{matrix} > \\ < \end{matrix}\right\}\delta\varphi(y')[\varphi(x') - \varphi(y)]$$

$$\Leftrightarrow \delta\varphi(x)\left\{\begin{matrix} > \\ < \end{matrix}\right\}\delta\varphi(y')$$

$$\Leftrightarrow \delta[\varphi(x) - \varphi(y')]\left\{\begin{matrix} > \\ < \end{matrix}\right\}0,$$

which is true because $\varphi(x) - \varphi(y') > 0$.

Proposition 2

Proof: Part 1 $\Rightarrow$ Part 2. Aczél (1966, p. 287) proves that an operation $\oplus$ is bisymmetric if and only if there is a strictly increasing function $\xrightarrow{\text{onto}}$ nonnegative real numbers and nonnegative constants μ, υ, and σ satisfying

$$\psi(x \oplus y) = \mu\psi(x) + \nu\psi(y) + \sigma.$$

Because $\psi(0) = 0$ and Equation 5, this implies $\sigma = 0$, and so

$$\psi(x \oplus y) = \mu\psi(x) + \nu\psi(y), \text{ with } \psi(x \oplus x) = \psi(x).$$

Setting $x = y$ in the above display yields $\mu + \upsilon = 1$, leading to the weighted average representation

$$\psi(x \oplus y) = \mu\psi(x) + (1 - \mu)\,\psi(y),\ 1 \geq \mu \geq 0, \text{ with}$$

$$\psi(x \oplus x) = \psi(x). \tag{A1}$$

Part 2 $\Rightarrow$ Part 1. Consider the left side of bisymmetry,

$$\begin{aligned}
&\psi((x \oplus y) \oplus (u \oplus v)) \\
&= \mu\psi(x \oplus y) + (1 - \mu)\,\psi(u \oplus v) \\
&= \mu[\mu\psi(x) + (1 - \mu)\,\psi(y))] + (1 - \mu)\,[\mu\psi(u) + (1 - \mu)\,\psi(v))] \\
&= \mu^2\psi(x) + \mu(1 - \mu)\,[\psi(y) + \psi(u)] + (1 - \mu)^2\psi(v) \\
&= \mu^2\psi(x) + \mu(1 - \mu)\,[\psi(u) + \psi(y)] + (1 - \mu)^2\psi(v) \\
&= \psi((x \oplus u) \oplus (y \oplus v)).
\end{aligned}$$

Taking ψ^{-1} proves bisymmetry.

Part 3. Given Equation 64, observe that

$$\psi(x \oplus 0) = \mu\psi(x)$$

$$\psi(0 \oplus y) = (1 - \mu)\,\psi(y),$$

so

$$\frac{\psi(x \oplus 0)}{\psi(0 \oplus x)} = \frac{\mu\psi(x)}{(1 - \mu)\,\psi(x)} = \frac{\mu}{1 - \mu} = \rho.$$

(*Appendices continue*)

Proposition 3

Proof: Part 1. Assume $\delta = 0$. By Proposition 2 above with $\Theta = \Psi$, and because $\Psi(x, u)$ is additive, as is $\psi(x \oplus u)$, so they are proportionate, and so $\Psi(x, u)$ satisfies bisymmetry and $\rho = \gamma$.

Part 2. Assume $\delta = 1$. By the same argument as in Part 1, we know that $\Phi(x, u)$ is proportional to $\psi(x \oplus u)$. Then from Equations 8, 10, and 40, for all $x \in X$

$$\begin{aligned}\rho &= \frac{\psi(x \oplus 0)}{\psi(0 \oplus x)} \\ &= \frac{\ln[1 + \Psi(x, 0)]}{\ln[1 + \Psi(0, x)]} \\ &= \frac{\ln[1 + \gamma\Psi(0, x)]}{\ln[1 + \Psi(0, x)]}.\end{aligned}$$

Setting $Z := \Psi(0, x)$, this is equivalent to, for all Z,

$$\begin{aligned}\rho &= \frac{\ln(1 + \gamma Z)}{\ln(1 + Z)} \\ &\Leftrightarrow \rho\ln(1 + Z) = \ln(1 + \gamma Z) \\ &\Leftrightarrow \ln(1 + \gamma Z)^{\rho} = \ln(1 + Z) \\ &\Leftrightarrow (1 + Z)^{\rho} = 1 + \gamma Z \\ &\Leftrightarrow \rho = \gamma = 1,\end{aligned}$$

which implies $\mu = 1/2$. So by Equation 64, this is equivalent to

$$x \oplus y = y \oplus x \Leftrightarrow (x, y) \sim (y, x),$$

that is, commutativity.

