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RESEARCH ARTICLE

Utility of gambling I: entropy modified linear weighted utility

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Abstract Behavioral axioms about preference orderings among gambles and their joint receipt lead to numerical representations consisting of a subjective utility term plus a term depending upon the events and the subjective weights. The results here are for uncertain alternatives, in much the same sense as Savage's usage. Several open problems are described. Results for the risky case are in a second article.

Keywords Duplex decomposition · Entropy · Functional equations · Joint receipt · Linear weighted utility · Segregation · Utility of gambling

JEL Classification Numbers C91 · D46 · D81

Early in the history of "modern" utility theory, the utility of gambling was recognized to consist of one or more phenomena that were difficult to handle theoretically. For example Ramsey (1931), p. 172, remarked that the method of establishing beliefs in terms of bets is "...inexact...partly because the person may have a special eagerness or reluctance to bet, because he either enjoys or dislikes excitement...The difficulty is like that of separating two different co-operating forces." This 1931 essay was actually dated 1926. Over two decades later von Neumann and Morgenstern (1947), p. 28,

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remarked: "Since (our axioms) secure that the necessary construction can be carried out, concepts like a 'specific utility of gambling' cannot be formulated free of contradiction on this level." Adjoined is the footnote: "This may seem to be a paradoxical assertion. But anybody who has seriously tried to axiomatize that elusive concept, will probably concur with it." More recently, Le Menestrel (2001), p. 249, argues that "...any specific utility or disutility for gambling must be excluded from expected utility because such a theory is consequential while a pleasure or displeasure for gambling is a matter of process, not of consequences." One can view the present article as an attempt to provide such a process model.

This issue has remained largely a backwater with most theories of utility ruling it out by incorporating, in one way or another, some version of idempotence: a gamble that attaches the same consequence x to each of the chance events constituting an event partition of a chance "experiment" is perceived as indifferent to receiving x with certainty. That indifference means that no utility or disutility accrues either to the events themselves or to the execution of the experiment, as such.

Some discussion of the utility of gambling has appeared in the utility literature and some ad hoc ideas have been presented that in many cases focused on the risky cases.[1] Typically these involved special modifications of the expected utility representation. Summaries of references can be found in Conlisk (1993) from an economic perspective and in Luce and Marley (2000) from a more, though not entirely, psychological perspective. Pope (1985) and (1995) takes a quite different perspective. She treats the issue from a temporal perspective about when uncertainty is resolved, and she postulates a general decomposition somewhat formally similar to (15) below. Two recent contributions are Diecidue et al. (2004) and Yang and Qiu (2005). In line with statements in Conlisk (1993), all these contributions tend not to be axiomatic, not to be mathematically very general, and not to apply to uncertain[2] alternatives. To the best of our knowledge, Meginniss (1976) was the first author to arrive at a sensible theory incorporating what amounts to a concept of a utility of gambling for risky gambles. We take that up more in Ng et al. (Ng, C.T., Luce, R.D., Marley, A.A.J.: Utility of gambling: extending the approach of Meginniss (1976), submitted) where we list other relevant references.

A second vexing problem, also recognized early on, is another usual assumption of most theories that the chance events are themselves neutral and without any inherent value, as may be postulated, for example, regarding the draw of a colored ball from an urn. However, this is not true for many important real-world situations such as travel by airplane in which some of the chance events, such as the trip being terminated in a crash, are themselves of (negative) value. Such a value is independent of any bet—e.g., insurance on the flight—that is placed on the trip. Or a more innocent example is the common decision about whether or not to carry an umbrella. Here, in addition to the umbrella gamble, is the fact that the event of rain may itself be negative (or often positive for a farmer) independent of whether one elects to carry the umbrella or not. Keep in mind that such "consequences" are not arbitrarily assigned to the events; they

[1] Those for which the possible events are simply replaced by probabilities, which is often described as there being a probability distribution associated with the possible consequences of the gamble.

[2] Those where the events have no readily agreed upon probabilities.

are inherent. We do not encompass this phenomenon within our present framework, but Ng et al. (2007) develop a possible approach to it that draws upon the present results.

The representations of this article are additive over joint receipt. Our axiomatization yields a 2×2 classification. The rows are whether or not the subjective weights are finitely additive (FA) and the columns are whether a distribution property, called segregation, (37), or a non-rational decomposition of binary gambles into simpler binary ones, called duplex decomposition, (57), is satisfied. The case of non-FA and segregation yields rank-dependent utility (see Luce 2000) with no utility of gambling (UofG) except for a constant when the local state space is maximal. The other three cases yield linear weighted utility (LWU), which is SEU when the weights are FA, plus a common general UofG expression, (49). These representations, as given by Theorems 13 and 16, are summarized in Table 1 of Sect. 7.

As noted earlier, a second article, Luce et al. (2007), uses these results to find results in the case of pure risk.

1 Basic notation, definitions, and assumptions

Let X denote a set of *pure consequences*—ones for which chance or uncertainty plays no role. The adjective "pure" is used to distinguish such consequences from those that are, in fact, gambles; see below. Let X include a distinguished element e that is interpreted as representing *no change from the status quo*.

Let $\mathfrak{B}$ be a set of events (sets) containing, with any two events, also their union and difference, thus also their intersection and the empty element $\varnothing$. In mathematical terms we say that $\mathfrak{B}$ is a Boolean ring. Let $\mathfrak{B}^* = \mathfrak{B} \backslash \{\varnothing\}$.

Let $\mathbf{E}$ be a finite "experiment" or "chance phenomenon". Let $(C_1, \ldots, C_n)$, $C_i \in \mathfrak{B}$, $C_i \cap C_j = \varnothing$ if $i \neq j$, denote the partition defining $\mathbf{E}$. The "universal" event for the experiment $\mathbf{E}$ is the union $\Omega = \bigcup_{i=1}^{n} C_i$. We place quotes around "universal" because the event Ω is only universal for the purpose of a particular local chance experiment $\mathbf{E}$, not in the global sense of a state space Savage (1954). Thus, we admit the possibility of many experiments each having its own Ω.

