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On the utility of gambling: extending the approach of Meginniss (1976)

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Summary. J. R. Meginniss modified expected utility to accommodate a concept of the utility of gambling that led to a representation composed of a utility expectation term plus an entropy of degree κ term. He imposed several apparently strong assumptions. One of these is that a number of unknown generating functions are identical. A second is that he assumed he was working with given probabilities. Here we follow his general framework but weaken considerably those assumptions. Our problem is reduced to solving some functional equations induced by gamble decomposition. From the solutions, we obtain the representation of the utility function. Further axiomatic restrictions are imposed that lead ultimately to Meginniss' earlier result.

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1. Background

Within the axiomatic theories of utility developed over the past sixty years, mention has been made of the problem of the utility of gambling itself. For example, both seminal articles of Ramsey [16] and of von Neumann and Morgenstern [18] mention it as a problem. But beyond a few, more-or-less discursive treatments of the problem, little has been done on it. Some of the attempts to address the problem are summarized by Luce and Marley in [7]; but, for the most part, they have not been axiomatic in nature. A notable, but largely ignored[1], attempt is a short article by Meginniss [12] in which he arrived, in a partially axiomatic fashion, at two possible forms for the utility of gambling. Our realization of the power of his approach has led to the current attempt to generalize it.

We have completed three further articles, aimed more at an economic and

[1] One person, David Wolpert, when alerted to Meginniss [12] said "... it's amazing! He derives Tsallis entropy over a decade before Tsallis popularized it. Typewritten, with penciled-in corrections; it's like finding a hieroglyphic tablet saying '$E = mc^2$'." (personal communication, October 28, 2004). As was reported in the book of Aczél, & Daróczy [1], what has been called the Tsallis entropy was first arrived at by Havrda, & Charvát [4].

psychological audience, in which we combine one aspect of Meginniss' approach, called gamble decomposition (see Section 2.1) with the axiomatic approach of Luce and Marley in [7] based on the concepts of joint receipts and kernel equivalents of gambles. The first article (Luce, Ng, Marley & Aczél, [9]) deals with uncertain gambles (those where the events have no readily agreed upon probabilities), the second (Luce, Ng, Marley & Aczél, [10]) with risky gambles (where the possible events are simply replaced by probabilities). The third article (Ng, Luce & Marley, [15]) uses mathematical results on *inset entropy* to extend the results to cases where events have value in and of themselves, independent of any consequences associated with them.

1.1. Notation

We denote a set of pure consequences – ones for which chance or uncertainty plays no role – by X. Within that set we assume that there exists a distinguished element $e \in X$ that is interpreted as meaning *no change from the status quo*.

Let $\mathcal{E}_{\mathbf{E}}$ be an algebra of events arising from an "experiment" or "chance phenomenon" $\mathbf{E}$ with the "universal" event denoted by $\mathbf{\Omega_E}$ and a typical event denoted by $C \in \mathcal{E}_{\mathbf{E}}$. Let $\mathbf{C}_n = (C_1, \ldots, C_i, \ldots, C_n)$, where $C_i \cap C_j = \varnothing$ if $i \neq j$, be the underlying partition for a local experiment $\mathbf{E}$. Thus the local universal event is related to the partition as $\Omega_{\mathbf{E}} = \bigcup_{i=1}^{n} C_i$. We assume that all such local universal events are non-empty. A (consequence, event) pair (x, C) is called a *branch* of a *first order gamble*, with the latter being a $2n$-tuple composed of n branches based on a partition of $\Omega_{\mathbf{E}}$. So we may write a typical first-order gamble with n consequences as

$$\mathbf{g}_n = (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n). \tag{1.1}$$

The set of first-order gambles (of any finite length n) is denoted $\mathcal{G}_1$. When the value of n is not playing an explicit role, we sometimes omit it.

A *second-order gamble* is a first-order one in which at least one of the consequences x_i is replaced by a first-order gamble. The set of second-order gambles is denoted $\mathcal{G}_2$. Let $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_1 \cup X$.

We assume that over $\mathcal{G}$ there exists a *preference order* $\succsim$ that is a weak order. As usual, $\precsim$ denotes the converse of $\succsim$, $\sim$ denotes the corresponding indifference relation: $\sim := \succsim \cap \precsim$, and $\succ$ denote strict preference: $\succ := \succsim \cap \neg \sim$.

We assume that for each gamble $\mathbf{g} \in \mathcal{G}$, the set X is sufficiently rich that it contains an element, denoted $CE(\mathbf{g})$, such that $CE(\mathbf{g}) \sim \mathbf{g}$. This pure consequence is called a *certainty equivalent* of the gamble.

We do *not* assume the property of idempotence, i.e.,

$$(x, C_1; \ldots; x, C_i; \ldots; x, C_n) \sim x.$$

Foregoing idempotence is one – maybe, the – way to allow for utility of gambling.

Most theories either assume or derive idempotence.

We say that the gamble (1.1) is *ranked* if the consequences are numbered in order of preference, i.e.,

$$x_1 \succsim x_2 \succsim \cdots \succsim x_n. \tag{1.2}$$

The ranking is often assumed not to matter in so far as behavior is concerned in the sense that any permutation of the branches yields an indifferent gamble. This implies, in particular, that

$$(x, C_1; \ldots; x, C_i; \ldots; x, C_n) \sim (x, C_{\pi(1)}; \ldots; x, C_{\pi(i)}; \ldots; x, C_{\pi(n)}) \tag{1.3}$$

for all permutations π. This is a weak form of symmetry which we shall assume at some point. Some properties, such as upper gamble decomposition in Section 2.1, and utility representations are most neatly written in ranked form.

1.2. Ranked additive representations

A utility function U is an order-preserving mapping from X onto a real interval with the property that $U(e) = 0$. Given such a U, the existence of certainty equivalents induces the utility function on gambles, but not its form, for instance, as being additive over the branches. No loss of generality results from using the same notation U over consequences and over gambles. Following Luce and Marley [8], we begin with the general ranked additive utility (RAU) representation for $x_1 \succsim x_2 \succsim \cdots \succsim x_n$,

$$U(x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) = \sum_{i=1}^{n} L_i\left(U(x_i), \mathbf{C}_n\right), \tag{1.4}$$

where U over gambles and over consequences is onto some non-trivial interval containing 0, and if $C_i = \varnothing$, then $L_i\left(U(x_i), \mathbf{C}_n\right) = 0$. This article investigates restrictions on this representation that allow us to say something about the utility of gambling, per se.

1.3. Meginniss' model

Meginniss' [12] model involved some strong constraints on the form (1.4) which in Section 2 we attempt to weaken in several ways. One of his constraints was that he worked with given probabilities and conditional ones, whereas we are formulating matters using events. When such probabilities are given, we may write $p_i = \Pr(C_i \,|\Omega_{\mathbf{E}})$ and $P = (p_1, p_2, \ldots, p_n)$. Thus, $\sum\limits_{i=1}^{n} p_i = 1$. Further, he assumed that the event dependence is only on the local event C_i relative to $\Omega_{\mathbf{E}}$ and on the length n, i.e.,

$$L_i\left(U(x_i), \mathbf{C}_n\right) = f_{i,n}\left(U(x_i), p_i\right). \tag{1.5}$$

He assumed that $f_{i,n}$ is continuous in each variable and strictly increasing in the first one for each $p_i > 0$. He later concentrated on the special case $f_{i,n} = f$.

We generalize some of these assumptions.

In his proof he invoked the following version of the "reduction of compound gambles"[2] for a gamble with $n = 3$

$$(x_1, p_1; x_2, p_2; x_3, p_3) \sim (x_1, p_1; (x_2, p_2/(p_2+p_3); x_3, p_3/(p_2+p_3)), p_2+p_3). \quad (1.6)$$

He also restricted himself to the unranked case. His assumptions led to the functional equation

$$\begin{aligned} &f\left(U(x_2), p_2\right) + f\left(U(x_3), p_3\right) \\ &\quad = f\left(f\left(U(x_2), p_2/(p_2+p_3)\right) + f\left(U(x_3), p_3/(p_2+p_3)\right), p_2+p_3\right), \end{aligned} \quad (1.7)$$

which he called the "recursive property" of f. Assuming that f is differentiable he derived the following form for U which has two numerical parameters a and κ : if $\mathbf{g}_n = (x_1, p_1; \ldots; x_i, p_i; \ldots;\ x_n, p_n)$,

$$U(\mathbf{g}_n) = WU_{P^\kappa}(\mathbf{g}_n) + aH^{(\kappa)}(P) \quad (\kappa > 0), \quad (1.8)$$

where

$$WU_{P^\kappa}(\mathbf{g}_n) = \sum_{i=1}^{n} U(x_i) p_i^\kappa \quad (\kappa > 0), \quad (1.9)$$

$$H^{(\kappa)}(P) = \begin{cases} \frac{1}{\ln 2}\left(-\sum\limits_{i=1}^{n} p_i \ln p_i\right), & \kappa = 1 \\ \frac{1}{1-2^{1-\kappa}}\left(1 - \sum\limits_{i=1}^{n} p_i^\kappa\right), & \kappa \neq 1, \kappa > 0. \end{cases} \quad (1.10)$$

Ng ([13], equation (5.1.1) and Theorem 5.8) arrived at (1.7) independently from a different source, and solved the equation under weaker assumptions than differentiability. Ebanks [2] generalizes Meginniss' [12] in an explicit application to utility rules for gambles.

