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Comment

A solution to a problem raised in Luce and Marley (2005)

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Abstract

Luce and Marley [2005. Ranked additive utility representations of gambles: Old and new axiomatizations. *Journal of Risk and Uncertainty*, *30*, 21–62] examined various relations between mathematical forms for the utility of joint receipt $\oplus$ of gambles and for the utility of uncertain gambles. Their assumptions lead to a bisymmetry functional equation which, when the gambles are ranked, is defined on a restricted domain. Maksa [1999. Solution of generalized bisymmetry type equations without surjectivity. *Aequationes Mathematicae*, *57*, 50–74] solved the general case and Kocsis [2007. A bisymmetry equation on restricted domain. *Aequationes Mathematicae*, *73*, 280–284] presents the solution for the ranked case. The latter solution allows us to solve open problem 5 in Luce and Marley (2005) by showing that the assumptions of their Theorem 19 for an order-preserving ranked additive utility (RAU) representation U imply that U is a ranked weighted utility (RWU) representation that is additive over $\oplus$.
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Luce and Marley (2005) examined various relations between mathematical forms for the utility of joint receipt $\oplus$ of gambles, including as a special case the pure consequences $x, y \in X$, and for the utility of uncertain gambles, including the binary cases $(x, C_1; y, C_2)$, where the consequence x is received if the event C_1 occurs and y if C_2 occurs. Because this Note relies heavily on the notation and results of that paper, we summarize the relevant material in Appendix A.

In [Thm. 19][1] and its Corollary, Luce and Marley explore ranked and unranked additive order preserving utility representations U onto $I = [0, \infty[$ (Appendix A, (6) and (7)) under the assumptions that

$$U(x \oplus y) = F(U(x), U(y)),$$

where $F : I \times I \xrightarrow{\text{onto}} I$ and is symmetric and strictly increasing in each variable, and

$$U(x, C_1; y, C_2) = G(U(x), U(y)),$$

where $G : I \times I \xrightarrow{\text{onto}} I$ and is strictly increasing in each variable and, in the ranked case, $x \succsim y$, where $\succsim$ is the preference relation over pure consequences and gambles. For convenience at this stage we suppress the dependence of G on $\overrightarrow{\mathbf{C}}_2 = (C_1, C_2)$. Because each of F and G is strictly increasing in each variable and map intervals onto intervals, each must be continuous in each variable. Also, they assume *component summing* [Def. 17], which leads to the bisymmetry functional equation [(47)]: for $X_1, X_2, Y_1, Y_2 \in [0, k[$, $k > 0$, where $[0, k[$ is the range of U,

$$F(G(X_1, X_2), G(Y_1, Y_2)) = G(F(X_1, Y_1), F(X_2, Y_2)), \tag{1}$$

with, in the ranked case, $X_1 \geqslant X_2$, $Y_1 \geqslant Y_2$.

Without the restriction $X_1 \geqslant X_2$, $Y_1 \geqslant Y_2$, the general case of (1) has been dealt with by Maksa (1999), and Kocsis (2007) presents the solution for the ranked case. Here, we show that the latter solution allows us to solve open problem 5 in Luce and Marley (2005) by showing that the assumptions of [Thm. 19] for an order-preserving ranked additive utility (RAU) representation U, [Def. 1], imply that U is a ranked weighted utility (RWU) representation, [Def. 2], that is additive over $\oplus$ [Def. 16].

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[1] Any citation [...] is to Luce and Marley (2005), which is summarized in Appendix A.

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As discussed in Section 2, this result, especially the additivity of $\oplus$ in the context of the RWU representation, may be inconsistent with some interpretations of data.

1. The Result

First we summarize the Kocsis (2007) result, then use it to solve the above open problem.

1.1. Kocsis (2007)

Kocsis (2007) solves the ordered problem of (1) when the arguments lie on an interval[2] I and F and G are continuous and strictly increasing in each variable. It is: there is a continuous and strictly monotonic function $\alpha : I \to \mathbb{R}$ and positive numbers a, d_1, d_2 and real numbers b, c with

$$(2a-1)c = (d_1 + d_2 - 1)b \tag{2}$$

such that

$$F(r,s) = \alpha^{-1}(a\alpha(r) + a\alpha(s) + b), \quad (r,s) \in I^2 \tag{3}$$

$$G(r,s) = \alpha^{-1}(d_1\alpha(r) + d_2\alpha(s) + c), \quad (r,s) \in I^2, \ r \geqslant s. \tag{4}$$

