Aequationes Math. 78 (2009) 321–328
0001-9054/09/030321-8
DOI 10.1007/s00010-009-2972-z

Aequationes Mathematicae

A functional equation proof of the distributive-triples theorem

R. Duncan Luce

Summary. A 1987 article studied the fact that some attributes can be measured in two ways: via concatenations and via decomposition in a conjoint fashion. The measures were shown to be the same provided a distribution law is satisfied. The proof was algebraic and indirect. A direct proof is provided using functional equation arguments.

Mathematics Subject Classification (2000). 06F99, 39B72.

Keywords. Distributive triple, functional equation, Pexider equation.

1. Classical foundations of dimensional representations

1.1. A typical qualitative description

A situation often encountered in classical physical measurement and sometimes in the behavioral sciences is typified by the case of

$$\text{mass} = \text{volume} \times \text{substance}, \tag{1}$$

with domains $\mathcal{M}, \mathcal{V}$, and $\mathcal{S}$. The substances are assumed to be homogeneous. In this case, mass has two decompositions. The first is the conjoint one $\langle \mathcal{V} \times \mathcal{S}, \succsim_M \rangle$ underlying the usual multiplicative relation, where $\succsim_M$ is a qualitative weak ordering determined by placing masses on the two scales of an equal-arm balance. A standard monotonicity assumption induces orderings $\succsim_V$ on volumes and $\succsim_S$ on homogenous substances. The second is a concatenation structure for the masses, $\langle \mathcal{M}, \succsim_M, \circ_M \rangle$, which formalizes the idea of placing two masses on the same pan of the balance. And there is a volume concatenation structure $\langle \mathcal{V}, \succsim_V, \circ_V \rangle$.

When certain axioms are satisfied, each structure gives rise to a measure of mass: From $\circ_M$, as m_1 a homomorphic mapping onto the nonnegative real numbers

This work was supported in part by National Science Foundation grant SES-0452756 to the University of California, Irvine. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

$\langle \mathbb{R}_+, \geq, + \rangle$ for the mass concatenation, and from the conjoint structure, m_2. Of course, we hardly expect two different measures of mass, one from concatenation and the other from the conjoint one. The question is what insures that they are the same.

1.2. Distributive triples: linking the two structures

The answer has been shown to be the following linking distributive law: For $a, b \in \mathcal{M}$, $u, v \in \mathcal{V}$, and $s \in \mathcal{S}$

$$\begin{aligned} & a \sim (u, s) \text{ and } b \sim (v, s) \\ & \Rightarrow a \circ_M b \sim (u, s) \circ_M (v, s) \sim (u \circ_V v, s). \end{aligned} \tag{2}$$

Summaries of these issues were stated in Section 10.7 of Krantz, Luce, Suppes, and Tversky (1971) for less general linking laws, and more generally and insightfully 19 years later in Section 20.2.7 of Luce, Krantz, Suppes, and Tversky (1990) based on work of Luce (1978, 1987), Narens and Luce (1983), and Narens (1985). However, the mathematics there is fairly complex, abstract algebra, whereas what I propose here is a good deal simpler and, perhaps, more accessible mathematically.

The resulting classical representation is, indeed, the existence of a single measure of mass, Γ_M, on $\mathcal{M}$ that is additive over $\circ_M$, a measure Γ_V on $\mathcal{V}$ that is additive over $\circ_V$, and a measure Γ_S on $\mathcal{S}$ such that

$$\Gamma_M(u, s) = \Gamma_V(u)\Gamma_S(s). \tag{3}$$

is a conjoint representation. The usual physical notation is, of course, $m(u, s) = V(u)\rho(s)$. We choose a less specific notation because the modeling is not restricted to the mass, volume, and density case.

It is easy to show that (2) is a necessary condition for (3).