Observe that the first term, $WU_{P^\kappa}(\mathbf{g}_n)$, is the subjective weighted utility based on the weights p_i^κ. For $\kappa = 1$ this is just the expected utility

$$EU(\mathbf{g}_n) = \sum_{i=1}^{n} U(x_i) p_i.$$

The second term for the case of $\kappa = 1$ is the Shannon information entropy measure and for $\kappa \neq 1$ it is the entropy of degree κ, often written $\frac{1}{(1-2^{1-\kappa})} \sum\limits_{i=1}^{n} (p_i - p_i^\kappa)$ (see

[2] The history of the term "reduction of compound gambles" is both very old and somewhat obscure, and we do not know who first used it. The concept has played a role, at least in implicit form, throughout the entire history of random variables, and it was used without comment by Bernoulli, the founder of modern utility theory. (Taken from a personal comment, February 10, 2005, from Mark Machina.)

Ch. 6, Aczél & Daróczy, [1], and for a more recent source with many references to the literature, see Ebanks, Sahoo, & Sander, [3]).

Without being aware of Meginniss' article, Yang and Qiu [19] proposed the $\kappa = 1$ model for lotteries and explored some of its properties and applied it to some of the well-known anomalies which we discuss in Luce et al. [10].

2. A more general formulation

We explore various generalizations of Meginniss' approach. In particular, we no longer assume that all $f_{i,n}$ are the same nor do we restrict attention to risky gambles (i.e., known probabilities), or to the unranked formulation.

2.1. Gamble decomposition

Although (1.6) was taken for granted because Meginniss accepted without comment the reduction of compound gambles property, we do not make that assumption which is not even well defined unless one has objective probabilities. In lieu of that, we define explicitly here the concept of gamble decomposition which has received some attention in fairly general contexts by Liu [5], Luce [6], Luce and Marley [7], and Marley and Luce [11], but usually restricted to just gains, i.e., $\succsim e$, or to just losses, $\precsim e$.

To that end, if $\mathbf{g}_n$ is a general ranked gamble of size n, then define the following subgambles of it: for $i = 1, \ldots, n$,

$$\mathbf{g}_{n,-i} := (x_1, C_1; \ldots; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \ldots; x_n, C_n). \tag{2.1}$$

Note that $\mathbf{g}_{n,-i}$ is ranked and is missing branch i, and so it has $n-1$ branches and is based on the sub-experiment with the local universal event $\Omega_{\mathbf{E}} \backslash C_i$.

We assume that

$$D_U := \{U(x) \,|\, x \in X\} =]\iota_*, \iota^*[\tag{2.2}$$

for some $-\infty \leq \iota_* < 0 < \iota^* \leq \infty$. As $U(\mathbf{g}) = U(CE(\mathbf{g}))$, $U(\mathbf{g}) \in D_U$ for all $\mathbf{g}$.

Definition 1. Within the domain of second-order compound gambles, gamble decomposition of type i, $i = 1, \ldots, n$, holds if for $\mathbf{g}_n$ satisfying $x_1 \succsim x_2 \succsim \cdots \succsim x_n$,

$$\mathbf{g}_n \sim \begin{cases} (\mathbf{g}_{n,-i}, \Omega_{\mathbf{E}} \backslash C_i; x_i, C_i), & \mathbf{g}_{n,-i} \succ x_i \\ (x_i, C_i; \mathbf{g}_{n,-i}, \Omega_{\mathbf{E}} \backslash C_i), & \mathbf{g}_{n,-i} \precsim x_i \end{cases}, \tag{2.3}$$

where $(\mathbf{g}_{n,-i}, \Omega_{\mathbf{E}} \backslash C_i; x_i, C_i)$ and $(x_i, C_i; \mathbf{g}_{n,-i}, \Omega_{\mathbf{E}} \backslash C_i)$ are compound binary gambles. Upper gamble decomposition (UGD) refers to $i = 1$ and lower gamble decomposition (LGD) to $i = n$.

In this language, Meginniss [12] arrived at his functional equation (1.7) from symmetry and upper gamble decomposition for $n = 3$.

In the next two sections we develop the functional equations that arise from our general assumptions with either LGD or UGD which are sufficient for present purposes.

2.2. Functional equations for UGD

Consider the RAU model given by (1.4). For ranked consequences, (1.2), and the corresponding ordered partition $\mathbf{C}_n$, we assume that, in a parallel to Meginniss' notation, the term $L_i\left(U(x_i), \mathbf{C}_n\right)$ has the special form

$$L_i\left(U(x_i), \mathbf{C}_n\right) = f_{i,n}(U(x_i), W_{\Omega_{\mathbf{E}}}(C_i)), \tag{2.4}$$

where we assume that $W_{\Omega_{\mathbf{E}}}$, a weighting function, takes values in $[0,1]$; and

$$f_{i,n}(u, 0) = 0, \quad (u \in D_U). \tag{2.5}$$

We now develop the functional equation for UGD when $n = 3$. There are two cases, namely $x_1 \succsim \mathbf{g}_{3,-1}$, and $\mathbf{g}_{3,-1} \succ x_1$.

When $x_1 \succsim \mathbf{g}_{3,-1}$, UGD (2.3) reduces to

$$(x_1, C_1; x_2, C_2; x_3, C_3) \sim (x_1, C_1; (x_2, C_2; x_3, C_3), C_2 \cup C_3),$$

and so from (1.4) and (2.4), with $\Omega_{\mathbf{E}} = \bigcup_{i=1}^{3} C_i$, we obtain

$$\begin{aligned}\sum_{i=1}^{3} f_{i,3}\big(U(x_i), W_{\Omega_{\mathbf{E}}}(C_i)\big) = {} & f_{1,2}\big(U(x_1), W_{\Omega_{\mathbf{E}}}(C_1)\big) \\ & + f_{2,2}\big[f_{1,2}\big(U(x_2), W_{C_2 \cup C_3}(C_2)\big) \\ & + f_{2,2}\big(U(x_3), W_{C_2 \cup C_3}(C_3)\big), W_{\Omega_{\mathbf{E}}}(C_2 \cup C_3)\big].\end{aligned} \tag{2.6}$$

To simplify the notation, let

$$\begin{aligned} u_i &= U(x_i),\ w_i = W_{\Omega_{\mathbf{E}}}(C_i), \\ t &= W_{\Omega_{\mathbf{E}}}(C_2 \cup C_3), \\ r &= W_{C_2 \cup C_3}(C_2),\ s = W_{C_2 \cup C_3}(C_3), \\ f &= f_{1,3},\ g = f_{2,3},\ h = f_{3,3},\ k = f_{1,2},\ l = f_{2,2}. \end{aligned} \tag{2.7}$$

Note that $w_i, r, s, t \in [0,1]$. The functional equation (2.6) can be written as

$$g(u_2, w_2) + h(u_3, w_3) - l(k(u_2, r) + l(u_3, s), t) = k(u_1, w_1) - f(u_1, w_1). \tag{2.8}$$

The above equation came from the case $x_1 \succsim \mathbf{g}_{3,-1}$. So it holds if $u_1 \geq k(u_2, r) + l(u_3, s)$.

We must also consider the case $\mathbf{g}_{3,-1} \succ x_1$. It leads to the equation

$$g(u_2, w_2) + h(u_3, w_3) - k(k(u_2, r) + l(u_3, s), t) = l(u_1, w_1) - f(u_1, w_1) \tag{2.9}$$

which holds if $u_1 < k(u_2, r) + l(u_3, s)$.

Recall that $r = W_{C_2 \cup C_3}(C_2)$, $s = W_{C_2 \cup C_3}(C_3)$ in (2.8). If W is perceived as a conditional probability (finitely additive) measure, then $r + s = 1$. If it comes from a subadditive measure, then we anticipate only $r + s \leq 1$.

For the weighting function W we shall assume that there exists an order-reversing and self-inverting homeomorphism $\Gamma : [0,1] \to [0,1]$ such that

$$\begin{aligned} &\text{(i):} \quad W_{C\cup D}(D) = \Gamma(W_{C\cup D}(C)) \qquad (C \cup D \neq \varnothing), \\ &\text{(ii):} \qquad W_C(\varnothing) = 0, \;\; W_C(C) = 1 \qquad\quad (C \neq \varnothing) \end{aligned}$$

for all disjoint events C, D; and that there exists a continuous function

$$K : [0,1] \times [0,1] \xrightarrow{\text{onto}} [0,1]$$

satisfying

$$K(w, 1) = w, \quad (w \in [0,1]) \tag{2.10}$$

through which W is transitive:

$$W_{C\cup D\cup E}(C) = K(W_{C\cup D}(C), W_{C\cup D\cup E}(C \cup D)) \tag{2.11}$$

for all disjoint events C, D and E (with $C \cup D \neq \varnothing$).