1.2. Extension to the ranked case of Theorem 19, Luce and Marley (2005)

With $\mathscr{D}_+$ the set of gambles, as defined in Appendix A, we define the *equivalence*, $\approx$, of any two binary operations $\circledcirc_1$, $\circledcirc_2$ on $\mathscr{D}_+$ by

$$\circledcirc_1 \approx \circledcirc_2 \iff \forall g, h \in \mathscr{D}_+ \quad (g \circledcirc_1 h \sim g \circledcirc_2 h).$$

Theorem 1. *Suppose that U is an order-preserving RAU representation*, [*Def.* 1], *on the non-negative real interval* $[0, \infty[$, *that joint receipts are strictly increasing in each variable*, *and that $e \in X$ is an identity of $\oplus$ over $X \times X$. Then*, *the following are equivalent*:

1. *U is decomposable over joint receipts*, [*Def.* 18], *and*, *for gambles with the same event partition*, *$\oplus \approx \divideontimes$ is satisfied*, *where $\divideontimes$ denotes component summing*, [*Def.* 17].
2. *U is a RWU representation*, [*Def.* 2], *of gambles and is additive over* $\oplus$, [*Def.* 16].

The proof is in Appendix B. Note that it follows that $U(e) = 0$.

2. Discussion

As discussed in Luce and Marley (2005) for the parallel result in the unranked case (Corollary to [Thm. 19]), the above theorem is surprisingly strong and seems at first a bit disquieting. The property of component summing of gambles on the same ordered partition seems, on its face, highly innocent. Yet, in the presence of decomposability of $\oplus$ and $\oplus \approx \divideontimes$, it implies not only that RAU reduces to RWU, which is fine, but also that joint receipts are additive, which may not be so fine, especially for money where $x \oplus y = x + y$ implies that U has to be proportional to money. Most of the empirical literature, except for an occasional linear approximation for small sums of money by, among others, Michael H. Birnbaum, has concluded that with money gambles and assuming that some form of RWU holds, then U is non-linear with money. However, that conclusion is based on data-fitting, not the evaluation of critical behavioral (axiomatic) properties, and in such data-fitting there is almost always a trade-off between utility and probability functions. Traditionally, the St. Petersburg paradox has been interpreted as strong evidence against the proportionality of utility to money. However, as was argued in Section 2.3 of Luce, Ng, Marley, and Aczél (2007), the gamble discussed in the paradox is so preposterous—both from the perspective of finding any one willing to sell one or from that of a buyer, who knows that no seller has infinite resources to pay it off—that no empirical conclusions really can be drawn from it.

[2] He used X, but we have used that symbol for the set of riskless consequences.

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Appendix A. Notation, definitions and results from Luce and Marley (2005)

A.1. Notation

Let X denote the set of pure consequences for which chance or uncertainty plays no role. A distinguished element $e \in X$ is interpreted to mean *no change from the status quo*. We assume a *preference order* $\succsim$ exists over X and that it is a weak order. Let $\sim$ denote the corresponding indifference relation. We confine our attention to *gains*, i.e., where $x \succsim e$ for each $x \in X$ (or, equally, *losses*, i.e., $x \precsim e$ for each $x \in X$). A typical *first-order ranked gamble of gains g* with n consequences is of the form

$$g = (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) \equiv (\ldots; x_i, C_i; \ldots),$$

where $x_i \succsim e$ are consequences (of gains) with

$$x_1 \succsim \cdots \succsim x_i \succsim \cdots \succsim x_n \succsim e \tag{5}$$

and $\mathbf{C}_n = (C_1, \ldots, C_i, \ldots, C_n)$ is a partition of some "universal" event $C(n) = \bigcup_{i=1}^n C_i$. In this ordered case, there is an order induced on the underlying partition $\mathbf{C}_n$, which we emphasize by the vector notation $\overrightarrow{\mathbf{C}}_n$.

We assume X is so rich that for any first-order gamble g, there exists $CE(g) \in X$, called the *certainty equivalent* of g, such that $CE(g) \sim g$. Thus, the preference order $\succsim$ can be extended to the domain of gains $\mathscr{D}_+$ that consists of X and all first-order gambles. For some results we need to expand $\mathscr{D}_+$ to include *second-order gambles* in which some of the x_i are replaced by first-order gambles.

We explore utility representations U onto real intervals of the form $I = [0, \kappa[$, where $\kappa \in]0, \infty]$, that meet various, increasingly stronger, restrictions. Two conditions that are common to all representations that we consider are

$$g \succsim h \text{ iff } U(g) \geqslant U(h), \tag{6}$$

$$U(e) = 0. \tag{7}$$

We refer to these as *order-preserving representations*.

