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PURITY, RESISTANCE, and INNOCENCE IN UTILITY THEORY*

ABSTRACT. This note addresses 3 issues that seem to pervade much of economic thought about individual decisions among uncertain alternatives: (1) Restricting primitives to just orderings of first-order gambles and not admitting, e.g., compound acts or joint receipt of consequences and gambles. (2) Great resistance to experimental findings that strongly suggest that most current theories fail descriptively. (3) Taking for granted the innocence of some assumptions when, in fact, they are not innocent, e.g., that constant acts are idempotent. My conclusion is that these restrictions have greatly inhibited developments in utility theory.

KEY WORDS: compound gambles, constant acts, experimental findings, innocent assumptions, joint receipt, restricting primitives

In a long career, I have interacted often with the axiomatic and other developments of utility theory. It began early when, as a graduate student in mathematics, I first encountered the appendix of the 1947 edition of von Neumann and Morgenstern's *The Theory of Games and Economic Behavior* where the first really axiomatic version of expected utility was presented. It was soon followed by reading and re-reading Savage (1954) leading up to the treatment given in *Games and Decisions* (Luce and Raiffa, 1957/1989). My interest was sustained during the preparation of a chapter by Luce and Suppes (1965) and during the work leading to the three volumes of the *Foundations of Measurement* (Krantz et al., 1971/2007; Suppes et al., 1989/2007; Luce et al., 1990/2007), and greatly increased by work during the 1980s and 90s on rank-dependent theories

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that led to my monograph *Utility of Gains and Losses* (Luce, 2000). During that exposure to the many theoretical ideas and experimental evaluations of them, I have drawn a few bits of, what I like to think of as, wisdom, some of which I shared at a round table organized by Mark Machina who, because of illness, was unable to be present and selected Edie Karni to chair it.

1. PURITY OF PRIMITIVES

The fathers—there were no mothers in this field in mid century—of utility theory based their work on preference orderings over one type of risky or uncertain alternative (gamble) or another. A sharp distinction was made between consequences, which are valued, and events or, in the case of risk, probabilities, which are evaluated in terms of subjective (or descriptive) probabilities or, more generally, weights. In essence, the theory explores how the two factors tradeoff.

Never mind that some events have a great deal of inherent value, e.g., in choosing how to travel from home to Rome and return, some possibilities—such as a crash—are quite negative, such as serious injury or death, whether or not I placed a money bet (insurance) on the outcomes of the trip. Also, the classical theories make no attempt to incorporate other, quite natural primitives, such as an operation of joint receipt, which I have exploited.

Purity seems to be that one *should* limit attention to the following four ideas:

1. An unbelievably large (Savage) *state space* of valueless events.

Go back to his definition and see what it entails for even a fairly simple experiment, let alone the set of decisions that you face in, say, the course of a year. It is 10^n where n is some quite large number. In practice, all examples and all experiments are based on *local* "universal events." The Savage state space closely resembles Kolmorgorov's (1933/1956) axiomatization of probability theory which begins with a single sample space and its subsets. It is never clear in such axiomatizations what one is supposed to do if some source of chance is either added or subtracted from the original set, and yet such changes are ubiquitous—the elimination of some chancy or uncertain things, the addition of new ones.

2. An even more unbelievably large set of *acts*, i.e., maps of finite support from the state space to pure consequences.

In practice, relatively simple acts are what people contemplate, not the sort of thing Savage postulated.

3. The classical theories of decisions under uncertainty tended to deal with what may be called first-order gambles, i.e., mappings from events to pure consequences. One does not admit gambles whose consequences are, themselves, gambles.

In practice, life is full of compound gambles aside from first-order gambles or lotteries: Flights through hubs, automobile trips, etc. For example, my trip from Orange County, CA, to Rome involved a change of plane in Chicago. Exceptions to this are found in the study of risk, where the "natural" reduction of compound risks to first-order ones is assumed.