So, in (2.8), $w_2 = K(r,t)$, $w_3 = K(s,t)$, $s = \Gamma(r)$ and $t = \Gamma(w_1)$. As the variable u_1 does not appear on the left side of this equation, the right side can only be a function, say β, of $t = \Gamma(w_1)$. That is,

$$k(u_1, w_1) - f(u_1, w_1) =: \beta(\Gamma(w_1)) = \beta(t). \tag{2.12}$$

In light of (2.5), we have $\beta(1) = 0$. Under the above assumptions, (2.8) is reduced to

$$\begin{gathered} g(u_2, K(r,t)) + h(u_3, K(s,t)) = l(k(u_2, r) + l(u_3, s), t) + \beta(t) \\ \big(\iota^* > u_2 \geq u_3 > \iota_*, \quad r, t \in [0,1], \quad s = \Gamma(r)\big). \end{gathered} \tag{2.13}$$

We assume also that g, h, k, l are continuous and strictly increasing in the first variable, u, with the exception that when the second variable, w, is at the boundary 0. K is supposed to be continuous and strictly increasing in each variable, with exception at the boundary points 0 and 1.

This equation, (2.13), will be treated first. A parallel one coming from (2.9) will be considered afterwards.

2.3. Functional equations for LGD

We now develop the functional equation for LGD when $n = 3$. There are two cases, namely $\mathbf{g}_{3,-3} \succ x_3$ and $x_3 \succsim \mathbf{g}_{3,-3}$.

When $\mathbf{g}_{3,-3} \succ x_3$, we have

$$\sum_{i=1}^{3} f_{i,3}\left(U(x_i), W_\Omega(C_i)\right) = U(\mathbf{g}_3)$$
$$= f_{1,2}\big[f_{1,2}(U(x_1), W_{C_1\cup C_2}(C_1)) + f_{2,2}(U(x_2), W_{C_1\cup C_2}(C_2)), W_\Omega(C_1\cup C_2)\big]$$
$$+ f_{2,2}\left(U(x_3), W_\Omega(C_3)\right). \tag{2.14}$$

With the notations

$$\begin{aligned} u_i &= U(x_i),\ w_i = W_\Omega(C_i),\\ t &= W_\Omega(C_1\cup C_2),\\ r &= W_{C_1\cup C_2}(C_1),\ s = W_{C_1\cup C_2}(C_2),\\ f &= f_{1,3}, g = f_{2,3}, h = f_{3,3}, k = f_{1,2}, l = f_{2,2} \end{aligned}$$

(2.14) becomes

$$f(u_1, w_1) + g(u_2, w_2) + h(u_3, w_3) = k[k(u_1, r) + l(u_2, s), t] + l(u_3, w_3)$$

which holds if $k(u_1, r) + l(u_2, s) \geq u_3$. Arguing as before that this implies that $l(u_3, w_3) - h(u_3, w_3)$ is independent of u_3, we obtain that

$$l(u_3, w_3) - h(u_3, w_3) =: \beta^*(\Gamma(w_3)) = \beta^*(t), \tag{2.15}$$

with $\beta^*(1) = 0$ and the functional equation becomes

$$\begin{gathered} f(u_1, K(r,t)) + g(u_2, K(s,t)) = k[k(u_1, r) + l(u_2, s), t] + \beta^*(t)\\ \big(\iota^* > u_1 \geq u_2 > \iota_*,\quad r,t\in[0,1],\quad s = \Gamma(r)\big). \end{gathered} \tag{2.16}$$

This is formally very similar to (2.13) from UGD, with a parallel equation arising when $x_3 \succsim \mathbf{g}_{3,-3}$.

3. The results for UGD

3.1. A reduction of (2.13)

Let

$$D_{RS} := \{(r,s)\,|\,r\in[0,1], s = \Gamma(r)\}. \tag{3.1}$$

To get rid of some subscripts in (2.13) we write u for u_2 and v for u_3 and rewrite it as

$$\begin{gathered} g(u, K(r,t)) + h(v, K(s,t)) = l(k(u,r) + l(v,s), t) + \beta(t)\\ (u\geq v,\ u,v\in D_U,\ t\in[0,1],\ (r,s)\in D_{RS}). \end{gathered} \tag{3.2}$$

Let

$$I := \{k(u,r) + l(v,s)\,|\,u\geq v,\ u,v\in D_U,\ (r,s)\in D_{RS}\} \subseteq D_U. \tag{3.3}$$

Proposition 2. *The solution of* (3.2)*, or equivalently* (2.13)*, has the representation*

$$K(r,t) = \alpha^{-1}(\alpha(r)\alpha(t)) \quad (r,t \in [0,1]), \tag{3.4}$$

where α is an increasing homeomorphism mapping $[0,1]$ *onto* $[0,1]$*, and, for $u \in D_U$, $w, r, t \in [0,1]$,*

$$\begin{aligned} g(u,w) &= \theta_1(u)\alpha(w) + g_0(w), \\ h(u,w) &= \theta_2(u)\alpha(w) + h_0(w), \\ k(u,r) &= \theta_1(u)\alpha(r)/\sigma + k_0(r), \\ l(u,t) &= \theta_2(u)\alpha(t)/\sigma + l_0(t), \end{aligned} \tag{3.5}$$

where $\sigma > 0$ is a constant, θ_1 and θ_2 are continuous and strictly increasing on D_U, $\theta_1(0) = \theta_2(0) = 0$,

$$\theta_2(z) = \sigma^2 z \ (z \in I), \tag{3.6}$$

and that g_0, h_0, k_0, l_0 are functions satisfying

$$g_0(K(r,t)) + h_0(K(s,t)) = \sigma\alpha(t)[k_0(r) + l_0(s)] + l_0(t) + \beta(t) \tag{3.7}$$

for all $(r,s) \in D_{RS}$, $t \in [0,1]$.

If we let

$$F_1 := g_0 \circ \alpha^{-1}, \quad F_2 := h_0 \circ \alpha^{-1}, \tag{3.8}$$

and use the form of $K(r,t)$ established in (3.4), then (3.7) takes the form

$$F_1(\alpha(r)\alpha(t)) + F_2(\alpha(s)\alpha(t)) = \sigma\alpha(t)[k_0(r) + l_0(s)] + l_0(t) + \beta(t) \tag{3.9}$$

for all $(r,s) \in D_{RS}$, $t \in [0,1]$.

All proofs are in the appendix.

3.2. Solution of (3.7)

Proposition 3. *The solution of* (3.7) *is given by either*

$$\begin{aligned} g_0(w) &= [\sigma k_0(1) + l_0(1)]\alpha(w) - c\alpha(w)\ln\alpha(w), \\ h_0(w) &= (1+\sigma)l_0(1)\alpha(w) - c\alpha(w)\ln\alpha(w), \\ k_0(r) &= \beta(\Gamma(r)) + k_0(1)\alpha(r) - \frac{c}{\sigma}\alpha(r)\ln\alpha(r) - \frac{c}{\sigma}(1-\sigma)\alpha(\Gamma(r))\ln\alpha(\Gamma(r)), \\ l_0(t) &= -c\alpha(t)\ln\alpha(t) + l_0(1)\alpha(t) - \beta(t), \end{aligned} \tag{3.10}$$

where $c \neq 0$, α and Γ satisfy $\alpha(r) + \alpha(\Gamma(r)) = 1$; or by

$$g_0(w) = [\sigma k_0(1) + l_0(1)]\alpha(w) + \frac{c}{\rho - 1}(\alpha(w) - \alpha(w)^\rho),$$

$$h_0(w) = (1+\sigma)l_0(1)\alpha(w) + \frac{c}{\rho - 1}(\alpha(w) - \alpha(w)^\rho),$$

$$k_0(r) = k_0(1)\alpha(r) + \left[\frac{l_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)}\right][\alpha(r) + \alpha(\Gamma(r)) - 1] \tag{3.11}$$
$$- \frac{c}{\rho-1}(\alpha(\Gamma(r)) - \alpha(\Gamma(r))^\rho) + \beta(\Gamma(r)),$$
$$l_0(t) = l_0(1)\alpha(t) + \frac{c}{\rho-1}(\alpha(t) - \alpha(t)^\rho) - \beta(t),$$

where positive $\rho \neq 1$, α *and* Γ *satisfy* $c[\alpha(r)^\rho + \alpha(\Gamma(r))^\rho - 1] = 0$.

Remark 4. Although the forms of $f_{1,2}, f_{2,2}$ contain terms of the rather arbitrary function β, in computing the form of $U(\mathbf{g}_2)$ the β terms cancel out and have no effect (cf. (6.60) and (6.61)). By UGD (2.3), we see that it has no effect on the form of $U(\mathbf{g}_n)$ for all $n \geq 2$. As our primary interest is with U, there is no loss in assuming $\beta = 0$ and consequently, in view of (2.12), $k = f$, i.e. $f_{1,2} = f_{1,3}$.

3.3. Form of U for UGD

It was mentioned earlier that the ranking is often assumed not to matter in so far as behavior is concerned in the sense that any permutation of the branches yields an indifferent gamble. Let us call this property *branch-symmetry*. The Meginniss model, where he assumes that (2.4) holds with $f_{i,n} = f$ for all i, n, clearly invoked the branch-symmetry assumption. We shall see what forms U will have if we take the weaker assumption (1.3).

