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Torgerson's conjecture and Luce's magnitude production representation imply an empirically false property

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ABSTRACT

The prediction presented is based upon the empirically well sustained magnitude production representation that arose in both of Luce's global psychophysical theories for subjective intensity of binary and unary continua coupled with Torgerson's (1961) conjecture that respondents fail to distinguish subjective differences from subjective ratios. When applied to eqisections and fractionation the conjecture implies that the cognitive distortion function of the magnitude production representation is the identity function, which is firmly rejected by existing data.

1. Modeling background

The result arrived at here is based on the method of *magnitude production* (Stevens, 1975) but using the mathematical representation that was derived from testable and tested behavioral properties (Luce, 2002; Luce, 2004; Luce, 2008; Luce, in press). The experimental tests were reported for audition by Steingrimsson and Luce (2005a,b, 2006, 2007), for brightness by Steingrimsson (2009, 2010, in preparation a), and for contrast by Steingrimsson (in preparation b, in preparation c).

Let X denote the set of signals identified by the physical intensity (not dB) presented to the respondent less the corresponding threshold intensity. When a respondent is asked to produce the signal $x_p \in X$ that stands in the subjective ratio p to a given signal $x \in X$, the theory derives the following representation. There exist a strictly increasing, psychophysical ratio scale ψ over X, a strictly increasing cognitive distortion function W over positive numbers, and a parameter $\rho \in X$, called a reference signal, such that together they satisfy the following magnitude production constraint

$$W(p) = \frac{\psi(x_p) - \psi(\rho)}{\psi(x) - \psi(\rho)}. \tag{1}$$

In some cases the reference signal ρ is explicitly specified by the experimenter and in others it must be respondent generated and simply is a parameter to be estimated.

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2. When are fractionations and equisections the same?

This is a very natural question to ask if one takes seriously, as many psychophysicists seem to have, Torgerson's (1961) conjecture that functionally people fail to distinguish between questions about subjective signal differences from questions about subjective signal ratios. Currently Luce, Steingrimsson, and Narens (in preparation) are directly addressing that conjecture in an ordinal fashion. Here I simply note that the question leads to a prediction that existing data say is wrong, thereby rendering Torgerson's conjecture very suspect because (1) is very well supported empirically.

Suppose that m and n are integers such that $1 \le m \le n$, and that x, y are signals such that $x < y$. When the respondent is asked to report the signal $r_{m,n}$ that divides the interval (x, y) so that $(x, r_{m,n})$ is in the "ratio" $\frac{m}{n}$ to (x, y), then from (1) we assume that this means

$$W\left(\frac{m}{n}\right) = \frac{\psi(r_{m,n}) - \psi(x)}{\psi(y) - \psi(x)}. \tag{2}$$

This is called *fractionation* (Stevens, 1975, p. 157). Note that in such an experiment the experimenter explicitly states the reference signal x.

When the respondent is asked to partition the interval (x, y) into n subjectively "equal" intervals, then the interval of the first m of these equal ones, called $e_{m,n}$, must satisfy

$$(n-m)\left(\psi(e_{m,n}) - \psi(x)\right) = m\left(\psi(y) - \psi(e_{m,n})\right)$$
$$\Leftrightarrow \frac{m}{n-m} = \frac{\psi(e_{m,n}) - \psi(x)}{\psi(y) - \psi(e_{m,n})}. \tag{3}$$

The special case where $m = 1$ and $n = 2$ is called *bisection* and the general case *equisection* (Stevens, 1975, p. 154, 158).

According to Torgerson's (1961) conjecture that people fail to distinguish subjective ratios from subjective differences, the following simple condition must be met:

$$r_{m,n} = e_{m,n}, \quad \text{for all } m, n \text{ integers satisfying } 1 \le m \le n. \tag{4}$$

Proposition 1. *If (2), (3) and (4) hold, then the cognitive number distortion function W must, in fact, be the identity function:*

$$W(p) = p \quad \text{for all real } p \text{ in } [0, 1]. \tag{5}$$

The proof is in the Appendix.