1.3. Goals of this note

We continue to assume that the conjoint structure has a multiplicative conjoint representation

$$\Psi_M(a) = \Psi_V(u)\Psi_S(s). \tag{4}$$

And we assume that the mass and volume concatenation structures both satisfy Hölder's axioms that are sufficient for an additive representation over the positive quadrant of objects ordered higher than the identity e. A representation Φ_M is said to be **additive (over the concatenation $\circ_M$)** iff for all $a, b \in \mathcal{M}$,

$$\Phi_M(a \circ_M b) = \Phi_M(a) + \Phi_M(b). \tag{5}$$

Similarly, Φ_V is additive over $\circ_V$ iff for all $u, v \in \mathcal{V}$

$$\Phi_V(u \circ_V v) = \Phi_V(u) + \Phi_B(v). \tag{6}$$

Basically the issue is whether we can find common functions Γ_M, Γ_V, and Γ_S that satisfy (3).

This note establishes two new things:

1. By invoking the well-known solutions to the Pexider equation, we obtain the classic result of a single measure Γ_M that satisfies both additivity of concatenation, (5), and multiplicative conjointness, (3), using a fairly straightforward functional equation argument.
2. Attempting to look at more general concatenation, such as the p-additive form, does not really lead to anything new. This finding is very different from some studies of the representation of structures of uncertain alternatives where the results have differed in significant ways from the additive case (see the several cited collaborative papers with Ng and others).

2. The distributive-triples theorem

Theorem 1. *Assume that $\langle \mathcal{M}, \succsim_M, \circ_M \rangle$ and $\langle \mathcal{V}, \succsim_V, \circ_V \rangle$ each have an additive, order preserving representation, Φ_M and Φ_V, respectively, onto $\langle \mathbb{R}_+, \geq, + \rangle$ and that $\langle \mathcal{V} \times \mathcal{S}, \succsim_V, \circ_V \rangle$ has a multiplicative, order preserving representation $\Psi_M = \Psi_V \Psi_S$ with Ψ_M and Ψ_V both onto $\langle \mathbb{R}_+, \geq, + \rangle$ and Ψ_S onto $\langle \mathbb{R}_+, \geq \rangle$ such that for all $v \in \mathcal{V}$ and $s \in \mathcal{S}$, $\Psi_M(v,s) = \Psi_V(v)\Psi_S(s)$. Then there exist additive representations Γ_M and Γ_V and a conjoint one $\Gamma_M = \Gamma_V \Gamma_S$ such that (3) is satisfied.*

A new, somewhat simpler and transparent, proof is provided in Appendix A.

3. Generalized additive representations of concatenation

3.1. Definition of generalized additivity

Suppose that for some measure Θ_M over $\mathcal{M}$ we have

$$\Theta_M(a \circ_M b) = H_M[\Theta_M(a), \Theta_M(b)], \tag{7}$$

where

$$H_M : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$$

is strictly increasing in each variable. The function H_M is said to satisfy **generalized additivity** iff for some φ_M

$$\varphi_M\left[\Theta_M(a \circ_M b)\right] = \varphi_M\left[\Theta_M(a)\right] + \varphi\left[\Theta_M(b)\right]. \tag{8}$$

However, we then simply work with $\Phi_M := \varphi_M \Theta_M$ which is additive in the usual sense.

A parallel structure holds over volume concatenations.

Such a generalization may seem of no importance, so I give an example of (7) that has played a useful role in utility theory.

3.2. The definition of p-additive

Starting with Luce's (2000) summary of earlier results and in a series of articles about decision making under uncertainty (Luce, 2009; Luce, Ng, Marley, & Aczél, 2008a,b; Ng, Luce, & Marley, 2008, 2009), we have focused on the fact that if one permits a concatenation structure to be mapped into the full real structure $\langle \mathbb{R}_+, \geq, +, \cdot \rangle$, not just into the substructure $\langle \mathbb{R}_+, \geq, + \rangle$, other possible representations exist than just additive ones. In particular:

A representation Ψ is **p-additive (over the concatenation** $\circ$ with domain $\mathcal{X}$**)** if there exists constant δ such that for $a, b \in \mathcal{X}$

$$\Psi(a \circ b) = \Psi(a) + \Psi(b) + \delta\Psi(a)\Psi(b). \tag{9}$$

By working with $\delta\Psi$, we see that there is no loss of generality in assuming $\delta = -1, 0, 1$. Clearly, $\delta = 0$ is the additive case just explored. Ng, Luce, and Marley (2009) show the range of Ψ for $\delta = -1$ is $]-\infty, 1[$, whereas for $\delta = 0$ the range is $]-\infty, \infty[$, and for $\delta = 1$ it is $]-1, \infty[$.

For $\delta \neq 0$, the transformation

$$\Phi(a) := \ln\left[1 + \delta\Psi(a)\right] \tag{10}$$

is easily seen to be additive.

Thus, for $\delta \neq 0$, in the kinds of cases we have in mind of the positive quadrant, we see that $1 + \delta\Psi(a) > 1$, which is not possible for $\delta = -1$. Appendix B explores the more general p-additive cases, but nothing new arises.

4. Conclusions

The standard result of a single "mass" representation for concatenation and conjoint views of mass is again proved, but with a simpler proof using functional equations and the known solution to the Pexider equation. Second, unlike the utility of uncertain alternatives, a generalized additive form such as p-additivity gives nothing really new. Given the interesting new results in the utility case, this negative finding was disappointing.

Acknowledgment. I thank Professor János Aczél for some advice on how to obtain general solutions to the key functional equation (18), Professor A. A. J. Marley, consultant to the NSF grant, for his very careful critical reading of multiple versions and key corrections, and to a referee for a prompt, very favorable response and pointing out four typos.

5. Appendices

5.1. Appendix A: New proof of Theorem 1

Because Ψ_M and Φ_M are each order preserving of mass ordering for there must exist a strictly increasing function f

$$\Psi_M(a) = f(\Phi_M(a)). \tag{11}$$

Similarly there exists an increasing function g :

$$\Psi_V(u) = g(\Phi_V(u)). \tag{12}$$

Because f and g are strictly monotonic on intervals to intervals, they are continuous. Moreover,

$$\begin{aligned} \Psi_M(a \circ_M b) &= f(\Phi_M(a \circ_M b)) \qquad (11) \\ &= f(\Phi_M(a) + \Phi_M(b)) \qquad (5), \end{aligned}$$

and

$$\begin{aligned} \Psi_V(u \circ_V v)\Psi_S(s) &= g(\Phi_V(u \circ_V v)\Psi_S(s) \qquad (12) \\ &= g(\Phi_V(u) + \Phi_V(v))\Psi_S(s). \quad (6) \end{aligned}$$

Therefore, by (2) and (4),

$$f(\Phi_M(a) + \Phi_M(b)) = g(\Phi_V(u) + \Phi_V(v))\Psi_S(s), \tag{13}$$

where $a \sim (u, s)$ and $b \sim (v, s)$. By (4), (11), and (12)

$$\begin{aligned} g(\Phi_V(u)) &= \Psi_V(u) \\ &= \frac{\Psi_M(a)}{\Psi_S(s)} \\ &= \frac{f(\Phi_M(a))}{\Psi_S(s)}. \end{aligned} \tag{14}$$

Let

$$x := \Phi_M(a),\ x' := \Phi_M(b),\ y = \Psi_S(s), \tag{15}$$

So, by (14)

$$\Psi_V(u) = \frac{f(x)}{y}, \tag{16}$$

and a parallel argument gives

$$\Psi_V(v) = \frac{f(x')}{y}. \tag{17}$$

Using (11) and (12) and that $a \sim (u,s)$ and $b \sim (v,s)$, any x, x' pair can be achieved because

$$f(x) = \Psi_V(u)\Psi_S(s),\ f(x') = \Psi_V(v)\Psi_S(s)$$
$$\Rightarrow \frac{f(x')}{f(x)} = \frac{\Psi_V(v)}{\Psi_V(u)},$$

which can take on any positive value because Ψ_V is onto R_+. By (13), (16), and (17),

$$\begin{aligned} f(x+x') &= g\left(g^{-1}\Psi_V(u) + g^{-1}\Psi_V(v)\right) y \\ &= g\left(g^{-1}\left(\frac{f(x)}{y}\right) + g^{-1}\left(\frac{f(x')}{y}\right)\right) y. \end{aligned}$$