Theorem 5. *U determined by k and l as given by* (3.5), (3.6), (3.10) *and* (3.11) *is compatible with* (1.3) *for* $n = 2$ *if and only if*

$$\theta_1 = \theta_2 =: \theta, \quad k_0(1) = l_0(1). \tag{3.12}$$

With that, the possible representations of $U(\mathbf{g}_{3,-1}) = U(x_2, C_2; x_3, C_3)$ *are*

$$U(\mathbf{g}_{3,-1}) = \frac{1}{\sigma}\theta(U(x_2))\alpha(r) + \frac{1}{\sigma}\theta(U(x_3))\alpha(\Gamma(r))$$
$$- \frac{c}{\sigma}[\alpha(r)\ln\alpha(r) + \alpha(\Gamma(r))\ln\alpha(\Gamma(r))] + l_0(1) \tag{3.13}$$

and

$$U(\mathbf{g}_{3,-1}) = \frac{1}{\sigma}\theta(U(x_2))\alpha(r) + \frac{1}{\sigma}\theta(U(x_3))\alpha(\Gamma(r))$$
$$+ \left[\frac{(1+\sigma)}{\sigma}l_0(1) + \frac{c}{\sigma(\rho-1)}\right][\alpha(r) + \alpha(\Gamma(r)) - 1] + l_0(1). \tag{3.14}$$

Here, according to (3.6), θ *satisfies*

$$\theta(z) = \sigma^2 z, \forall z \in I. \tag{3.15}$$

Remark 6. Under (3.12), it turns out that (2.9) is satisfied. In essence, (1.3) is natural for (2.9) to hold in conjunction with (2.8).

We consider the property of *certainty*:

$$(x_1, \varnothing; \ldots; x_i, \Omega_{\mathbf{E}}; \ldots; x_n, \varnothing) \sim x_i, \quad (\Omega_{\mathbf{E}} \neq \varnothing). \tag{3.16}$$

Thus, x_i is the certainty equivalent of this "gamble."

Theorem 7. *U determined by k and l as given by* (3.5), (3.6), (3.10) *and* (3.11), *is compatible with* (1.3) *for $n = 2, 3$, or, as an alternative, compatible with* (1.3) *and certainty* (3.16) *for $n = 2$, if and only if*

$$\theta_1(u) = \theta_2(u) = u \quad (\forall u \in D_U), \tag{3.17}$$

$$\sigma = 1, \quad k_0(1) = l_0(1) = 0. \tag{3.18}$$

With that, for $n = 2, 3$, the possible representations of $U(\mathbf{g}_n)$ are

$$U(\mathbf{g}_n) = \sum_{i=1}^{n} U(x_i)\alpha(w_i) - c\sum_{i=1}^{n} \alpha(w_i) \ln \alpha(w_i), \tag{3.19}$$

where $\sum_{i=1}^{n} \alpha(w_i) = 1$; and

$$U(\mathbf{g}_n) = \sum_{i=1}^{n} U(x_i)\alpha(w_i) + \frac{c}{\rho - 1}\left[\sum_{i=1}^{n} \alpha(w_i) - 1\right], \tag{3.20}$$

where $c\Big[\sum_{i=1}^{n} \alpha(w_i)^{\rho} - 1\Big] = 0$.

The representation obtained, (3.19) and (3.20), easily extends to the unranked case, and to gambles of length $n > 3$ by induction, provided that gamble decomposition is assumed for all length $n \geq 3$.

Remark 8. (i) As in Meginniss, assume that (2.4) holds with $f_{i,n} = f$ for all i, n, that $K(r, t) = rt$, and that $\Gamma(r) = 1 - r$. Then from (3.4), with (6.3) and (6.4), we see that $\alpha(w) = w^{\kappa}$ for some constant $\kappa > 0$. In the first family, we have $\kappa = 1$ because of the condition $\alpha(r) + \alpha(1 - r) = 1$. So the solution, taking θ the identity map, is a form of subjective expected utility (SEU), using the weight w, plus an entropy term. The second family gives his second solution by setting $\rho = 1/\kappa$, $\kappa \neq 1$. When $c = 0$ and $\rho = \kappa = 1$ it reduces just to SEU. (ii) Our model allows for the conditional weighting function $W_{\Omega}(\cdot)$ to be non-additive and non-complementary. For instance, we could have $w_i = p_i^{\tau}$ for some additive complete probability distribution $(p_i)_{i=1}^{n}$ and positive $\tau \neq 1$. If so, we may take $\alpha(w) = w^{\kappa}$, $\Gamma(r) = (1 - r^{1/\tau})^{\tau}$ and $\rho = 1/(\tau\kappa)$.

4. The results for LGD

A simple adaptation of the proofs of Proposition 2 and Proposition 3 solves (2.16) and the result is stated below.

Proposition 9. *The solution of* (2.16) *is given by either*

$$\begin{aligned} f(u,w) &= \theta_1(u)\alpha(w) + (1+\sigma)k_0(1)\alpha(w) - c\alpha(w)\ln\alpha(w),\\ g(u,w) &= \theta_2(u)\alpha(w) + [\sigma l_0(1) + k_0(1)]\alpha(w) - c\alpha(w)\ln\alpha(w),\\ k(u,t) &= \theta_1(u)\alpha(t)/\sigma - c\alpha(t)\ln\alpha(t) + k_0(1)\alpha(t) - \beta^*(t),\\ l(u,r) &= \theta_2(u)\alpha(r)/\sigma + \beta^*(\Gamma(r)) + l_0(1)\alpha(r)\\ &\quad - \frac{c}{\sigma}\alpha(r)\ln\alpha(r) - \frac{c}{\sigma}(1-\sigma)\alpha(\Gamma(r))\ln\alpha(\Gamma(r)), \end{aligned} \tag{4.1}$$

where $c \neq 0$, α *and* Γ *satisfy* $\alpha(r) + \alpha(\Gamma(r)) = 1$; *or by*

$$\begin{aligned} f(u,w) &= \theta_1(u)\alpha(w) + (1+\sigma)k_0(1)\alpha(w) + \frac{c}{\rho-1}(\alpha(w) - \alpha(w)^\rho),\\ g(u,w) &= \theta_2(u)\alpha(w) + [\sigma l_0(1) + k_0(1)]\alpha(w) + \frac{c}{\rho-1}(\alpha(w) - \alpha(w)^\rho),\\ k(u,t) &= \theta_1(u)\alpha(t)/\sigma + k_0(1)\alpha(t) + \frac{c}{\rho-1}(\alpha(t) - \alpha(t)^\rho) - \beta^*(t),\\ l(u,r) &= \theta_2(u)\alpha(r)/\sigma + l_0(1)\alpha(r) + \left[\frac{k_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)}\right][\alpha(r) + \alpha(\Gamma(r)) - 1]\\ &\quad - \frac{c}{\rho-1}(\alpha(\Gamma(r)) - \alpha(\Gamma(r))^\rho) + \beta^*(\Gamma(r)), \end{aligned} \tag{4.2}$$

where positive $\rho \neq 1$, α *and* Γ *satisfy* $c[\alpha(r)^\rho + \alpha(\Gamma(r))^\rho - 1] = 0$.

Here, $\beta^*(1) = 0$ *and* θ_1 *satisfies the linearity requirement* (3.6)*:* $\theta_1(z) = \sigma^2 z$.

In a parallel to Theorems 5 and 7 for UGD, we get the following for LGD.

Theorem 10. *U determined by k and l as given by* (4.1) *and* (4.2) *is compatible with* (1.3) *for* $n = 2$ *if and only if*

$$\theta_1 = \theta_2 =: \theta, \quad k_0(1) = l_0(1). \tag{4.3}$$

With (4.3)*, the possible solutions are*

$$\begin{aligned} U(\mathbf{g}_{3,-3}) &= \frac{1}{\sigma}\theta(U(x_1))\alpha(r) + \frac{1}{\sigma}\theta(U(x_2))\alpha(\Gamma(r))\\ &\quad - \frac{c}{\sigma}\alpha(r)\ln\alpha(r) - \frac{c}{\sigma}\alpha(\Gamma(r))\ln\alpha(\Gamma(r)) + l_0(1) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} U(\mathbf{g}_{3,-3}) &= \frac{1}{\sigma}\theta(U(x_1))\alpha(r) + \frac{1}{\sigma}\theta(U(x_2))\alpha(\Gamma(r))\\ &\quad + \left[\frac{(1+\sigma)}{\sigma}l_0(1) + \frac{c}{\sigma(\rho-1)}\right][\alpha(r) + \alpha(\Gamma(r)) - 1] + l_0(1). \end{aligned} \tag{4.5}$$

Here, θ *satisfies* $\theta(z) = \sigma^2 z, \forall x \in I$.

The resulting U is compatible with (1.3) *for* $n = 3$*, or, alternatively, with the certainty property* (3.16) *for* $n = 2$*, if, and only if,* $\sigma = 1$ *and* $l_0(1) = 0$*. In that case, LGD also leads to the representation* (3.19) *and* (3.20)*.*

5. Conclusions

This article pursues Meginniss' [12] approach for developing a theory of utility that admits a notion of a utility for gambling. He assumed given probabilities over the event partitions, accepted the property of gamble decomposition as an immediate consequence of reduction of compounded lotteries, and assumed the general nature of the utility representation. The latter was that the utility of an unranked gamble was additive over the branches, with the utility of each branch a function of the associated consequence and of its probability of occurring. Moreover, he assumed that exactly the same function holds for each branch.