A *first-order gamble* consists of n, $1 \leq n < \infty$, (pure consequence, event) pairs, each of which is called a *branch* of the first-order gamble. Thus, replacing the parentheses of the branches by semicolons, we may write such a gamble as

$$g_{[n]} = (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n). \tag{1}$$

Whenever we use Ω along with $g_{[n]}$, it is implicit that $\Omega = \cup_{i=1}^{n} C_i$ is the universal event underlying the gamble. We sometimes suppress the subscript and write g in place of $g_{[n]}$.

A *second-order gamble* consists of (consequence, event) pairs, also called *branches*, where by a "consequence" we mean either a pure one or a first-order gamble, and at least one consequence is a first-order gamble. From here on, a *gamble* means either a pure consequence, a first-order gamble, or a second-order gamble, and the unmodified term *consequence* means either a pure consequence or a first-order gamble in a second-order gamble.

In addition we assume that a person can have the joint receipt of pure consequences and gambles, which operation is denoted by $\oplus$. We assume that $\oplus$ is commutative and associative, and that no change from the status quo, e, is an identity of $\oplus$. We assume that for any $x, y \in X$, $x \oplus y \in X$, i.e., X is closed under $\oplus$.

Let $\mathcal{G}$ denote the closure of all gambles under $\oplus$.

Assume that a decision maker has a *preference order* $\succsim_{\mathcal{G}}$ over $\mathcal{G}$, which we usually denote just as $\succsim$, and that it is a weak order. As usual, $\precsim$ denotes the converse of $\succsim$ and $\sim$ denotes the corresponding indifference relation: $\sim := \succsim \cap \precsim$ and $\succ := \succsim \setminus \sim$.

The gambles $g \succsim e$ are called *gains*, and those with $g \precsim e$ are *losses*.

We will also assume that joint receipt $\oplus$ is *monotonic* in the sense that, for all gambles $f, f', g \in \mathcal{G}$,

$$f \succsim f' \Leftrightarrow f \oplus g \succsim f' \oplus g. \tag{2}$$

This was empirically explored in Cho and Fisher (2000) and was not rejected.

We assume that for each gamble $g \in \mathcal{G}$, the set X is sufficiently rich that it contains an element, denoted $CE(g)$, such that $CE(g) \sim g$. This pure consequence is called a *certainty equivalent* of the gamble.

We also assume the property of *certainty*:

$$(x_1, \varnothing; \ldots; x_{i-1}, \varnothing; x_i, \Omega; x_{i+1}, \varnothing; \ldots; x_n, \varnothing) \sim x_i; \tag{3}$$

and the property of *expansibility*:

$$\begin{aligned}(x_1, C_1; \ldots; x_{i-1}, C_{i-1}; x_i, \varnothing; x_{i+1}, C_{i+1}; \ldots; x_n, C_n) \\ \sim (x_1, C_1; \ldots; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \ldots; x_n, C_n).\end{aligned} \tag{4}$$

A gamble (1) is *ranked* when the consequences are numbered in order of preference, i.e.,

$$x_1 \succsim x_2 \succsim \ldots \succsim x_n \tag{5}$$

and the corresponding event partition is treated as an ordered n-tuple with that induced order of indices.

We assume that any permutation of the branches yields an indifferent gamble, i.e., for any permutation π,

$$(x_1, C_1; x_2, C_2; \ldots; x_n, C_n) \sim (x_{\pi(1)}, C_{\pi(1)}; x_{\pi(2)}, C_{\pi(2)}; \ldots; x_{\pi(n)}, C_{\pi(n)}). \tag{6}$$

Ranking is essential in formulating some properties, such as consequence co-monotonicity below. When we assume the ranked form, we explicitly say so. Otherwise, we do not assume that the consequences are ranked.

2 Two aspects of numerical representations

2.1 Order preservation

We call a mapping $U : \mathcal{G} \xrightarrow{onto}]\alpha, \beta[\subseteq \mathbb{R} :=]-\infty, \infty[$ a (utility) *representation* of gambles if it is order preserving, i.e., if for $f, g \in \mathcal{G}$

$$f \succsim g \text{ iff } U(f) \geq U(g), \tag{7}$$

and if

$$U(e) = 0. \tag{8}$$

2.2 Additivity of joint receipt

A representation is *additive* over $\oplus$ if for $f, g \in \mathcal{G}$,

$$U(f \oplus g) = U(f) + U(g). \tag{9}$$

In this article, we focus on additive representations. One set of assumptions that gives rise to an additive utility representation is that $\langle \mathcal{G}, \oplus, \succsim, e \rangle$ satisfies the usual axioms for extensive measurement, e.g., Krantz et al. (1971), Chaps. 3, 6.

Because $f \oplus e \sim f$, (8) follows from (9). So, $0 \in]\alpha, \beta[$. Because X is closed under $\oplus$, for any positive integer n we may define nx in the usual fashion: $1x := x$ and for $n > 1$, $nx := (n-1)x \oplus x$. Thus, if U is additive over $\oplus$, $U(nx) = nU(x)$, and so U is unbounded. This proves, for U additive over $\oplus$, that $]\alpha, \beta[= \mathbb{R}$. Together with (7), it implies the sometimes assumed structural property of solvability, i.e., for $x \succsim y$, there exists z such that $x \sim z \oplus y$. Such z is unique up to order indifference and will be denoted as $x \ominus y$. In short,

$$x \ominus y \sim z \Leftrightarrow x \sim z \oplus y. \tag{10}$$

2.3 Additivity and St Petersburg Paradox

Let $M \subseteq X$ denote the set of possible money amounts. Let $\succsim_M$ and U_M denote the specializations of $\succsim_X$ and U to M, respectively. Assume that $0, 1 \in M$ and

$$x \succsim_M y \Longleftrightarrow x \geq y \quad (x, y \in M), \tag{11}$$

$$x \oplus y = x + y \quad (x, y \in M). \tag{12}$$

So from (9) and (12),

$$U_M(x + y) = U_M(x \oplus y) = U_M(x) + U_M(y) \quad (x, y \in M). \tag{13}$$

Assuming countable density of M, then the fact that U_M is order preserving and that the Cauchy equation (13) is satisfied implies

$$U_M(x) = kx \quad (k := U_M(1) > 0), \tag{14}$$

i.e., the utility is proportional to money.