Defining $g_y(x) := g^{-1}\left(\frac{f(x)}{y}\right)$, we have the familiar Cauchy equation

$$g_y(x+x') = g_y(x) + g_y(x'),$$

which is well known to have the continuous solution

$$g_y(x) = c(y)x \Longleftrightarrow g(c(y)x) = \frac{f(x)}{y}. \tag{18}$$

It is easily verified that for any $\gamma > 0$, the functions

$$f(x) = \beta_1\beta_2 x^\gamma,\ c(y) = \left(\frac{\beta_1}{y}\right)^{1/\gamma},\ g(z) = \beta_2 z^\gamma \tag{19}$$

form a family of solutions to (18).

Moreover, as we now show, this is the unique continuous family. Taking the logarithm of (18) we have

$$\begin{aligned} \ln f(x) - \ln y &= \ln g(c(y)x) \\ &= \ln g\left((\exp)\left(\ln c(y) + \ln x\right)\right). \end{aligned} \tag{20}$$

Denote

$$F := \ln g \exp,\ z := \ln c(y),\ w := \ln x,$$
$$G(z) := -\ln c^{-1}\left(e^z\right) = -\ln y,\ H(w) := \ln f\left(e^w\right),$$

then (20) becomes

$$F(z+w) = G(z) + H(w), \tag{21}$$

which is the additive version of Pexider's equation (Aczél, 1966, p. 141–142). By Theorem 3.1 of Aczél, the general solution is

$$F(t) = \theta(t) + a + b, \quad G(t) = \theta(t) + a, \quad H(t) = \theta(t) + b,$$

where a, b are constants and θ is some solution to the Cauchy equation. In our case, the continuous, increasing one, this is of the form

$$\theta(t) = \gamma t,$$

where the constant $\gamma > 0$. Retracing our several steps shows that this solution to (18) is the power family (19).

Thus f and g must both be power functions with the same exponent, call it $1/\gamma_0$. And so

$$\Psi_M(a) = \beta_1\beta_2 \left(\Phi_M(a)\right)^{1/\gamma_0}$$
$$\Psi_V(u) = \beta_2 \left(\Phi_V(u)\right)^{1/\gamma_0}.$$

So by (4)

$$\Psi_M(a)^{\gamma_0} = \Psi_V(u)^{\gamma_0}\Psi_S(s)^{\gamma_0}$$
$$\iff (\beta_1\beta_2)^{\gamma_0}\,\Phi_M(a) = \beta_2^{\gamma_0}\Phi_V(u)\Psi_S(s)^{\gamma_0}.$$

Defining

$$\Gamma_M(a) := (\beta_1\beta_2)^{\gamma_0}\,\Phi_M(a),\ \Gamma_V(u) := \beta_2^{\gamma_0}\Phi_V(u),\ \Gamma_S(s) := \Psi_S(s)^{\gamma_0},$$

we have that

$$\Gamma_M(a) = \Gamma_V(u)\Gamma_S(s)$$

where Γ_M and Γ_V are both additive, thus proving the Theorem.

5.2. Appendix B: The most general p-additive case

In the text, the ranges of $1 + \delta\Psi$ were restricted to be > 1. However, the full domains and ranges of (9) are shown in Table 1.