We have relaxed his assumptions by retaining rank dependence in formulating upper gamble decomposition, permitting different functions on each branch, and replacing probabilities by, in general, non-additive weights on events. With these less stringent assumptions, we arrived at a functional equation, (2.13), and solved it under several assumptions about the weights. The form of the solution is given in Proposition 2 and Proposition 3. Theorem 7 gives the utility representation when we add symmetry (1.3) and certainty (3.16) to the assumptions. As previously noted, the representation obtained, (3.19) and (3.20), easily extends to the unranked case and to gambles of length $n > 3$ by induction, provided that gamble decomposition is assumed for that length. It leads towards Meginniss' results.

6. Appendix

6.1. A uniqueness theorem

The next two theorems follow easily from Ng [13], Lemma 2.1 and Theorem 1.0.

Theorem 11. (1) Additive form. *Let X and Y be real intervals, $T : X \times Y \to \mathbb{R}$ be continuous, and consider the functional equation*

$$\begin{aligned} &f(x) + g(y) = h(T(x, y)) \quad (x \in X,\, y \in Y), \\ &f : X \to \mathbb{R},\ g : Y \to \mathbb{R},\ h : T(X \times Y) \to \mathbb{R}. \end{aligned} \tag{6.1}$$

If (6.1) *has a solution (f_0, g_0, h_0) with continuous, nonconstant f_0 and g_0, then*

$$f = \gamma f_0 + \beta_1,\ g = \gamma g_0 + \beta_2,\ h = \gamma h_0 + \beta_1 + \beta_2, \tag{6.2}$$

where β_1, β_2, γ are arbitrary constants, give the general solutions (f, g, h) with continuous f and g.

(2) Multiplicative form. *Let X and Y be real intervals, $T : X \times Y \to \mathbb{R}$ be continuous, and consider the functional equation*

$$\begin{gathered} f(x)g(y) = h(T(x,y)) \quad (x \in X, y \in Y), \\ f : X \to]0,\infty[,\ g : Y \to]0,\infty[,\ h : T(X \times Y) \to]0,\infty[. \end{gathered} \tag{6.3}$$

If (6.3) has a solution (f_0, g_0, h_0) with continuous, nonconstant f_0 and g_0, then

$$f = \beta_1 f_0^\gamma,\ g = \beta_2 g_0^\gamma,\ h = \beta_1\beta_2 h_0^\gamma, \tag{6.4}$$

where $\beta_1 > 0, \beta_2 > 0, \gamma$ are arbitrary constants, give the general solutions (f, g, h) with continuous f and g.

Theorem 12. *Let X and Y be non-degenerate real intervals, $T : X \times Y \to \mathbb{R}$ be continuous and strictly monotonic in each variable, and let (f, g, h) be a solution of the equation (6.3). If f is continuous and not strictly monotonic, then both f and g are constant.*

6.2. Proof of Proposition 2

Recalling the definition of D_{RS}, (3.1), let $D^\circ_{RS} := D_{RS} \cap (\,]0,1[\times]0,1[)$. Let $x \in D_U$, $(r,s) \in D^\circ_{RS}$, $t \in]0,1]$ be temporarily fixed and consider (3.2) holding for all (u, v) in the rectangle

$$D_x := [x, \iota^*[\times]\iota_*, x].$$

We are about to apply Theorem 11 in additive form. In so doing, we take $T(u,v) := k(u,r) + l(v,s)$, $\tilde{f}_0(u) := k(u,r)$, $\tilde{g}_0(v) := l(v,s)$ and $\tilde{h}_0(z) := z$ for all $u \in X := [x, \iota^*[$, $v \in Y :=]\iota_*, x]$ and $z \in T(D_x)$ as a particular solution. By the assumptions on k and l, $\tilde{f}_0$, on X, and $\tilde{g}_0$, on Y, are continuous and nonconstant. Because $\tilde{f}(u) := g(u, K(r,t))$, $\tilde{g}(v) := h(v, K(s,t))$ and $\tilde{h}(z) := l(z,t) + \beta(t)$ is another solution with continuous $\tilde{f}$ and $\tilde{g}$, Theorem 11 asserts that there exist constants γ and β_i (constant with respect to the variables u, v, but may depend on the fixed x, r, s, t) such that

$$\begin{aligned} g(u, K(r,t)) &= \gamma(x,r,s,t)k(u,r) + \beta_1(x,r,s,t) \quad (u \in [x, \iota^*[), \\ h(v, K(s,t)) &= \gamma(x,r,s,t)l(v,s) + \beta_2(x,r,s,t) \quad (v \in]\iota_*, x]), \\ l(z,t) + \beta(t) &= \gamma(x,r,s,t)z + \beta_3(x,r,s,t) \quad (z \in T(D_x)), \\ \beta_3 &= \beta_1 + \beta_2. \end{aligned} \tag{6.5}$$

We compare the first equation at two distinct values of x and conclude that γ and β_1 do not depend on x, and that the equation holds for all $u \in D_U$. Similarly β_2 does not depend on x, and the second equation holds for all $v \in D_U$. This justifies the writing of γ and β_i without x and we get

$$\begin{aligned} g(u,K(r,t)) &= \gamma(r,s,t)k(u,r) + \beta_1(r,s,t) \quad (u \in D_U),\\ h(v,K(s,t)) &= \gamma(r,s,t)l(v,s) + \beta_2(r,s,t) \quad (v \in D_U),\\ l(z,t) &= \gamma(r,s,t)z + \beta_3(r,s,t) - \beta(t) \quad (z \in \cup_{x\in D_U} T(D_x)),\\ \beta_3 &= \beta_1 + \beta_2. \end{aligned} \tag{6.6}$$

We observe that (i) $\cup_{x\in D_U} T(D_x) = \{k(u,r) + l(v,s) \,|\, u,v \in D_U,\ u \ge v\} =: I_{(r,s)}$ is a proper open interval for each $(r,s) \in D^\circ_{RS}$, (ii) for each fixed (r_0,s_0) there exists a small neighborhood N of (r_0,s_0) relative to D°_{RS} such that $\cap_{(r,s)\in N} I_{(r,s)}$ still contains a proper interval. The latter and the third equation in (6.6) show that γ and β_3 do not depend on (r,s) locally at each $(r_0,s_0) \in D^\circ_{RS}$. We also observe that γ and β's are continuous functions. The local constancy of γ and β_3 in (r,s) implies that they are constant in (r,s) on the connected space D°_{RS}. That is, they do not depend on (r,s). With that, the third equation of (6.6) holds for all z in $\{k(u,r) + l(v,s) \,|\, u,v \in D_U, u \ge v, (r,s) \in D^\circ_{RS}\}$. By continuity, it holds for all $z \in I$. Comparing the first two equations we see that β_1 is only a function of (r,t), and that β_2 is only a function of (s,t). Thus (6.6) is strengthened to

$$\begin{aligned} g(u,K(r,t)) &= \gamma(t)k(u,r) + \beta_1(r,t) \quad (u \in D_U, 0<r<1, t>0),\\ h(v,K(s,t)) &= \gamma(t)l(v,s) + \beta_2(s,t) \quad (v \in D_U, 0<s<1, t>0),\\ l(z,t) &= \gamma(t)z + \beta_3(t) - \beta(t) \quad (z \in I, t>0),\\ \beta_3(t) &= \beta_1(r,t) + \beta_2(s,t) \quad ((r,s) \in D^\circ_{RS}, t>0). \end{aligned} \tag{6.7}$$

Let us define

$$\begin{aligned} g_0(t) &:= g(0,t), \quad g_1(u,t) := g(u,t) - g_0(t),\\ k_0(t) &:= k(0,t), \quad k_1(u,t) := k(u,t) - k_0(t),\\ h_0(t) &:= h(0,t), \quad h_1(v,t) := h(v,t) - h_0(t),\\ l_0(t) &:= l(0,t), \quad l_1(v,t) := l(v,t) - l_0(t). \end{aligned} \tag{6.8}$$

By the assumptions made on g,h,k,l, it follows that for each fixed $t>0$, g_1,h_1,k_1,l_1 are continuous and strictly increasing in the first variable. Moreover, we have $g_1(0,\cdot) = h_1(0,\cdot) = k_1(0,\cdot) = l_1(0,\cdot) = 0$. We get from the first equation of (6.7) that

$$g_1(u,K(r,t)) = \gamma(t)k_1(u,r) \quad (u \in D_U, 0<r<1, t>0). \tag{6.9}$$

Because g_1,k_1 have positive values for $u>0$, we get $\gamma(t)>0$ for $t>0$. Putting in (6.9) $t=1$ and using (2.10) $K(r,1)=r$, we get $g_1(u,r) = \gamma(1)k_1(u,r)$ for $0<r<1$. By continuity, the relation extends to the full domain:

$$g_1 = \gamma(1)k_1. \tag{6.10}$$

Let

$$\sigma := \gamma(1), \quad \alpha := \gamma/\sigma. \tag{6.11}$$

Then $\sigma>0$ and α is continuous and positive on $]0,1]$ with $\alpha(1)=1$. From the last equation in (3.5) and because of the boundary condition (2.5) $l(z,0)=0$ we get

$$\lim_{t\to 0} \alpha(t) = 0. \tag{6.12}$$

So α is non-constant on $]0,1]$. Using (6.10) to eliminate g_1, (6.9) gives

$$k_1(u, K(r,t)) = \alpha(t)k_1(u,r) \quad (u \in D_U, 0 < r < 1, t > 0). \tag{6.13}$$