Economists usually consider (14) undesirable mainly because of Bernoulli's St. Petersburg Paradox in which a decision maker is faced with a lottery in which a coin is flipped until a head appears. If that is on the nth flip, the payoff is \$$2^n$, which event occurs with probability $1/2^n$, and so the expected value of the lottery is ∞. Such a lottery is totally unreal, first, because no vendor of lotteries would consider selling one that has an expected payout of ∞, which means that on any realization there is some small chance that bankruptcy will occur, and, second, because no gambler when deciding what price to buy the lottery would ignore the fact of possible bankruptcy and non-payment. These considerations seem to us to override the usual non-linear utility explanation.

We defer to Luce et al. (2007) the explanation of various paradoxes using representations for risky gambles related to those developed here for uncertain gambles, which explanations are compatible with utility being of the form (14).

3 Assumptions about gambles and their joint receipt

3.1 Kernel equivalents and elements of chance

Following Luce and Marley (2000), any gamble for which every consequence is no change from the status quo, e, i.e., gambles of the form $(e, C_1; e, C_2; \ldots; e, C_n)$, is called an *element of chance.* For any gamble $g_{[n]} = (g_1, C_1; g_2, C_2; \ldots; g_n, C_n)$, its *kernel equivalent,* denoted $KE(g_{[n]})$, is defined to be the pure consequence solution, which we assume exists, to the following indifference

$$g_{[n]} \sim KE(g_{[n]}) \oplus (e, C_1; e, C_2; \ldots; e, C_n). \tag{15}$$

Note that, because $KE(g_{[n]})$ is a pure consequence, the right hand expression involves only one realization of the experiment.

If U is an additive representation, then applying U to (15) yields

$$U(g_{[n]}) = U(KE(g_{[n]})) + U(e, C_1; e, C_2; \ldots; e, C_n). \tag{16}$$

The utility of an element of chance is a possible measure of the utility of gambling. Our purpose here is to discover something about its mathematical form.

Definition 1 The **kernel equivalents are idempotent** if for any gamble $g_{[n]}(x)$ all of whose consequences are x,

$$KE(g_{[n]}(x)) \sim x. \tag{17}$$

The **elements of chance are e-idempotent** if

$$e \sim (e, C_1; e, C_2; \ldots; e, C_n). \tag{18}$$

The former is referred to as *KE-idempotence,* which sometimes arises naturally as, e.g., in Proposition 10.

Traditional theories of utility typically assume or prove $e-$idempotence from other assumptions, in which case KE is the certainty equivalent. We explicitly do *not* assume idempotence in this article. To some this seems counter intuitive, but, for example, in a drought region many people will not satisfy

$$e \sim (e, rain; e, \neg rain).$$

Suppose that C_i, $i = 1, \ldots, n$, form a partition of a universal event Ω and that C_i' is the same partition but arising from an independent realization of the underlying experiment.[3] We assume that

$$(e, C_1; e, C_2; \ldots; e, C_n) \sim (e, C_1'; e, C_2'; \ldots; e, C_n'). \tag{19}$$

Although, Luce and Marley (2000), derived a number of properties about such a decomposition into KEs and elements of chance, they had no principled way of getting results about the utility of elements of chance. This article offers one remedy for that incompleteness.

3.2 Co-monotonicity

We assume that the following property is satisfied:

Definition 2 Let

$$g_{[n]} = (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n),$$
$$g_{[n]}' = (x_1, C_1; \ldots; x_i', C_i; \ldots; x_n, C_n),$$

be ranked gambles in which all $C_i \neq \varnothing$. **Consequence co-monotonicity** holds if

$$x_i \succsim x_i' \Leftrightarrow g_{[n]} \succsim g_{[n]}'. \tag{20}$$

In the above definition, the x_i may be first-order gambles as well as pure consequences. Below, we mostly omit the adjective "consequence" before co-monotonicity.

Note that both monotonicity of $\oplus$, (2), and co-monotonicity of gambles, as defined here, imply the strict ordering case in which $\succsim$ is replaced by $\succ$.

[3] In some important uncertain situations, such as a job interview, independent realizations cannot be effected. The theory does not apply to such cases unless, of course, hypothetical realizations are acceptable in eliciting data.

By the existence of certainty equivalents and co-monotonicity of binary gambles, a binary second-order gamble, sometimes called a compound gamble, can be replaced by a binary first-order gamble of pure consequences as follows: For any gambles f, g, we have $f \sim CE(f)$ and $g \sim CE(g)$, and by co-monotonicity and expansibility, (4),

$$(f, C; g, \Omega \backslash C) \sim (CE(f), C; CE(g), \Omega \backslash C).$$

Thus, if a condition, such as simple joint receipt decomposability, Def. 3 below, holds for binary gambles of pure consequences, then provided that the above conditions and the monotonicity of joint receipt, Def. 2, hold, the condition will also hold for binary compound gambles. We use this fact extensively in the proofs without comment.