A.2. Additive and weighted utility representations

Definition 1. An order-preserving representation $U : D_+ \xrightarrow{\text{onto}} I \subseteq \mathbb{R}_+$ is a *ranked additive utility* (*RAU*) one iff, for all $x_i \in X$ satisfying (5) and for every corresponding ordered partition $\overrightarrow{\mathbf{C}}_n$ of $C(n) = \bigcup_{i=1}^n C_i$, there exist strictly increasing functions $L_i(\cdot, \overrightarrow{\mathbf{C}}_n) : I \xrightarrow{\text{onto}} I$, with the following properties:

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=1}^{n} L_i(U(x_i), \overrightarrow{\mathbf{C}}_n), \tag{8}$$

$$L_i(0, \overrightarrow{\mathbf{C}}_n) = 0,$$

$$C_i = \varnothing \text{ implies } L_i(Z, \overrightarrow{\mathbf{C}}_n) = 0 \quad (Z \in I).$$

It is an *additive utility* (*AU*) *representation* iff the functions $L_i(\cdot, \overrightarrow{\mathbf{C}}_n)$ are the same for every ordering of the partition C_n of $C(n)$.

Definition 2. An order-preserving representation $U : \mathscr{D}_+ \xrightarrow{\text{onto}} I \subseteq \mathbb{R}_+$ is a *ranked weighted utility* (*RWU*) one iff there exists weights $S_i(\overrightarrow{\mathbf{C}}_n)$ assigned to each index $i = 1, \ldots, n$ and possibly dependent on the entire ordered partition $\overrightarrow{\mathbf{C}}_n$, where $0 \leqslant S_i(\overrightarrow{\mathbf{C}}_n)$ and $S_i(\overrightarrow{\mathbf{C}}_n) = 0$ iff $C_i = \varnothing$, such that, for (5) holding,

$$U(\ldots; x_i, C_i; \ldots) = \sum_{i=1}^{n} U(x_i) S_i(\overrightarrow{\mathbf{C}}_n). \tag{9}$$

If the ranking is immaterial, it is called *weighted utility* (*WU*).

A.3. Joint receipts

With X the set of pure consequences, for $x, y \in X$, $x \oplus y \in X$ represents receiving or having both x and y, and when f, g are gambles, $f \oplus g$ means having or receiving both gambles.

Definition 16. The operation $\oplus$ has a *generalized additive representation* $U : \mathscr{D}_+ \xrightarrow{\text{onto}} \mathbb{R}_+ := [0, \infty[$ iff (6), (7), and there exists a strictly increasing function φ such that:

$$U(f \oplus g) = \varphi^{-1}(\varphi(U(f)) + \varphi(U(g))). \tag{10}$$

It is called *additive* if φ is the identity.

A.3.1. Dependent gambles and component summing

Luce and Marley (2005) discuss the idea of experimentally totally dependent gambles. First, let $\overrightarrow{\mathbf{C}}_n$ and $\overrightarrow{\mathbf{D}}_m$ be two ordered event partitions, and let

$$f = (\ldots; x_i, C_i; \ldots), \quad g = (\ldots; y_j, D_j; \ldots). \tag{11}$$

Definition 17. Let f and g be of the form (11) but with $m = n$ and $\overrightarrow{\mathbf{C}}_n = \overrightarrow{\mathbf{D}}_n$ for a single realization of the underlying experiment. Then *component summing*, denoted $f \circledast g$, of the gambles f and g is defined by

$$f \circledast g := (x_1 \oplus y_1, C_1; \ldots; x_i \oplus y_i, C_i; \ldots; x_n \oplus y_n, C_n). \tag{12}$$

A.4. Theorem 19 and its corollary

Definition 18. U is said to be *decomposable over joint receipt* iff there is a function $F : I \times I \xrightarrow{\text{onto}} I$, with F strictly increasing in each variable, such that:

$$U(f \oplus g) = F(U(f), U(h)) \quad (f, g \in \mathscr{D}_+). \tag{13}$$

Theorem 19. *Suppose that U is a RAU order-preserving representation, Definition* 1, *on the non-negative real interval* $[0, \infty[$, *and that joint receipts are strictly increasing in each variable. If U is decomposable over joint receipts, Definition* 18, *e is an identity of $\oplus$, and for gambles with the same event partition $\oplus \approx \circledast$ is satisfied, where $\circledast$ denotes component summing, Definition* 17, *then $\oplus$ has a generalized additive form, Definition* 16.

The following corollary says that if one assumes AU rather than RAU, then the result is much stronger.