First, let us allow u to vary only in $]0, \iota^*[$. Because K is strictly increasing in each variable, by Theorem 12 in multiplicative form, α, being non-constant and satisfying (6.12), must be strictly increasing on $]0,1]$. Moreover, $k_1(u,r)$ must be strictly monotonic over all $r \in]0,1[$ for each fixed $u > 0$. By Theorem 11 in multiplicative form, applied to (6.13) with $T := K$ and with $u > 0$ as a parameter, we get that, for some constant (which may depend on u) $c^*(u) > 0$,

$$\begin{aligned} k_1(u,r) &= c^*(u)\psi(r) \quad (u \in D_U, u > 0, 0 < r < 1), \\ \psi(K(r,t)) &= \alpha(t)\psi(r) \quad\ \ (0 < r < 1, t > 0), \end{aligned} \tag{6.14}$$

where $\psi(r) := k_1(u_0, r) > 0$ for some fixed $u_0 > 0$. Next, we consider (6.13) and allow u to vary in $]\iota_*, 0[$. As in the first case, we obtain

$$\begin{aligned} k_1(u,r) &= c_*(u)\tilde{\psi}(r) \quad (u \in D_U, u < 0, 0 < r < 1), \\ \tilde{\psi}(K(r,t)) &= \alpha(t)\tilde{\psi}(r) \quad\ \ (0 < r < 1, t > 0), \end{aligned} \tag{6.15}$$

where $\tilde{\psi}(r) := k_1(\tilde{u}_0, r)$ for some fixed $\tilde{u}_0 < 0$ and $c_*(u) > 0$. Comparing the second equation of (6.14) and of (6.15) we get that $\tilde{\psi} = \tilde{a}\psi$ for some constant $\tilde{a} < 0$. Defining $c : D_U \to \mathbb{R}$ by

$$c(u) = c^*(u), \text{ if } u > 0, \quad c(u) = \tilde{a}c_*(u), \text{ if } u < 0, \quad c(0) = 0, \tag{6.16}$$

we combine (6.14) and (6.15) and get

$$\begin{aligned} k_1(u,r) &= c(u)\psi(r) \quad (u \in D_U, 0 < r < 1), \\ \psi(K(r,t)) &= \alpha(t)\psi(r) \quad (0 < r < 1, t > 0). \end{aligned} \tag{6.17}$$

This solves (6.13).

Just as the first equation of (6.7) gives (6.10) and (6.17), the second equation of (6.7) yields

$$h_1 = \gamma(1) l_1 \tag{6.18}$$

and

$$\begin{aligned} l_1(v,s) &= d(v)\Psi(s) \quad (v \in D_U, 0 < s < 1), \\ \Psi(K(s,t)) &= \alpha(t)\Psi(s) \quad (0 < s < 1, t > 0) \end{aligned} \tag{6.19}$$

for some functions d, Ψ with $d(0) = 0$ and $\Psi(s) > 0$.

The third equation in (6.7) yields

$$l_1(z,t) = \gamma(t)z + \beta_3(t) - \beta(t) - l_0(t) \quad (z \in I,\ t \in]0,1]).$$

Comparing it with the first equation in (6.19) we get that, for some constant $a > 0$,

$$d(z) = az\ (z \in I), \quad \Psi(s) = (1/a)\gamma(s), \quad l_0(s) = \beta_3(s) - \beta(s). \tag{6.20}$$

We compare the last equation of (6.14) and of (6.19) using Theorem 11 and get

$$\psi = b\Psi \tag{6.21}$$

for some constant $b > 0$. Let

$$\theta_1(u) := \sigma^2 bc(u)/a, \quad \theta_2(v) := \sigma^2 d(v)/a. \tag{6.22}$$

The third equation of (6.7), the relations (6.8), (6.10), (6.17)–(6.22) give (3.5) and

$$\alpha(K(r,t)) = \alpha(t)\alpha(r) \quad (0 < r < 1, t \in]0,1]) \tag{6.23}$$

first over all $r, s \in]0,1[$ and $t \in]0,1]$. Because of (6.12) and that α is continuous, strictly increasing on $]0,1]$ with $\alpha(1) = 1$, it can be extended to an increasing homeomorphism on $[0,1]$ by defining $\alpha(0) = 0$. With that, (6.23) is extended to (3.4) and (3.5) holds on the full domains by continuity.

With the above necessary forms of the functions, direct substitution shows that (2.13) is reduced to (3.7). □

6.3. Proof of Proposition 3

The proof is preceded by three lemmas.

Lemma 13. *Let $\alpha_1, \alpha_2 :]0,1[\to]0,1[$ be homeomorphisms. Let $f :]0,1[\to \mathbb{R}$ be a continuous solution of the functional equation*

$$f(\alpha_1(r)x) + f(\alpha_2(r)x) = f(x) \quad (r, x \in]0,1[). \tag{6.24}$$

Then either $f = 0$ or

$$f(x) = cx^\rho \quad \text{and} \quad \alpha_1(r)^\rho + \alpha_2(r)^\rho = 1 \quad (r, x \in]0,1[) \tag{6.25}$$

for some constants $c \neq 0$, $\rho > 0$.

Proof. Letting $y := \alpha_2(r)x$ and $q := \alpha_1 \circ \alpha_2^{-1}$, the equation takes the form

$$f(xq(y/x)) = f(x) - f(y) \quad (y, x \in]0,1[, y < x), \tag{6.26}$$

where q is again a homeomorphism on $]0,1[$. The only constant function f satisfying the equation is $f = 0$. Assume now that f is nonconstant. By continuity, there exist a_1, a_2 in $]0,1[$, with $a_1 < a_2$, such that f is nonconstant on each of the sub-intervals $]0, a_1[$, $]a_1, a_2[$ and $]a_2, 1[$. By inspection, for each fixed $\mu \in]0,1[$, the map f_μ defined by $f_\mu(x) := f(\mu x)$ is again a continuous solution of (6.26). Applying Theorem 11 in additive form to the equation with $T(x,y) = xq(y/x)$ over the two rectangular sub-domains $D_i := \{(x,y) \mid x, y \in]0,1[, y < a_i < x\}$, $(i = 1,2)$, we get that there exist constants $\phi(\mu, a_i)$, $\psi_1(\mu, a_i)$ and $\psi_2(\mu, a_i)$ such that

$$\begin{aligned} f(\mu x) &= \phi(\mu, a_i) f(x) + \psi_1(\mu, a_i) && (x \in]a_i, 1[), \\ f(\mu y) &= \phi(\mu, a_i) f(y) - \psi_2(\mu, a_i) && (y \in]0, a_i[), \\ f(\mu u) &= \phi(\mu, a_i) f(u) + \psi_1(\mu, a_i) + \psi_2(\mu, a_i) && (u \in T(D_i)). \end{aligned} \tag{6.27}$$

The consistency of the first equation holding for a_1 and for a_2 and the non-constancy of f on the common interval $]a_2, 1[$ yields $\phi(\mu, a_1) = \phi(\mu, a_2) =: \phi(\mu)$ and $\psi_1(\mu, a_1) = \psi_1(\mu, a_2)$. Similarly, from the second equation for a_1 and for a_2 we get also $\psi_2(\mu, a_1) = \psi_2(\mu, a_2)$. Comparing the first equation for a_1 with the second for a_2 over the common interval $]a_1, a_2[$ on which f is non-constant, we get that $\psi_1(\mu, a_1) = -\psi_2(\mu, a_2)$. The third equation thus becomes $f(\mu u) = \phi(\mu) f(u)$. When this third equation is compared with the first two, we get $\psi_i = 0$. The union of the equations of (6.27) is, therefore, simply

$$f(\mu t) = \phi(\mu) f(t) \quad (\mu, t \in]0, 1[).$$

This is a Pexider equation, and so f has the form

$$f(t) = ct^\rho \quad (t \in]0, 1[)$$

for some constants $c \neq 0, \rho \neq 0$. Putting it back into (6.24) we arrive at $\alpha_1(r)^\rho + \alpha_2(r)^\rho = 1$ as the necessary and sufficient condition for the equation to hold. This condition itself rules out the use of negative ρ values and we have proved (6.25).□

Lemma 14. *Let α_1, α_2 be homeomorphisms on $]0, 1[$. Let $f :]0, 1[\to \mathbb{R}$ be a solution satisfying the functional equation*

$$f(\alpha_1(r)x) + f(\alpha_2(r)x) = xm(r) + f(x) \quad (r, x \in]0, 1[) \tag{6.28}$$

for some function $m :]0, 1[\to \mathbb{R}$. If f has continuous first derivative on $]0, 1[$, then either

$$f(x) = c_1 x \quad (x \in]0, 1[) \tag{6.29}$$

for some constant c_1; or

$$f(x) = -cx \ln x + c_1 x \quad \text{and} \quad \alpha_1(r) + \alpha_2(r) = 1 \quad (r, x \in]0, 1[) \tag{6.30}$$

for some constants $c \neq 0$ and c_1; or

$$f(x) = -\frac{c}{\rho - 1} x^\rho + c_1 x \quad \text{and} \quad \alpha_1(r)^\rho + \alpha_2(r)^\rho = 1 \quad (r, x \in]0, 1[) \tag{6.31}$$

for some constants $c \neq 0$, $\rho > 0$ and $\rho \neq 1$, and c_1.