3.3 Separable representations of unitary gambles

We make the following assumption about branches: for $\Omega \in \mathcal{B}^*$, $C, D \in \mathcal{B}$ with $C, D \subseteq \Omega$, and $x \succ e$, $y \succ e$,

$$\begin{aligned} &KE(x, C; e, \Omega \backslash C) \succsim KE(x, D; e, \Omega \backslash D) \\ &\quad \Leftrightarrow KE(y, C; e, \Omega \backslash C) \succsim KE(y, D; e, \Omega \backslash D). \end{aligned} \tag{21}$$

In terms of this we define an ordering of branches, $\succsim_\Omega$, by: for any $x \succ e$, and so for all,

$$(x, C) \succsim_\Omega (x, D) \Leftrightarrow KE(x, C; e, \Omega \backslash C) \succsim KE(x, D; e, \Omega \backslash D),$$

and in an abuse of notation we also write

$$C \succsim_\Omega D \Leftrightarrow KE(x, C; e, \Omega \backslash C) \succsim KE(x, D; e, \Omega \backslash D). \tag{22}$$

Note that because $\succsim$ is a weak order, so is $\succsim_\Omega$. For losses, $x \prec e$,

$$C \succsim_\Omega D \Leftrightarrow KE(x, C; e, \Omega \backslash C) \precsim KE(x, D; e, \Omega \backslash D).$$

A binary gamble with one of the consequences equal to e, e.g., $(x, C; e, \Omega \backslash C)$ is called *unitary*. Assume properties for the kernel equivalent $KE(x, C; e, \Omega \backslash C)$ that are necessary and sufficient for the existence of a family of order preserving functions $W_\Omega : \{C \,|C \in \mathcal{B}, C \subseteq \Omega\} \overset{onto}{\longrightarrow} [0, 1]$, $(\Omega \in \mathcal{B}^*)$, and a representation $U^* : X \longrightarrow \mathbb{R}$, such that, for all unitary binary gambles,

$$U^*(KE(x, C; e, \Omega \backslash C)) = U^*(x) W_\Omega(C). \tag{23}$$

Such a multiplicative conjoint representation is called *separable*. The major properties needed to construct a separable representation are monotonicity in the consequence x, which, in this context, is equivalent to co-monotonicity, Def. 2, an appropriate form of branch monotonicity, e.g., (22), the Thomsen condition[4] for the branch (x, C), an Archimedean property,[5] and a form of solvability,[6] (Krantz et al. 1971), Chap. 6.

3.4 Simple joint-receipt decomposability

The following property modifies a definition of Luce (1996) to the non-idempotent case; see also Luce (2000), p. 153:

Definition 3 Simple[7] joint-receipt (JR-)decomposability is satisfied if for all disjoint events C, D with independent realizations C', D', and for all gambles f and g,

$$(f \oplus g, C; e, D) \oplus (e, C'; e, D') \sim (f, C; e, D) \oplus (g, C'; e, D'). \tag{24}$$

Four observations:

- Because, on the right, C', D' are independent realizations of C, D this is not a "rational" accounting principle. But keep in mind that we have assumed, (19), that $(e, C'; e, D') \sim (e, C; e, D)$.
- Using (15) and (24),

$$\begin{aligned}&[KE(f \oplus g, C; e, D) \oplus (e, C; e, D)] \oplus (e, C'; e, D')\\ &\quad \sim [KE(f, C; e, D) \oplus (e, C; e, D)] \oplus \left[KE(g, C'; e, D') \oplus (e, C'; e, D')\right].\end{aligned}$$

  As $\oplus$ is commutative, associative, and monotonic, it follows that

$$KE(f \oplus g, C; e, D) \sim KE(f, C; e, D) \oplus KE(g, C'; e, D'), \tag{25}$$

  i.e., KEs satisfy ordinary simple joint-receipt decomposability Luce (1996).
- We are not aware of any empirical tests of either (24) or (25).

3.5 Relating U and U^*

Proposition 4 *Suppose that U is additive over $\oplus$, (9), that $U^* W_{C \cup D}$ is a separable representation, (23), of the kernel equivalents of the family of unitary gambles*

[4] Suppose P and Q are sets and $\succsim$ is a weak order over $P \times Q$. The Thomsen condition asserts for all $p_1, p_2, p_3 \in P, q_1, q_2, q_3 \in Q$, $[(p_1, q_2) \sim (p_2, q_3)]$ & $[(p_2, q_1) \sim (p_3, q_2)] \Longrightarrow (p_1, q_1) \sim (p_3, q_3)$. In the current application, $P = X$, $Q = \{C \mid C \in \mathcal{B}, C \subseteq \Omega\}$.

[5] This is the assertion that every bounded standard sequence is finite, where a left standard sequence is defined recursively by $(p_n, q_1) \sim (p_{n+1}, q_2)$, for any p_0 and for any fixed pair $q_1, q_2 \in Q$ such that $(p_0, q_1) \succ (p_0, q_2)$ for any x. A right standard sequence is defined similarly. A standard sequence is either a left or right standard one.

[6] If $(\overline{p}, q_1) \succsim (p_2, q_2) \succsim (\underline{p}, q_1)$, then there exists p_1 such that $(p_1, q_1) \sim (p_2, q_2)$, and similarly for the other component.

[7] There is a generalization which is useful in the case of p-additive joint receipt; see Luce (2000).

based on the event $C \cup D$, and that the kernel equivalents of gambles satisfy simple joint-receipt decomposability, (25). Then there exists $\kappa > 0$ such that

$$U(KE(x, C; e, D)) = U(x)W_{C\cup D}(C)^{\kappa} = U(x)S_{C\cup D}(C), \tag{26}$$

where

$$S_{C\cup D}(C) := W_{C\cup D}(C)^{\kappa}. \tag{27}$$

All proofs of propositions, lemmas and theorems are given in the Appendix.