The issue of whether or not W is the identity function is not new. It arose when Narens (1996) attempted to formalize what he then believed Stevens (1975) must have meant when he claimed that magnitude methods lead to ratio scales of subjective intensity. Narens arrived at two properties: production commutativity and a multiplicative property, both involving compound magnitude productions. Ellermeier and Faulhammer (2000), Steingrimsson and Luce (2007), and Zimmer (2005) empirically tested both hypotheses, and they all found good evidence favoring the commutativity property and they all firmly rejected the multiplicative property.

One thing to note is that when W is the identity function, then finding the fraction $y_{m,n}$ followed by the further fraction $(y_{m,n})_{m',n'}$ implies, using (2), that

$$(y_{m,n})_{m',n'} = y_{mm',nn'}.$$

That is false when W is not the identity because

$$W\left(\frac{m}{n}\right) W\left(\frac{m'}{n'}\right) \ne W\left(\frac{mm'}{nn'}\right).$$

Of course, equisections should exhibit such a regularity.

The Steingrimsson and Luce article pointed out that the culprit seemed to be Narens' assumption that $W(1) = 1$ which in the context of his theory led to W being the identity function. Indeed, they formulated a behavioral property, which they confirmed to hold for loudness, that implies that the weighting function W is well approximated by the Prelec function

$$W(p) = W(1) \exp\left[-\omega(-\ln p)^{\mu}\right], \quad W(1), \omega, \mu > 0. \tag{6}$$

Unpublished brightness data Steingrimsson (in preparation a) also leads to the same Prelec function.

Therefore, the rejection of $r_{m,n} = e_{m,n}$ means that Torgerson's conjecture – that people fail to distinguish subjective differences and ratios – is probably false. We will report other more direct ordinal properties of the conjecture for which data are now being collected.

3. Summary

Coupling the empirically well sustained representation (1) that Luce (2004) derived from several behavioral invariances with Torgerson's (1961) conjecture that functionally people fail to distinguish sensory subjective differences from sensory subjective ratios, I addressed the question of when are equisections and fractionations not different. The answer is when, and only when, the cognitive function W over numbers is the identity function. And that has been soundly rejected by a good deal of data for loudness and brightness, thereby casting either Torgerson's conjecture or the extensively sustained magnitude production representation (1) into doubt.

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Appendix

Proposition 1.

Proof. We may rewrite (3) as

$$\begin{aligned}
\frac{m}{n-m} &= \left(\frac{\psi(e_{m,n}) - \psi(x)}{\psi(y) - \psi(e_{m,n})}\right)\left(\frac{\psi(y) - \psi(x)}{\psi(y) - \psi(x)}\right) \\
&= \left(\frac{\psi(e_{m,n}) - \psi(x)}{\psi(y) - \psi(x)}\right)\left(\frac{\psi(y) - \psi(x)}{\psi(y) - \psi(e_{m,n})}\right) \\
&= \left(\frac{\psi(e_{m,n}) - \psi(x)}{\psi(y) - \psi(x)}\right) \\
&\quad \times \left(\frac{\psi(y) - \psi(x)}{\psi(y) - \psi(x) - \left[\psi(e_{m,n}) - \psi(x)\right]}\right).
\end{aligned}$$

Using (1), (2) and (4), i.e., $r_{m,n} = e_{m,n}$, then the above expression becomes

$$\begin{aligned}
\frac{m/n}{1 - m/n} &= \frac{m}{n-m} \\
&= \left(\frac{\psi(r_{m,n}) - \psi(x)}{\psi(y) - \psi(x)}\right) \\
&\quad \times \left(\frac{\psi(y) - \psi(x)}{\psi(y) - \psi(x) - \left[\psi(r_{m,n}) - \psi(x)\right]}\right) \\
&= W\left(\frac{m}{n}\right) \frac{1}{\frac{\psi(y)-\psi(x)}{\psi(y)-\psi(x)} - \frac{\psi(r_{m,n})-\psi(x)}{\psi(y)-\psi(x)}} \\
&= \frac{W\left(\frac{m}{n}\right)}{1 - W\left(\frac{m}{n}\right)} \\
&\Leftrightarrow W\left(\frac{m}{n}\right) = \frac{m}{n}.
\end{aligned} \tag{7}$$

Because this holds for all rational numbers in [0, 1], which are dense in the real numbers, and because W is strictly increasing, it is well known that (5) must be true. □

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