Proposition 4

Proof. By the definition of $\oplus$

$$\begin{aligned}((x, y), u) &\sim ((x, y) \oplus u, (x, y) \oplus u), \\ &\sim (x, (y, u)) \sim (x \oplus (y, u)), (x \oplus (y, u)).\end{aligned}$$

So, by monotonicity

$$\begin{aligned}&(x, y) \oplus u \sim (x \oplus (y,u)) \\ &\Leftrightarrow ((x \oplus y) \oplus u, (x \oplus y) \oplus u) \sim (x \oplus (y \oplus u), x \oplus (y \oplus u)).\end{aligned}$$

And using monotonicity again

$$(x \oplus y) \oplus u \sim x \oplus (y \oplus u).$$

Proposition 5

Proof. Assume associativity (Equation 23), then because Proposition 2 is satisfied, repeated use of Equation 36 yields

$$\begin{aligned}&(x, (y, z)) \sim ((x, y), z) \\ &\Leftrightarrow x \oplus (y \oplus z) = (x \oplus y) \oplus z \\ &\Leftrightarrow \psi(x \oplus (y \oplus z)) = \psi((x \oplus y) \oplus z) \\ &\Leftrightarrow \psi(\psi^{-1}[\mu\psi(x) + (1 - \mu)\,\psi(y \oplus z)]) = \psi(\psi^{-1}[\mu\psi((x \oplus y) \\ &\qquad + (1 - \mu)\,\psi(z))]) \\ &\Leftrightarrow \mu\psi(x) + (1 - \mu)\,\psi(y \oplus z) \sim \mu\psi(x \oplus y + (1 - \mu)\,\psi(z)) \\ &\Leftrightarrow \mu\psi(x) + (1 - \mu)\,\psi(\psi^{-1}[\mu\psi(y) + (1 - \mu)\,\psi(z)]) \\ &\qquad = \mu\psi(\psi^{-1}[\mu\psi(x) + (1 - \mu)\,\psi(y)]) + (1 - \mu)\,\psi(z) \\ &\Leftrightarrow \mu\psi(x) + (1 - \mu)\,\mu\psi(y) + (1 - \mu)^2\psi(z) \\ &\qquad = \mu^2\psi(x) + \mu(1 - \mu)\,\psi(y) + (1 - \mu)\,\psi(z),\end{aligned}$$

which is clearly equivalent to

$$\mu = \mu^2 \Leftrightarrow \mu = 1$$

and

$$(1 - \mu) = (1 - \mu)^2 \Leftrightarrow 1 - \mu = 1 \Leftrightarrow \mu = 0,$$

a contradiction. So associativity cannot hold.

Proposition 6

Proof. Given $x \neq y$,

$$\psi[x \oplus y] = \mu\psi(x) + (1 - \mu)\,\psi(y),$$

and Equation 43, that is,

$$\kappa x \oplus \kappa y = \kappa^{\beta}(x \oplus y),\ \kappa > 0,$$

we have

$$\begin{aligned}&\mu\psi(\kappa x) + (1 - \mu)\,\psi(\kappa y) \\ &= \psi(\kappa x \oplus \kappa y) \\ &= \kappa^{\beta}\psi(x \oplus y) \\ &= \kappa^{\beta}[\mu\psi(x) + (1 - \mu)\,\psi(y)].\end{aligned}$$

Setting $y = 0$, this is equivalent to

$$\psi(\kappa x) = \kappa^{\beta}\psi(x),$$

which for ψ strictly increasing and onto is known from Aczél (1966, p. 15) to be equivalent to ψ being a power function (Equation 44).

Proposition 7

Proof. Because the p-additive representation of $\odot$ (Equation 24) is equivalent to

$$\begin{aligned}&\psi_{\odot}(x \odot y) = \psi_{\odot}(x) + \psi_{\odot}(y) \ \text{ if } \delta = 0, \\ &\ln[1 + \delta\psi_{\odot}(x \odot y] = \ln[1 + \delta\psi_{\odot}(x)] + \ln[1 + \delta\psi_{\odot}(y)] \ \text{ if } \delta \neq 0,\end{aligned}$$

then Aczél (1966, p. 287) implies that $\odot$ satisfies bisymmetry.

(*Appendices continue*)

Appendix B

Estimation Issues for the Binary Representation

Some Simplifications

Proposition 3, together with the existing data showing that bisymmetry holds and that commutativity does not hold, implies that the representation must be additive.

Steingrimsson and Luce (2007) argued empirically that W has the Prelec (1998) form:

$$W(p) = W(1)\begin{cases} \exp[-\omega(-\ln p)^{\mu}] \ (0 < p \leq 1) \\ \exp[\omega'(\ln p)^{\mu'}] \ (1 < p) \end{cases}. \tag{B1}$$

Note that this specializes to a power function when $\mu = \mu' = 1$. In particular, their data supported $\mu = 1$ for six of six respondents but $\mu' = 1$ for three of five respondents. For the other two, they tested and supported the behavioral property called *double reduction invariance* that Aczél and Luce (2007) showed to be equivalent to Equation B1. Previous conditions of Prelec (1998) and Luce (2001) worked only for the case $W(1) = 1$, which restriction was shown not to be satisfied by Ellermeier and Faulhammer (2000), Zimmer (2005), and Steingrimsson and Luce (2007), who not only presented new data but summarized the entire family of results.