Three observations:

- The pair (U^*, W) was introduced in Sect. 3.3 to serve as a separable representation, (23). In such separable representations, W is only unique up to a positive power. By (26) we see that $(U, W^{\kappa}) = (U, S_{\Omega})$, with U additive, can be, and usually will be, used to retire the role of (U^*, W).
- By certainty, (3), $e \sim (x, \varnothing; e, \Omega)$ and $x \sim (x, \Omega; e, \varnothing)$, so from (26),

$$U(KE(x, \varnothing; e, \Omega)) = U(e) = 0 = U(x)S_{\Omega}(\varnothing),$$

  and

$$U(KE(x, \Omega; e, \varnothing)) = U(x) = U(x)S_{\Omega}(\Omega).$$

  Hence $S_{\Omega}(\varnothing) = 0$ and $S_{\Omega}(\Omega) = 1$. For $C \neq \varnothing$, (26) and co-monotonicity imply that $S_{\Omega}(C) > 0$.
- This shows that

$$S_{\Omega}(\varnothing) = 0, \quad S_{\Omega}(\Omega) = 1, \quad S_{\Omega}(C) > 0 \text{ if } C \neq \varnothing. \tag{28}$$

- The assumption that W_{Ω} is a mapping of $\{C \mid C \in \mathfrak{B}, C \subseteq \Omega\}$ onto $[0, 1]$ implies that the ring $\mathfrak{B}$ is rich. In particular it implies that for each $\Omega \in \mathfrak{B}^*$ and any size n, there exists a partition $(C_1, \ldots, C_n)$ of Ω with $C_i \in \mathfrak{B}^*$. Such a strong assumption seems needed for Proposition 4. The examples typically do not exhibit such richness, and so must implicitly be imbedded in a richer event space than is often made explicit.

3.6 The choice property

The following property, first formulated in Luce (1959/2005) and here derived from other assumptions, plays an important role in this article.

Definition 5 The family of weighting functions S_{Ω}, $\Omega \in \mathfrak{B}^*$, satisfies the **choice property** [8] if for all C, D, E *in* $\mathfrak{B}^*$ *such that* $C \subseteq D \subseteq E$,

$$S_E(C) = S_D(C)S_E(D). \tag{29}$$

[8] Luce (1959/2005) in the context of probabilities called this the choice axiom which is to be carefully distinguished from the mathematical axiom of choice. The latter is essential in getting (30) and (49).

Theorem 6 *Suppose that S_Ω satisfies the choice property. Then there exists an increasing function μ on $\mathfrak{B}$, positive on $\mathfrak{B}^*$ and $\mu(\varnothing) = 0$, such that the representation*

$$S_\Omega(C) = \mu(C)/\mu(\Omega) \tag{30}$$

holds for all $C, \Omega \in \mathfrak{B}^*$, $C \subseteq \Omega$.

In the case of *finitely additive weights*, i.e., for any $\Omega \in \mathfrak{B}^*$ and any disjoint $C, D \subseteq \Omega$,

$$S_\Omega(C \cup D) = S_\Omega(C) + S_\Omega(D), \tag{31}$$

the choice property representation (30) reduces to that stated in Luce (1959/2005). In the case of probabilities, this representation came to be known as the Bradley-Terry-Luce (BTL) representation.

3.7 Gamble recursions

3.7.1 Gamble decomposition

The following concept was first emphasized by Liu (1995) and used by Luce (2000) and by Luce and Marley (2005). In these earlier publications the property was invoked for gains only or for losses only. Here we accept it for general gambles and change the name from "gains decomposition" to "gamble decomposition." Suppose that $g_{[n]}$, $n \geq 2$, is a ranked gamble, i.e., with $x_1 \succsim x_2 \succsim \cdots \succsim x_n$, $C_i \in \mathfrak{B}^*$,

$$g_{[n]} := (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n). \tag{32}$$

Consider the following ranked subgamble of it that omits the branch (x_i, C_i):

$$g_{[n],-i} := (x_1, C_1; \ldots; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \ldots; x_n, C_n). \tag{33}$$

Definition 7 **Upper gamble decomposition (UGD)** holds if

$$g_{[n]} \sim \left(x_1, C_1; g_{[n],-1}, \cup_{j \neq 1} C_j\right), \tag{34}$$

and **lower gambles decomposition (LGD)** holds if

$$g_{[n]} \sim \left(g_{[n],-n}, \cup_{j \neq n} C_j; x_n, C_n\right), \tag{35}$$

where $\left(x_1, C_1; g_{[n],-1}, \cup_{j \neq 1} C_j\right)$ and $\left(g_{[n],-n}, \cup_{j \neq n} C_j; x_n, C_n\right)$ are compound binary gambles.

Note that the assumption here does not presume the ordering between x_1 and $g_{[n],-1}$ nor between x_n and $g_{[n],-n}$. This necessitates some delicacy in the proofs of the following results.

Comment: To realize such properties in the laboratory, one could proceed, for example, in the following fashion. The chance experiment underlying $g_{[n]}$ may be based on random draws from an opaque urn containing balls of n different colors, with incomplete information about the numbers of balls of each color. Each x_i is associated with a different color, which color in turn composes the event C_i. The chance experiment that involves the binary partition $\{C_i; \cup_{j \neq i} C_j\}$ would consist of an urn with the same number of the color C_i and with the balance of the balls all of single color different from that used with the general gamble. And the chance experiment underlying $g_{[n],-i}$ would be the original urn but with all the balls of color C_i simply removed. In the last two cases, the attachment of consequences is obvious.

3.7.2 Branching

We also introduce a condition that is, in some ways, very closely related to lower gamble decomposition. To that end, let

$$g_{[2]} := (x_1, C_1; x_2, C_2).$$

Definition 8 Branching holds if, for every ranked gamble $g_{[n]}$, (32),

$$\begin{aligned} g_{[n]} &\sim (g_{[2]}, C_1 \cup C_2; x_3, C_3; \ldots; x_n, C_n) \\ &\sim ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; \ldots; x_n, C_n). \end{aligned} \tag{36}$$

Here instead of separating the gamble into a single branch (x_i, C_i) and the remaining subgamble of size $n-1$, the separation is into a binary subgamble and the remaining $n-2$ branches of the original gamble. We do not presume the ordering satisfies $(x_1, C_1; x_2, C_2) \succsim x_3$, which, again, necessitates some delicacy in the proofs of the following results. In the case of $e-$idempotent gambles, it is easy to see from comonotonicity that the ordering is always satisfied, but with non-idempotent gambles, the element of chance term can cause a reversal of the order. Branching is another example of an indifference where each x_i arises under the same conditions provided we assume that for $(x_1, C_1; x_2, C_2)$ the conditioning on $C_1 \cup C_2$ means that x_1 arises when C_1 occurs and x_2 when C_2 occurs, where one or the other must occur. Sometimes such indifferences are called accounting ones because the “bottom line” for each side is the same.