Corollary to Theorem 19. *Suppose that U is an AU order-preserving representation, Definition* 1, *on the non-negative real interval* $[0, \infty[$, *that joint receipts are strictly increasing in each variable, and that e is an identity of $\oplus$. Then, the following are equivalent*:

1. *U is decomposable over joint receipts, Definition* 18, *and for gambles with the same event partition $\oplus \approx \circledast$ is satisfied, where $\circledast$ denotes component summing, Definition* 17.

2. *There exists* U^* *that both forms a WU representation*, *Definition* 2, *of gambles and is additive over* $\oplus$, *Definition* 16.

Appendix B. Proof of Theorem 1

Proof. 1 implies 2. Let F be the symmetric decomposable function for $\oplus$, then by the RAU form, (8), applied to $\oplus \approx \divideontimes$,

$$F\left(\sum_{i=1}^{n} L_i(U(x_i), \overrightarrow{\mathbf{C}}_n), \sum_{i=1}^{n} L_i(U(y_i), \overrightarrow{\mathbf{C}}_n)\right) = \sum_{i=1}^{n} L_i(F(U(x_i), U(y_i)), \overrightarrow{\mathbf{C}}_n). \tag{14}$$

Also, we have that $I = [0, \infty[$ and each of $F : I \times I \xrightarrow{\text{onto}} I$ and $L_i : I \xrightarrow{\text{onto}} I$, $i = 1, \ldots, n$, is strictly increasing in each variable. Since each maps onto the interval I, each must be continuous.

Consider the special case $x_2 = y_2 = x_3 = y_3 = \cdots = x_n = y_n = e$, abbreviate $L_1(., \overrightarrow{\mathbf{C}}_n)$ for the moment by $L_1(\cdot)$, denote $U(x_1) = r, U(y_1) = s$ and recall that $L_i(U(e), \overrightarrow{\mathbf{C}}_n) = L_i(0, \overrightarrow{\mathbf{C}}_n) = 0$. Thus, $F(L_1(r), L_1(s)) = L_1(F(r,s))$ with, under our assumptions, F and L_1 continuous. This is Eq. (7) on p. 62 of Aczél (1966), which has the general continuous solution

$$F(r,s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s)), \quad L_1(t) = \varphi(d_1\varphi^{-1}(t)). \tag{15}$$

Thus, F has a generalized additive representation (10). Returning to the general notation, $L_1(t, \overrightarrow{\mathbf{C}}_n) = \varphi(S_1(\overrightarrow{\mathbf{C}}_n)\varphi^{-1}(t))$, where the constant d_1 is written $S_1(\overrightarrow{\mathbf{C}}_n)$ to show the explicit dependence on $\overrightarrow{\mathbf{C}}_n$ and to conform to our earlier notation.

Now consider the special case $x_3 = y_3 = \cdots = x_n = y_n = e$, abbreviate $L_i(., \overrightarrow{\mathbf{C}}_n)$, $i = 1, 2$, for the moment by $L_i(\cdot)$, denote $U(x_i) = X_i$, $U(y_i) = Y_i$, $i = 1, 2$, and recall that $L_i(U(e), \overrightarrow{\mathbf{C}}_n) = L_i(0, \overrightarrow{\mathbf{C}}_n) = 0$. Then (14) becomes

$$\begin{aligned} &F(L_1(X_1) + L_2(X_2), L_1(Y_1) + L_2(Y_2)) \\ &\quad = L_1(F(X_1, Y_1)) + L_2(F(X_2, Y_2)) \\ &\quad\quad (X_1 \geqslant X_2, Y_1 \geqslant Y_2). \end{aligned}$$

Recall that Kocsis (2007) solved the more general form of this functional equation, namely

$$\begin{aligned} &F(G(X_1, X_2), G(Y_1, Y_2)) = G(F(X_1, Y_1), F(X_2, Y_2)) \\ &\quad (X_1 \geqslant X_2, Y_1 \geqslant Y_2). \end{aligned}$$

when the arguments lie in an interval I and F and G are continuous and strictly increasing in each variable and F is symmetric. As mentioned earlier, he showed that there is a continuous and strictly monotonic function $\alpha : I \to \mathbb{R}$ and positive numbers a, d_1, d_2, and real numbers b, c satisfying (2) and with F, G satisfying, respectively, (3) and (4).