4. A preference order $\succsim$ over the set of acts.

Do these four features capture all there is to decision making under uncertainty? I think not because, in practice, there is at least one other ubiquitous primitive, namely, *joint receipts*: Let f and g be two gambles (including pure alternatives as a special case), one can have or receive both of them, denoted by $f \oplus g$. Most shopping involves joint receipts as do portfolios of investments. And recall that fundamental physical measurement involves at least two kinds of structures, conjoint ones such as varying mass by varying the volume of different homogeneous substances, and concatenation ones such as placing two masses on the pan of a pan balance, and some laws connecting the two. Often these linking laws are a form of distribution (see *Foundations of Measurement*, Chaps. 10, 19, and 20). So, I strongly urge studying linked structures with a common ordering, in addition to studying each one in isolation.

2. RESISTANCE TO EXPERIMENTAL FINDINGS

If a theory manages to explain a lot of behavior that previous ones failed to, there seems to be substantial resistance—sometimes, irrationally strong—on the part of disciples to acknowledge the significance of data that unambiguously contradict their favorite theory. To my mind, the most vivid example is *rank-dependent utility (RDU)* (=*Choquet utility)*, including *cumulative prospect theory (CPT)* as a special case, which is, of course, able to accommodate a number of phenomena not encompassed by expected utility in the case of risk or subjective expected utility in the case of uncertainty. That is its purpose.

Let

$$g_n = \begin{pmatrix} C_1 & C_2 & \dots & C_i & \dots & C_n \\ x_1 & x_2 & \dots & x_i & \dots & x_n \end{pmatrix}$$

denote the gamble in which x_i is the consequence if event C_i occurs, where $C_i \cap C_j = \varnothing$ if $i \neq j$, and the consequences are ordered from best, x_1, to worst, x_n. Let $C(i) := \bigcup_{j=1}^{i} C_j$, $C_0 = \varnothing$. The RDU models all take the general form

$$U(g_n) = \sum_{i=1}^{n} U(x_i)\,[S(C(i)) - S(C(i-1)],$$

where U is an order preserving utility measure over pure consequences and gambles and S is a subjective weighting function over events that, in general, is not finitely additive and for which $S(C(n)) = 1$ and $S(C_0) = S(\varnothing) = 0$. Note: $C(n)$ denotes the "universal event" of a local gamble, not the master state space of Savage (1954) and his followers.

All these representations, which include cumulative prospect theory as a special case, exhibit *coalescing* (also called *event splitting* for reasons that will become apparent), namely:

$$\begin{pmatrix} C_1 & C_2 & \ldots & C_i & \ldots & C_n \\ x & x & \ldots & x_i & \ldots & x_n \end{pmatrix} \sim \begin{pmatrix} C_1 \cup C_2 & \ldots & C_i & \ldots & C_n \\ x & \ldots & x_i & \ldots & x_n \end{pmatrix} \tag{1}$$

Note that events C_1 and C_2 give rise to the same consequence, as does, of course, $C_1 \cup C_2$. So (1) seems to be almost trivially rational. Indeed, it is automatically satisfied if risky gambles are modeled as random variables, which many economists are prone to do.

Moreover, it is coalescing, properly applied, that allows one to define a sensible concept of dominance in the context of uncertain alternatives.

In experimental practice, however, an asymmetry in (1) plays a role not at all evident in the mathematics: going from left to right, coalescing, is trivially easy, no thought is required. But going from right to left, event splitting, is not self-evident. An event E can be partitioned in many ways, and some of these, but not most, may prove useful in simplifying a decision problem.