In terms of the urn example, the colors for 1 and 2 are not distinct for the event $C_1 \cup C_2$ whereas they are for $g_{[2]}$.

Note that for $n = 3$, the branching property is identical to lower gamble decomposition, LGD.

4 Representations of general gambles under segregation

4.1 Segregation

To this point, we know the representation of the binary unitary gambles, (26), but not that of general binary gambles. To get that, we need a further assumption. One possibility is dealt with in this section and another in Sect. 5.1. Both have been investigated empirically and some people satisfy one and some the other (Cho et al. 2002).

Recall the definition of subtraction, (10). Note that from this and the additivity of U over $\oplus$, (9), it follows immediately that

$$U(x \ominus y) = U(x) - U(y).$$

Definition 9 Gambles are said to satisfy **segregation** if for every pair of consequences $x \succsim y$, and every event partition (C, D) with an independent realization (C', D'),

$$(x, C; y, D) \sim (x \ominus y, C'; e, D') \oplus y. \tag{37}$$

Segregation, which is a highly rational property in the sense that the "bottom lines" on the two sides of the indifference are identical, is the natural generalization of the concept originally defined for idempotent lotteries in Kahneman and Tversky (1979). Mathematically, segregation is a type of distributivity.

Note also that if segregation holds for binary gambles, then it holds for their kernel equivalents, i.e.,

$$KE(x, C; y, D) \sim KE(x \ominus y, C'; e, D') \oplus y. \tag{38}$$

Empirical objections to segregation have been expressed by Birnbaum (1997) and discussed in the context of generalizations to rank-dependent theories by Luce (2003). The primary issue is the apparent existence of violations of binary co-monotonicity when $y = e$ and $\Pr(C\,|C \cup D)$ is large, over 0.85. A blending of the current ideas with those of Luce (2003) would be interesting.

4.2 The representation of binary gambles under segregation

The following result is very closely related to Theorem 4.4.6 of Luce (2000) coupled with segregation. The major difference is that the result there is stated for a somewhat more general and complicated case than the additive one.

With S_Ω defined by (27) and US_Ω denoting the product of U and S_Ω, the following is a key result for the case of segregation:

Proposition 10 *Suppose that a gamble can be decomposed into the joint receipt of a kernel equivalent and an element of chance, (15), that U is additive over $\oplus$, (9), that the kernel equivalents of unitary gambles have the separable representation US_Ω, (26), and that gambles satisfy segregation, (37), and so their KE also satisfy segregation. Then for f and g, with $f \succsim g$, that are first-order gambles or pure consequences*

$$U[KE(f, C; g, D)] = U(f)S_{C\cup D}(C) + U(g)\left[1 - S_{C\cup D}(C)\right]. \tag{39}$$

Thus, the general representation of the binary gambles is

$$U(f, C; g, D) = U(f)S_{C\cup D}(C) + U(g)\,[1 - S_{C\cup D}(C)] \\ + U(e, C; e, D). \tag{40}$$

Because the representations (39) and (40) are ranked, we have to be careful in later proofs where gamble decomposition and branching are involved. However, if S_Ω is finitely additive, and so in particular $S_{C\cup D}(C) + S_{C\cup D}(D) = S_\Omega(\Omega) = 1$, then $1 - S_{C\cup D}(C) = S_{C\cup D}(D)$ and the results are invariant under the permutation of f and g – i.e., they hold in the unranked case.

4.3 The two recursions under segregation

We next arrive at recursions corresponding to UGD and branching. Toward that end, we simplify notation by defining

$$H_n(C_1, \ldots, C_n) := U(e, C_1; \ldots; e, C_n). \tag{41}$$

Note that, by (6), H_n is symmetric with respect to the C_i, and by certainty, (3),

$$H_1(C) = U(e, C) = U(e) = 0. \tag{42}$$

Expansibility, (4), then yields that for each $n > 1$,

$$H_n(\varnothing, C_2, \ldots, C_n) = H_{n-1}(C_2, \ldots, C_n). \tag{43}$$

Proposition 11 *Suppose that the several assumptions of Proposition 10 are satisfied: decomposability into a KE and an element of chance, (15), U is additive over joint receipt $\oplus$, (9), the kernel equivalents of unitary gambles based on Ω have the separable representation US_Ω, (26), and segregation, (37). Assume that, for $n > 2$, upper gamble decomposition, Definition 7, and branching, (36), both hold. Then:*

1. *S_Ω satisfies the choice property, (29)**,** with the representation μ of (30) satisfying the following property: there exists a dimensional constant Δ such that for all disjoint events $C, D \subseteq \Omega$,*

$$\mu(C \cup D) = \mu(C) + \mu(D) + \Delta\mu(C)\mu(D) \tag{44}$$
$$S_\Omega(C \cup D) = S_\Omega(C) + S_\Omega(D) + \Delta\mu(\Omega)S_\Omega(C)S_\Omega(D). \tag{45}$$

2. *H_n satisfies the two recursions*

$$H_n(C_1, \ldots, C_n) = H_{n-1}(C_2, \ldots, C_n)[1 - S_\Omega(C_1)] \\ + H_2(C_1, \Omega\backslash C_1), \tag{46}$$

and

$$H_n(C_1, \ldots, C_n) = H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) + H_2(C_1, C_2) S_\Omega(C_1 \cup C_2). \tag{47}$$

The representation of μ in (44) is said to be *p-additive*. If Δ is 0 then, of course, μ satisfies finite additivity in (44) as does S_Ω in (45). The distinction between the additive and non-trivially p-additive cases, i.e., (45) with $\Delta \neq 0$, will prove to be important.