Proof. Differentiate (6.28) with respect to x. We get

$$\alpha_1(r) f'(\alpha_1(r)x) + \alpha_2(r) f'(\alpha_2(r)x) = m(r) + f'(x). \tag{6.32}$$

Multiply that by x and subtract the result from (6.28) while letting

$$g(t) := f(t) - tf'(t) \quad (t \in]0, 1[). \tag{6.33}$$

We get

$$g(\alpha_1(r)x) + g(\alpha_2(r)x) = g(x) \quad (r, x \in]0, 1[). \tag{6.34}$$

As we have assumed that f' is continuous, the continuity of g follows from (6.33) and Lemma 13 is applicable. So, either $g = 0$ or

$$g(x) = cx^\rho \quad \text{and} \quad \alpha_1(r)^\rho + \alpha_2(r)^\rho = 1 \quad (r, x \in]0, 1[) \tag{6.35}$$

for some $c \neq 0$, $\rho > 0$. Putting the obtained forms of g in (6.33) and integrating, we get the asserted necessary forms of f on $]0,1[$. □

Lemma 15. *Let α_1, α_2 be homeomorphisms on $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ be a continuous solution of the functional equation*

$$f(\alpha_1(r)x) + f(\alpha_2(r)x) = xm(r) + f(x) \quad (r, x \in [0,1]) \tag{6.36}$$

for some function $m : [0,1] \to \mathbb{R}$. Then either

$$f(x) = c_1 x \quad (x \in [0,1]) \tag{6.37}$$

for some constant c_1; or

$$f(x) = -cx \ln x + c_1 x \quad \text{and} \quad \alpha_1(r) + \alpha_2(r) = 1 \quad (r, x \in [0,1]) \tag{6.38}$$

for some constants $c \neq 0$ and c_1; or

$$f(x) = -\frac{c}{\rho - 1} x^\rho + c_1 x \quad \text{and} \quad \alpha_1(r)^\rho + \alpha_2(r)^\rho = 1 \quad (r, x \in [0,1]) \tag{6.39}$$

for some constants $c \neq 0$, $\rho > 0$ and $\rho \neq 1$, and c_1.

Proof. For arbitrary $r \in]0,1[$ we integrate (6.36) with respect to x from 0 to a general $y \in]0,1[$ to get

$$\frac{1}{\alpha_1(r)} \int_0^{\alpha_1(r)y} f + \frac{1}{\alpha_2(r)} \int_0^{\alpha_2(r)y} f = y^2 m(r)/2 + \int_0^y f \quad (r, y \in]0,1[).$$

Dividing the equation by y and letting

$$h(y) := \frac{1}{y} \int_0^y f \quad (y \in]0,1[), \tag{6.40}$$

we get

$$h(\alpha_1(r)y) + h(\alpha_2(r)y) = ym(r)/2 + h(y) \quad (r, y \in]0,1[). \tag{6.41}$$

By the definition of h, and by the continuity of f, we observe that h has a continuous first derivative. Thus we can apply Lemma 14 to (6.41) and arrive at the fact that h is necessarily of the forms $h(x) = C_1 x$, $-Cx \ln x + C_1 x$, or $-\frac{C}{\rho'-1} x^{\rho'} + C_1 x$ on $]0,1[$. In all cases we infer that h has a continuous second order derivative on $]0,1[$. Hence h' has a continuous first order derivative on $]0,1[$. By (6.40),

$$h'(y) := \frac{-1}{y^2} \int_0^y f + f(y)/y \quad (y \in]0,1[).$$

Thus h' having a continuous derivative on $]0,1[$ implies that f has a continuous derivative on $]0,1[$. With that, Lemma 14 is applicable to (6.39) on the interior of $[0,1]$ and we get the asserted forms (6.37)–(6.39) first over the interior. The continuity of all functions on $[0,1]$ finally extends their validity to the full $[0,1]$. □

We proceed to solve (3.7) via its equivalent form (3.9), where $\mathrm{D}_{RS} = \{(r,s) \mid r,s \in [0,1], s = \Gamma(r)\}$, Γ is a self-inverting order reversing homeomorphism of $[0,1]$ and α is an order preserving homeomorphism of $[0,1]$:

$$F_1(\alpha(r)\alpha(t)) + F_2(\alpha(\Gamma(r))\alpha(t)) = \sigma\alpha(t)[k_0(r) + l_0(\Gamma(r))] + (l_0+\beta)(t) \tag{6.42}$$

for all $r,t \in [0,1]$, where $F_1, F_2, k_0, l_0, \beta$ are continuous on $[0,1]$, vanishing at 0, and $\beta(1) = 0$. Letting $r = 1$ and $r = 0$, respectively, we get

$$\begin{aligned} F_1(\alpha(t)) &= \sigma k_0(1)\alpha(t) + (l_0+\beta)(t), \\ F_2(\alpha(t)) &= \sigma l_0(1)\alpha(t) + (l_0+\beta)(t). \end{aligned} \tag{6.43}$$

Letting

$$F := (l_0+\beta)\circ\alpha^{-1}, \quad \text{i.e.} \quad F(\alpha(t)) = (l_0+\beta)(t), \tag{6.44}$$

(6.43) becomes

$$\begin{aligned} F_1(t) &= \sigma k_0(1)t + F(t), \\ F_2(t) &= \sigma l_0(1)t + F(t). \end{aligned} \tag{6.45}$$

Replacing F_1, F_2 in (6.42) using (6.45) we get

$$\begin{aligned} F(\alpha(r)\alpha(t)) + \sigma k_0(1)\alpha(r)\alpha(t) + F(\alpha(\Gamma(r))\alpha(t)) + \sigma l_0(1)\alpha(\Gamma(r))\alpha(t) \\ = \sigma\alpha(t)[k_0(r) + l_0(\Gamma(r))] + F(\alpha(t)). \end{aligned} \tag{6.46}$$

Defining further

$$G(t) = F(t) - F(1)t \tag{6.47}$$

we have

$$G(1) = 0 \tag{6.48}$$

and (6.46) becomes

$$\begin{aligned} G(\alpha(r)\alpha(t)) + [F(1)+\sigma k_0(1)]\alpha(r)\alpha(t) + G(\alpha(\Gamma(r))\alpha(t)) + [F(1)+\sigma l_0(1)]\alpha(\Gamma(r))\alpha(t) \\ = \sigma\alpha(t)[k_0(r) + l_0(\Gamma(r))] + G(\alpha(t)) + F(1)\alpha(t). \end{aligned}$$

Letting $x := \alpha(t)$, this takes the form

$$G(\alpha(r)x) + G(\alpha(\Gamma(r))x) = xm(r) + G(x) \quad (r, x \in [0,1]), \tag{6.49}$$

where

$$\begin{aligned} m(r) := -[F(1)+\sigma k_0(1)]\alpha(r) - [F(1)+\sigma l_0(1)]\alpha(\Gamma(r)) \\ + \sigma[k_0(r) + l_0(\Gamma(r))] + F(1). \end{aligned}$$

By Lemma 15, noting (6.48), we get either

$$G(t) = 0, \tag{6.50}$$

or

$$G(t) = -ct\ln t \quad \text{and} \quad \alpha(r) + \alpha(\Gamma(r)) = 1 \quad (r,t \in [0,1]), \tag{6.51}$$

for some $c \neq 0$, or

$$G(t) = \frac{c}{\rho-1}(t - t^\rho) \quad \text{and} \quad \alpha(r)^\rho + \alpha(\Gamma(r))^\rho = 1 \quad (r,t \in [0,1]) \tag{6.52}$$

for some $c \neq 0, \rho > 0$ and $\rho \neq 1$.