Returning to our application, in (15) we already have the form

$$F(r,s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s)), \tag{16}$$

which would be identical to (3) with $I = [0, \infty[$ if we knew that $\alpha = \varphi^{-1}$ and $a = 1$, $b = 0$. In fact, a careful study of Kocsis derivation shows that, when (16) holds, his representation has exactly these properties, and in addition[3] $c = 0$. Thus, replacing α by φ^{-1}, since we also have $G(r,s) = L_1(r) + L_2(s)$, the representations (3) and (4) become

$$F(r,s) = \varphi(\varphi^{-1}(r) + \varphi^{-1}(s)), \quad (r,s) \in I^2 \tag{17}$$

and

$$L_1(r) + L_2(s) = \varphi(d_1\varphi^{-1}(r) + d_2\varphi^{-1}(s)), \quad (r,s) \in I^2, \ r \geqslant s. \tag{18}$$

However, we have $L_2(0) = \varphi(0) = 0$, and so (18) gives

$$L_1(r) = \varphi(d_1\varphi^{-1}(r)),$$

in agreement with (15), which, substituted back into (18), gives

$$\begin{aligned} &\varphi(d_1\varphi^{-1}(r)) + L_2(s) \\ &\quad = \varphi(d_1\varphi^{-1}(r) + d_2\varphi^{-1}(s)), \quad (r,s) \in I^2, \ r \geqslant s. \end{aligned} \tag{19}$$

Since both L_2 and φ are strictly increasing functions and d_2 is positive, there exists a strictly increasing function π such that $L_2(s) = \pi(d_2\varphi^{-1}(s))$, so (19) becomes

$$\begin{aligned} &\varphi(d_1\varphi^{-1}(r)) + \pi(d_2\varphi^{-1}(s)) \\ &\quad = \varphi(d_1\varphi^{-1}(r) + d_2\varphi^{-1}(s)), \quad (r,s) \in I^2, \ r \geqslant s. \end{aligned}$$

With $R = d_1\varphi^{-1}(r), S = d_2\varphi^{-1}(s)$, this becomes

$$\begin{aligned} &\varphi(R) + \pi(S) = \varphi(R + S) \\ &\quad \left(\varphi\left(\frac{R}{d_1}\right), \varphi\left(\frac{S}{d_2}\right)\right) \in I^2, R \geqslant S\frac{d_1}{d_2}, \end{aligned}$$

which is the well-known Pexider equation. Since φ and π are strictly increasing, the equation is defined on a region (non-empty connected open set) and thus has a unique extension whose solutions for strictly increasing functions with $\varphi(0) = \pi(0) = 0$ are known (Aczél, 1987) to be $\varphi(u) = \pi(u) = ku$, $(k > 0)$. But $L_1(r) = \varphi(d_1\varphi^{-1}(r))$, $L_2(s) = \pi(d_2\varphi^{-1}(s))$, so we obtain $L_1(r) = d_1 r$, $L_2(s) = d_2 s$.

Now consider the special case of (14) with $x_4 = y_4 = \cdots x_n = y_n = e$, abbreviate $L_3(\cdot, \overrightarrow{\mathbf{C}}_n)$ for the moment by

[3] The important facts to note are that our (16) implies that $\gamma = \alpha = \varphi^{-1}$ in (5) of Kocsis, which in turn leads to $K = H$ in a later functional equation. This restriction then gives $c = 0$ in the solution of that more restricted functional equation. Later, Kocsis defines the function $\chi = \alpha \circ \gamma^{-1}$ which in our case becomes the identity function, which in turn leads to $a = 1$, $b = 0$.

$L_3(\cdot)$ and denote $U(x_i) = X_i$, $U(y_i) = Y_i$, $i = 1, 2, 3$. Then (14) becomes: for $X_1 \geqslant X_2 \geqslant X_3$, $Y_1 \geqslant Y_2 \geqslant Y_3$,

$$F(L_1(X_1) + L_2(X_2) + L_3(X_3), L_1(Y_1) + L_2(Y_2) + L_3(Y_3)) \\ = L_1(F(X_1, Y_1)) + L_2(F(X_2, Y_2)) + L_3(F(X_3, Y_3)),$$

which with the above results reduces to

$$L_3(X_3) + L_3(Y_3) = L_3(X_3 + Y_3),$$

with general continuous solution $L_3(r) = d_3 r$. An inductive argument then gives that $L_i(r) = d_i r$ for all $i = 1, \ldots, n$.

Finally, with $d_i = S_i(\overrightarrow{\mathbf{C}}_n)$, we obtain: for $x_1 \succsim \cdots \succsim x_n \succsim e$,

$$U(x_1, C_1; \ldots; x_n, C_n) = \sum_{i=1}^{n} U(x_i) S_i(\overrightarrow{\mathbf{C}}_n),$$

i.e., RWU.

2 implies 1. An additive $\oplus$ is, of course, decomposable. The calculation for $\oplus \approx \circledast$ is routine. $\square$

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