4.4 Elements of chance under segregation

Lemma 12 *Suppose that μ is a function such that the weighting function S_Ω has the representation (30).*

1. *Suppose that H_n satisfies the branching equation (47), i.e.,*

$$H_n(C_1, C_2, C_3, \ldots, C_n) = H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) + H_2(C_1, C_2) \left(\frac{\mu(C_1 \cup C_2)}{\mu(\Omega)} \right). \tag{48}$$

Then there exists a function $h : \mathfrak{B} \to \mathbb{R}$, with $h(\varnothing) = 0$ and with the dimension of μU such that

$$H_n(C_1, C_2, \ldots, C_n) = \frac{1}{\mu(\Omega)} \left[h(\Omega) - \sum_{i=1}^{n} h(C_i) \right]. \tag{49}$$

2. *Suppose, in addition, that H_n satisfies the upper gamble decomposition equation (46) that arises under segregation, i.e.,*

$$H_n(C_1, C_2, \ldots, C_n) = H_{n-1}(C_2, \ldots, C_n) \left(1 - \frac{\mu(C_1)}{\mu(\Omega)} \right) + H_2(C_1, \Omega \backslash C_1). \tag{50}$$

Then, either
(a) $\Delta = 0$ *and therefore μ and S_Ω are finitely additive, or*
(b) $\Delta \neq 0$ *and so μ and S_Ω are non-trivially p-additive and there exists a constant A such that*

$$H_n(C_1, C_2, \ldots, C_n) = \begin{cases} 0, & \text{if } \Omega \text{ is not maximal} \\ A, & \text{if } \Omega \text{ is maximal} \end{cases},$$

where Ω being maximal means there does not exist $\Omega' \in \mathfrak{B}^*$ such that $\Omega' \supseteq \Omega$ and $\Omega' \neq \Omega$.

In Luce et al. (2007), where we add assumptions to deal with risky gambles, (49) is restricted to what are known as entropies of degree α, where $\alpha = 1$ is the well-known Shannon entropy. For this reason, we add *entropy-modified* to the names of the several utility expressions for kernel equivalents that are derived in Theorems 13 and 16.

In the following Theorem 13, the case corresponding to Part 2(a), above, where S_Ω is finitely additive, will be shown to involve the standard subjective expected utility (SEU) representation, i.e.,

$$SEU(KE(g_{[n]})) := \sum_{i=1}^{n} U(x_i) S_\Omega(C_i), \quad \left(\sum_{i=1}^{n} S_\Omega(C_i) = 1 \right). \tag{51}$$

And the cases corresponding to Part 2(b) will be shown to have the representation

$$RDU(g_{[n]}) + \begin{cases} 0, & \text{if } C(n) \text{ is not maximal} \\ A, & \text{if } C(n) \text{ is maximal} \end{cases}, \tag{52}$$

where

$$\begin{array}{ll} C(i) = \bigcup_{j=1}^{i} C_j, & (i \geq 1) \\ C(0) = \varnothing, & (i = 0) \end{array},$$

and

$$RDU(g_{[n]}) = \sum_{i=1}^{n} U(x_i) S_\Omega(C_i) \left[1 + \Delta\mu(\Omega) S_\Omega(C(i-1)) \right], \tag{53}$$

with S_Ω satisfying (30). We will call (53) an RDU representation, or simply RDU. Note that if $\mathfrak{B}$ is a ring with no maximal event, the elements of chance are $e-$ idempotent.

4.5 The main representation under segregation

Theorem 13 *Suppose that $n \geq 2$ and that the several assumptions of Proposition 10 are satisfied: decomposability into a KE and an element of chance, (15), U is additive over joint receipt $\oplus$, (9), the kernel equivalents of unitary gambles have the separable representation US_Ω, (26), and segregation, (37). Suppose that both upper gamble decomposition, Definition 7, and branching, Definition 8, are satisfied. Then there exists an increasing function μ, positive on $\mathfrak{B}^*$, $\mu(\varnothing) = 0$, such that*

$$S_\Omega(C) = \mu(C)/\mu(\Omega),$$

and, for all gambles $g_{[n]} \in \mathcal{G}$, either

(i) S_Ω, *and thus* μ, *are finitely additive and there is a function* $h : \mathfrak{B} \to \mathbb{R}$, $h(\varnothing) = 0$, *with the dimension of* μU, *for which*

$$U(g_{[n]}) = SEU(g_{[n]}) + \frac{1}{\mu(\Omega)}\left[h(\Omega) - \sum_{i=1}^{n} h(C_i)\right], \tag{54}$$

where $SEU(g_{[n]})$ *is given by (51), or*

(ii) S_Ω *is p-additive (with* $\Delta \neq 0$*) and there is a constant* A *with the dimension of* U *such that*

$$U(g_{[n]}) = RDU(g_{[n]}) + H, \tag{55}$$

where $RDU(g_{[n]})$ *is given by (53) and* H is given by (52), i.e., 0 *when* $C(n)$ *is not maximal and a constant* A *when* $C(n)$ *is maximal.*

Three observations:

- If we define $K : \mathfrak{B} \to \mathbb{R}$ by $K(C) = h(C)/\mu(C)$ for $C \in \mathfrak{B}^*$, and $K(\varnothing) = 0$, then, noting $S_\Omega(C_i) = \mu(C_i)/\mu(\Omega)$, the representation (54) takes the form

$$U(g_{[n]}) = SEU(g_{[n]}) + \left[K(\Omega) - \sum_{i=1}^{n} K(C_i)S_\Omega(C_i)\right]. \tag{56}$$

- We refer to the form for $U(g_{[n]})$ in Part (i), or equivalently (56), as *Generalized SEU (G-SEU)*, and that in Part (ii), when $C(n)$ is maximal, by RDU_A.
- It is noteworthy that when S_Ω is not finitely additive and the event is not maximal, RDU is not modified at all. There is considerable evidence against the coalescing property of unmodified RDU. See discussion in Marley and Luce (2005).