Referring back to (6.47) and (6.44), we see that

$$l_0(t) = G(\alpha(t)) + l_0(1)\alpha(t) - \beta(t). \tag{6.53}$$

With the determined forms of G by (6.50) – (6.52) we get that l_0 has the following corresponding forms:

$$\begin{aligned} l_0(t) &= l_0(1)\alpha(t) - \beta(t), \quad \text{or} \\ l_0(t) &= -c\alpha(t)\ln\alpha(t) + l_0(1)\alpha(t) - \beta(t), \quad \text{or} \\ l_0(t) &= \frac{c}{\rho-1}(\alpha(t) - \alpha(t)^\rho) + l_0(1)\alpha(t) - \beta(t). \end{aligned} \tag{6.54}$$

Putting the above forms into (6.43) we obtain

$$\begin{aligned} F_1(t) &= \sigma k_0(1)t + l_0(1)t, \quad \text{or} \\ F_1(t) &= \sigma k_0(1)t - ct\ln t + l_0(1)t, \quad \text{or} \\ F_1(t) &= \sigma k_0(1)t + \frac{c}{\rho-1}(t - t^\rho) + l_0(1)t. \end{aligned} \tag{6.55}$$

and

$$\begin{aligned} F_2(t) &= \sigma l_0(1)t + l_0(1)t, \quad \text{or} \\ F_2(t) &= \sigma l_0(1)t - ct\ln t + l_0(1)t, \quad \text{or} \\ F_2(t) &= \sigma l_0(1)t + \frac{c}{\rho-1}(t - t^\rho) + l_0(1)t. \end{aligned} \tag{6.56}$$

In (6.42), letting $t = 1$ we get

$$\sigma k_0(r) = -\sigma l_0(\Gamma(r)) + F_1(\alpha(r)) + F_2(\alpha(\Gamma(r))) - l_0(1). \tag{6.57}$$

Using (6.43) we have

$$\sigma k_0(r) = -\sigma l_0(\Gamma(r)) + \sigma k_0(1)\alpha(r) + (l_0+\beta)(r) + \sigma l_0(1)\alpha(\Gamma(r)) + (l_0+\beta)(\Gamma(r)) - l_0(1) \tag{6.58}$$

through which we get the form of k_0 from that of l_0:

$$\begin{aligned} \sigma k_0(r) &= \sigma\beta(\Gamma(r)) + [l_0(1) + \sigma k_0(1)]\alpha(r) + l_0(1)\alpha(\Gamma(r)) - l_0(1), \quad \text{or} \\ \sigma k_0(r) &= \sigma\beta(\Gamma(r)) + \sigma k_0(1)\alpha(r) \\ &\qquad - c\alpha(r)\ln\alpha(r) - c(1-\sigma)\alpha(\Gamma(r))\ln\alpha(\Gamma(r)), \quad \text{or} \\ \sigma k_0(r) &= \sigma\beta(\Gamma(r)) + [l_0(1) + \sigma k_0(1)]\alpha(r) + l_0(1)\alpha(\Gamma(r)) - l_0(1) \\ &\qquad + (1-\sigma)\frac{c}{\rho-1}(\alpha(\Gamma(r)) - \alpha(\Gamma(r))^\rho) + \frac{c}{\rho-1}(\alpha(r) - \alpha(r)^\rho). \end{aligned} \tag{6.59}$$

6.4. Proof of Theorem 5

Using the definitions of k, r, l, s of (2.7), the forms of k and l as given by (3.5), (3.6), (3.10) and (3.11) give

$$U(\mathbf{g}_{3,-1}) = U(x_2, C_2; x_3, C_3)$$
$$= \theta_1(U(x_2))\alpha(r)/\sigma + \theta_2(U(x_3))\alpha(\Gamma(r))/\sigma - \frac{c}{\sigma}\alpha(r)\ln\alpha(r)$$
$$- \frac{c}{\sigma}\alpha(\Gamma(r))\ln\alpha(\Gamma(r)) + k_0(1)\alpha(r) + l_0(1)\alpha(\Gamma(r)), \quad (6.60)$$

where $\alpha(r) + \alpha(\Gamma(r)) = 1$; and

$$U(\mathbf{g}_{3,-1}) = \theta_1(U(x_2))\alpha(r)/\sigma + \theta_2(U(x_3))\alpha(\Gamma(r))/\sigma$$
$$+ \left[\frac{l_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)}\right][\alpha(r) + \alpha(\Gamma(r)) - 1] + k_0(1)\alpha(r) + l_0(1)\alpha(\Gamma(r)), \quad (6.61)$$

where positive $\rho \neq 1$, $c[\alpha(r)^\rho + \alpha(\Gamma(r))^\rho - 1] = 0$.

For U given by (6.60) to be compatible with (1.3), the expression on the right hand side at $U(x_2) = U(x_3) =: u$ must be invariant under the substitution $r \mapsto \Gamma(r)$. Because $\theta_1(0) = \theta_2(0) = 0$, it is necessary and sufficient to have the invariance of both $\theta_1(u)\alpha(r) + \theta_2(u)\alpha(\Gamma(r))$ and $-\frac{c}{\sigma}\alpha(r)\ln\alpha(r) - \frac{c}{\sigma}\alpha(\Gamma(r))\ln\alpha(\Gamma(r)) + k_0(1)\alpha(r) + l_0(1)\alpha(\Gamma(r))$ under the substitution. Comparing the value of the first expression $\theta_1(u)\alpha(r) + \theta_2(u)\alpha(\Gamma(r))$ at $r = 0$, to its value at $r = \Gamma(0) = 1$, we get $\theta_2 = \theta_1$. The invariance of the second expression yields $k_0(1) = l_0(1)$. This proves (3.12) with which (6.60) reduces to (3.13).

The argument for (6.61) is the same and we arrive at (3.12) and (3.14).

6.5. Proof of Theorem 7

Using the definitions of f, g, h, w_i of (2.7), and remembering that $f = k$ (Remark 4), then $U(\mathbf{g}_3)$ given by (3.5), (3.6), (3.10) and (3.11) and, according to Theorem 5, (3.12), has the form:

$$U(\mathbf{g}_3) = \theta(U(x_1))\alpha(w_1)/\sigma + \theta(U(x_2))\alpha(w_2) + \theta(U(x_3))\alpha(w_3)$$
$$- \frac{c}{\sigma}\alpha(w_1)\ln\alpha(w_1) - c\alpha(w_2)\ln\alpha(w_2) - c\alpha(w_3)\ln\alpha(w_3)$$
$$- \frac{c}{\sigma}(1-\sigma)(1-\alpha(w_1))\ln((1-\alpha(w_1))$$
$$+ l_0(1) + \sigma l_0(1)\alpha(w_2) + \sigma l_0(1)\alpha(w_3), \quad (6.62)$$

where $\sum_{i=1}^{3}\alpha(w_i) = 1$; and

$$U(\mathbf{g}_3) = \theta(U(x_1))\alpha(w_1)/\sigma + \theta(U(x_2))\alpha(w_2) + \theta(U(x_3))\alpha(w_3) - \left[\frac{l_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)}\right]$$
$$+ \left[\frac{l_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)} - \frac{c}{(\rho-1)}\right]\alpha(\Gamma(w_1))$$
$$+ \left[l_0(1) + \frac{l_0(1)}{\sigma} + \frac{c}{\sigma(\rho-1)}\right]\alpha(w_1)$$
$$+ \left[(1+\sigma)l_0(1) + \frac{c}{\rho-1}\right]\left[\alpha(w_2) + \alpha(w_3)\right], \quad (6.63)$$

where positive $\rho \neq 1$, $c\,[\alpha(w_1)^\rho + \alpha(\Gamma(w_1))^\rho - 1] = 0$, $c\left[\sum_{i=1}^{3} \alpha(w_i)^\rho - 1\right] = 0$.

The parallel equation (2.9) is satisfied and the above representations for $U(\mathbf{g}_3)$ are valid for the case $\mathbf{g}_{3,-1} \precsim x_1$ as well as for the case $\mathbf{g}_{3,-1} \succsim x_1$.

The compatibility of $U(\mathbf{g}_3)$ with (1.3) requires the symmetry of the expressions on the right in w_1, w_2 and w_3 when $U(x_1) = U(x_2) = U(x_3) = u$ is held fixed. The symmetry of (6.62) in w_i forces $\sigma = 1$ and $l_0(1) = 0$. Consider the symmetry of (6.63). Because $\theta(0) = 0$, we get that $\theta(u)\alpha(w_1)/\sigma + \theta(u)\alpha(w_2) + \theta(u)\alpha(w_3)$ and the sum of the remaining terms are separately symmetric in the variables w_i. The former yields $\sigma = 1$. The latter then has value $2l_0(1)$ at $(w_1, w_2, w_3) = (0, 1, 0)$, and $l_0(1)$ at $(w_1, w_2, w_3) = (1, 0, 0)$. Symmetry yields $l_0(1) = 0$. This proves (3.18) for both families.

Considering (3.13) and (3.14) at $r = 1$ and using (3.18) we get

$$U(x, \varnothing; x, C_3) = \theta(U(x)), \quad \forall x \in X.$$

By the definition of I, $U(x, \varnothing; x, C_3)$ belongs to I. This implies that θ maps D_U into the subinterval I. On the other hand, $\theta(z) = z$ for all $z \in I$, and θ is strictly increasing and continuous on D_U. Therefore, $I = D_U$ and (3.17) holds.

With (3.17) and (3.18), (3.13), (3.14), (6.62) and (6.63) are reduced to (3.19) and (3.20) for $n = 2, 3$.

Under the alternative assumption of certainty, it is simple to verify that U given by (3.13) and (3.14), at $r = 1$, satisfies $U(x, \varnothing; x, C_3) = U(x)$, as implied by certainty, if and only if (3.17) and (3.18) hold. The remaining arguments are the same. □

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