Much data—a great deal due to Michael Birnbaum (for summaries, see Birnbaum, 1997 and Marley and Luce, 2005)—tells us that this property often fails. An example is found in Birnbaum (2007). For consequences $x_1 > x_2 > x_3$, consider the gamble

$$g = \begin{pmatrix} p & 1-2p & p \\ x_1 & x_2 & x_3 \end{pmatrix},$$

its event split into

$$\overline{g} = \begin{pmatrix} \frac{p}{2} & \frac{p}{2} & 1-2p & p \\ x_1 & x_1 & x_2 & x_3 \end{pmatrix}$$

and also its split into

$$\underline{g} = \begin{pmatrix} p & 1-2p & \frac{p}{2} & \frac{p}{2} \\ x_1 & x_2 & x_3 & x_3 \end{pmatrix}.$$

Birnbaum (2007) reports data (averaged over participants) for which the estimates show $\overline{g} \succ g \succ \underline{g}$. Note that coalescing implies that $g \sim \overline{g} \sim \underline{g}$.

Yet aficionados of CPT ignore this accumulation of negative data and hang onto their beloved representation. Such resistance to empirical reality strikes me as blocking progress. Twenty-five years of effort is, perhaps, enough and new ideas need to be explored.

3. INNOCENT ASSUMPTIONS

We have already discussed one very general example of what many people seem to think is an innocent assumption: A single master state space. What harm can that do? It encompasses everything that can go on. Yet, a small experimental design ends up with a state space in the billions or trillions. But you say: it is standard in foundations of classical probability. Yes, but is that really so innocent? In fact, it becomes very tricky to add new gambles or to subtract old ones. Further, as was pointed out, it makes it very difficult to explore compounding of gambles. I urge theorists to question just how innocent it really is.

A far more limited innocent assumption is *idempotence*, namely, that a gamble in which the same consequence x is attached to the events of a partition is indifferent to itself:

$$\begin{pmatrix} C_1 & C_2 & \ldots & C_i & \ldots & C_n \\ x & x & \ldots & x & \ldots & x \end{pmatrix} \sim x. \tag{2}$$

These are the constant acts of Savage (1954), about which some misgivings exist. In particular, recall the example of air travel where one of the C_i is the event of a serious crash. Suppose that you do not buy travel insurance, and so we may take x to be e, no change from the status quo. Are you really idempotent?

One argument for idempotence that has often been made is that it establishes a firm link between the utility over gambles and that over pure consequences. Yet that link is redundant if one assumes the simple property of *certainty*:

$$\begin{pmatrix} C(n) \\ x \end{pmatrix} \sim x. \tag{3}$$

Most theories during the 50 years after Savage's book either invoke or imply (2). For example, certainty plus coalescing implies idempotence.

During the FUR-XII meeting, I also reported on a beginning effort several of us are making to discover what might happen when idempotence is dropped—a utility of gambling. Both Ramsey (1931) and von Neumann and Morgenstern (1947) remarked on the difficulties of axiomatizing the concept. The first article to explore this with some mathematical rigor, although not axiomatically, was Meginniss (1976). There have been later formal, but non-axiomatic, approaches including a temporal analysis by Pope (1984, 1995). Meginniss served as an impetus for the much more general—uncertain as well as risky gambles—axiomatic articles of Luce et al. (in press a, b) and of Ng et al. (in press a, b). All of these rested on Luce and Marley (2000) who used joint receipt as the tool to separate out the term having to do with gambling per se. The major thrust of the results is that several forms arise that are standard utility expressions to which a utility of gambling term is added. In the case of risk, those terms are, in the simplest case, expected utility plus a constant times the Shannon (1948) entropy and, in a more complex one, a linear form involving p_i^ρ, $\rho \neq 1$, plus what is known as entropy of degree ρ (for a summary of axiomatizations of entropy see Aczél and Daróczy (1975). Ng, Marley, and I are currently working on a modification of the theory that involved a multiplicative, rather than additive, impact of the utility of gambling upon a more usual subjective weighting of the utility of consequences. Further work is to be hoped for because a number of interesting, perplexing open problems remain.

4. CONCLUSIONS

My conclusion is simple: purity, resistance, and innocence may possibly be virtues in some realms, but in my judgment they certainly have not been in studying decision making under uncertainty—utility theory. Of course, I have not proposed exactly how one should violate them in productive ways, but surely it must be carried out with sensitivity and selectivity.

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