A desirable goal is to discover a way to estimate the parameters of the Prelec equations using double reduction invariance data in such a way that we can test for $\mu = \mu' = 1$ without running a separate experiment. One possibility is outlined below in the section Prelec With $\mu = \mu' = 1$.

The Production Equation

Suppose that we collect magnitude production data for various values of x and p in which case the parameters to be estimated are β, $W(p)$, and ρ, which may depend upon p, and so I write (ρ, ρ). The current evidence is that ρ only changes with the sign of $p - 1$ (Luce, Steingrimsson, & Narens, 2010; Steingrimsson & Luce, 2007; Steingrimsson, Luce, & Narens, in press).

The basic symmetric production equation (Equation 17) is

$$\psi(x \circ_p \rho(p)) = W(p)\psi(x) + [1 - W(p)]\psi(\rho(p)). \tag{B2}$$

By Equation 46 and because the constant $(1 + \gamma)\alpha_r$ is common to all three power terms, it follows that

$$(x \circ_p \rho(p))^{\beta} = W(p)x^{\beta} + [1 - W(p)]\rho(p)^{\beta}. \tag{B3}$$

So to estimate parameters in Equation B3, collect data for several p and x values, and then use some optimization technique to find the best fitting β. The estimated slope gives an estimate of $W(p)$, and with that the intercept yields an estimate of $\rho(p)^{\beta}$ and so of $\rho(p)$.

Prelec With $\mu = \mu' = 1$

Proposition 12. Suppose that the theory of section A General Representation of Binary Sensory Intensities holds with W a Prelec function (Equation B1). If $\psi(x) = \alpha x^{\beta}$ and $\mu = \mu' = 1$, then with $y := x \circ_p \rho$

$$\left(\frac{y}{x}\right)^{\beta} \approx W(1)\begin{cases} p^{\omega}, \ (0 < p \leq 1) \\ p^{\omega'} \ (1 < p) \end{cases}. \tag{B4}$$

From the double reduction data and the estimate of β, this plot then yields estimates of $W(1)$, ω, and ω', and of course, we can evaluate whether the special case of a power Prelec function holds.

Several Observations

1. We cannot rule out a priori that the estimates of β with different ps will in fact be different; but if it turns out they are not different, then we can say β is a parameter of the respondent. Observation 3 below suggests that it may depend on whether $p > 1$ or $p \leq 1$, but that is not predicted by the theory.
2. We can develop a plot of $W(p)$ versus p and try to decide how well it is described by separate Prelec functions for $p > 1$ and for $p \leq 1$.
3. We can also develop a plot of $\rho(p)$ versus p and mainly see whether it is flat or varies systematically with p. Again, the distinction $p > 1$ and $p \leq 1$ may matter as in several other cases (e.g., Luce et al., 2010; Steingrimsson & Luce, 2007; Steingrimsson et al., in press).
4. And finally, from the double reduction invariance data, we can evaluate the adequacy of the Prelec function; we also can evaluate the special case of a power function, $\mu = \mu' = 1$.

(*Appendices continue*)

Appendix C

Estimation Issues for the Unary Representation

Case of $\delta = 0$

This the same situation as was treated for the 2-D case in Appendix B.

Case of $\delta \neq 0$

The model yields Equation 26 with $\delta = -1$ and $\delta = 1$ as well as Equation 27. If we define

$$\Phi(x) = 1 + \delta\varphi(x),$$

it follows from Equation 26 that

$$\Phi(x \odot y) = \Phi(x)\Phi(y) \tag{C1}$$

and from Equation 27 that

$$\begin{aligned}\Phi(x \bigcirc_p y) &= 1 + \delta\varphi(x \bigcirc_p y)\\ &= 1 + \delta\varphi(x)W(p) + \delta\varphi(y)[1 - W(p)]\\ &= [1 + \delta\varphi(x)]W(p) + [1 + \delta\varphi(y)][1 - W(p)] \qquad \text{(C2)}\\ &= \Phi(x)W(p) + \Phi(y)[1 - W(p)]\end{aligned}$$

If, as in the 2-D cases, the data support

$$\varphi(x) = \alpha x^{\beta},$$

then

$$\Phi(x) = 1 + \delta\alpha x^{\beta},$$

which leads to the following expression for Equation C1:

$$(x \oplus y)^{\beta} = x^{\beta} + y^{\beta} + \delta\alpha(xy)^{\beta},$$

but no change in the form (Equation B3). So the estimation of unknowns in the 1-D and 2-D cases is the same, but substantial differences exist in the expressions involving $\odot$ and $\oplus$, respectively. This difference warrants careful empirical study.

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