Table 1

δ_M	$f : \Phi_M \to \Psi_M$	δ_V	$g : \Phi_V \to \Psi_V$	δ_M, δ_V	Ψ_S
1	$]1,\infty[\nearrow]0,\infty[$	1	$]1,\infty[\nearrow]0,\infty[$	A: $1, 1$	$]0,\infty[$
1	$]1,\infty[\nearrow]0,\infty[$	-1	$]0,1[\searrow]0,1[$	B: $1,-1$	$]0,\infty[$
-1	$]0,1[\searrow]0,1[$	1	$]1,\infty[\nearrow]0,\infty[$	C: $-1, 1$	$]0,\infty[$
-1	$]0,1[\searrow]0,1[$	-1	$]0,1[\searrow]0,1[$	D: $-1,-1$	$]0,\infty[$

1. Cases C and D of Table 1 are ruled out because according to the unboundedness of Ψ_S and the fact that $g > 0$ means $g\Psi_S$ ranges over $]0,\infty[$ whereas according to Table 1, f ranges only over $]0,1[$.
2. Case B of Table 1 is ruled out because in (14) the ratio f/Ψ_S spans $]0,\infty[$ whereas $g(\Theta_V(u))$ ranges over only $]0,1[$.
3. Case A differs from the proof in Appendix A only in letting the Θ's be $]0,\infty[$ rather than $]1,\infty[$. Otherwise the argument is unchanged and the measures $1 + \delta_M\Psi_M$ and $1 + \delta_V\Psi_V$ have the range $]-\infty,\infty[$.

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966.

[2] D. H. Krantz, R. D. Luce, P. Suppes and A. Tversky, *Foundations of Measurement*, Vol. I, Academic Press, New York, 1971.

[3] R. D. Luce, *Dimensionally invariant numerical laws correspond to meaningful qualitative relations*, Philosophy of Science *45* (1978), 1–16.

[4] R. D. Luce, *Measurement structures with Archimedean ordered translation groups*, Order *4* (1987), 165–189.

[5] R. D. Luce, *Interpersonal comparisons of utility for 2 of 3 types of people*, Theory and Decision (2009, on their web site, printed version in press).

[6] R. D. Luce, D. H. Krantz, P. Suppes, and A. Tversky, *Foundations of Measurement*, Vol. III, Academic Press, New York, 1990.

[7] R. D. Luce and A. A. J. Marley, *On elements of chance*, Theory and Decision *49* (2000), 97–126.

[8] R. D. Luce and L. Narens, *Symmetry, scale types, and generalizations of classical physical measurement*, Journal of Mathematical Psychology *27* (1983), 44–85.

[9] R. D. Luce, C. T. Ng, A. A. J. Marley and J. Aczél, *Utility of gambling I: Entropy-modified linear weighted utility*, Economic Theory *36* (2008a), 1–33. See erratum at Luce's web site: *Utility of gambling I: entropy-modified linear weighted utility and utility of gambling II: risk, paradoxes, and data*.

[10] R. D. Luce, C. T. Ng, A. A. J. Marley and J. Aczél, *Utility of gambling II: Risk, paradoxes, and data, economic*, Economic Theory *36* (2008b), 165–187. See same errata as in (2008a).

[11] L. Narens, *Abstract Measurement Theory*, MA: M.I.T. Press, Cambridge, 1985.

[12] C. T. Ng, R. D. Luce and A. A. J. Marley, *On the utility of gambling: Extending the Approach of Meginniss*, Aequationes Mathematicae *76* (2008), 281–304.

[13] C. T. Ng, R. D. Luce and A. A. J. Marley, *Utility of gambling under p(olynomial)-additive joint receipt and segregation or duplex decomposition*, Journal of Mathematical Psychology *53* (2009), 273–286.

R. D. Luce
Institute for Mathematical Behavioral Sciences
Social Science Plaza
University of California
Irvine, CA, 92697-5100
USA
e-mail: rdluce@uci.edu

Manuscript received: March 17, 2009 and, in final form, June 14, 2009.