The above results hold under segregation. Significantly different and weaker results arise when segregation is replaced by the weaker condition of duplex decomposition, Sect. 5.

5 Representations of general gambles under duplex decomposition

5.1 Duplex decomposition

The following alternative to segregation has been fairly widely studied in the e-idempotent case:

Definition 14 Gambles are said to satisfy **duplex decomposition** if

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'), \tag{57}$$

where (C', D') is an independent realization of (C, D).

In contrast to segregation, duplex decomposition is hardly rational. On the left the respondent either ends up with x or with y, but not both. On the right exactly one of four consequences arises: e, x, y, or $x \oplus y$. Thus, the payoff structures are quite different.

Using the same argument as for simple joint-receipt, based on properties of $\oplus$ together with the fact that gambles are invariant under permutations, yields the original concept of duplex decomposition for kernel equivalents, i.e.,

$$KE(x, C; y, D) \sim KE(x, C; e, D) \oplus KE(e, C; y, D). \tag{58}$$

Some history: The concept of duplex decomposition in the e-idempotent case is due to Slovic and Lichtenstein (1968), who reported some supporting data based on judged prices for gambles of mixed gains and losses. Using PEST-determined[9] CEs, Cho et al. (1994) again supported it for mixed gambles with independent realizations. Karabatsos (2005) re-analyzed these data from a Bayesian perspective and provided even stronger support. So far as we know, no one has looked at duplex decomposition, (57), for gains (or losses) alone.

These empirical studies of (57) all presumed e-idempotence, i.e., no utility of gambling, and to the extent that presumption is false, they are flawed tests of our current version of duplex decomposition.

Assuming (58) holds, then for the additive representation U, (9), we have

$$U(KE(x, C; y, D)) = U(KE(x, C; e, D)) + U(KE(e, C; y, D)). \tag{59}$$

5.2 The representation under duplex decomposition

The following result is very closely related to Theorem 4.4.6 of Luce (2000) coupled with duplex decomposition and to Proposition 10 for the assumption of segregation. The difference lies in whether the weight on $U(y)$ is $S_\Omega(D)$ or $1 - S_\Omega(C)$. Because of that close similarity, we will abbreviate both the discussion and proofs considerably. For example, the proof of the following Proposition, which we omit, is slight modification of Proposition 10 using duplex decomposition in lieu of segregation.

Proposition 15 *Suppose that a gamble can be decomposed into the joint receipt of a kernel equivalent and an element of chance, (15), that U is additive over $\oplus$, (9), that the kernel equivalents of unitary gambles have the separable representation US_Ω, (26), and that gambles satisfy duplex decomposition, (57), and so their KEs satisfy (58). Then for f and g that are first-order gambles (including pure consequences)*

$$U[KE(f, C; g, D)] = U(f)S_{C\cup D}(C) + U(g)S_{C\cup D}(D). \tag{60}$$

[9] This well known procedure is widely used in psychophysics and it has been adapted to decision making. It involves a systematic comparison of, usually, money amounts with gambles where on successive trials the money amount is adjusted up or down depending upon the choice made. A stopping criterion of reversals is adopted. The weakness of the method in the utility context is that practical limitations lead to less demanding stopping rules than are used in psychophysics. That may result in the process terminating before it is close to the actual CE.

Thus, the general representation of the binary gambles is

$$U(f, C; g, D) = U(f)S_{C\cup D}(C) + U(g)S_{C\cup D}(D) + U(e, C; e, D). \tag{61}$$

Theorem 16 *Suppose that $n \geq 2$ and that the several assumptions of Lemma 15 are satisfied: decomposability into a KE and an element of chance, (15), U is additive over joint receipt $\oplus$, (9), the kernel equivalents of unitary gambles have the separable representation US_Ω, (26) and duplex decomposition, (57). Suppose that both upper gamble decomposition, Definition 7, and branching, Definition 8 hold. Then there exist an increasing function μ, positive on $\mathfrak{B}^*$, $\mu(\varnothing) = 0$, and a function $h : \mathfrak{B} \to \mathbb{R}$, $h(\varnothing) = 0$, and with the dimension of μU such that*

$$S_\Omega(C) = \mu(C)/\mu(\Omega), \tag{62}$$

and

$$U(g_{[n]}) = LWU\left(g_{[n]}\right) + \frac{1}{\mu(\Omega)}[h(\Omega) - \sum_{i=1}^{n} h(C_i)], \tag{63}$$

where LWU is given by

$$LWU\left(g_{[n]}\right) := \sum_{i=1}^{n} U(x_i)S_\Omega(C_i). \tag{64}$$

Observe that if we convert h to $K : \mathfrak{B} \to \mathbb{R}$ by $K(C) = h(C)/\mu(C)$ for $C \in \mathfrak{B}^*$, and $K(\varnothing) = 0$, then the representation (63) takes the equivalent form

$$U(g_{[n]}) = LWU(g_{[n]}) + \left[K(\Omega) - \sum_{i=1}^{n} K(C_i)S_\Omega(C_i)\right]. \tag{65}$$

This result, under duplex decomposition, is somewhat similar to that under segregation, Theorem 13, but with differences. When S_Ω is finitely additive, they are identical. But when S_Ω is p-additive, but not finitely additive, the form of $U(e, C_1; \ldots; e, C_n)$ under segregation is greatly restricted and that of kernel equivalents is RDU, not LWU.

Unlike the idempotent case where each property leads to quite strong results, rank-dependent utility versus linear weight utility (Luce and Marley 2005), in the non-idempotent case segregation yields far tighter results than does duplex decomposition.

The UofG term on the right of (54), (56), (63) and (65) is quite general and does not involve specific functional relations between h and μ, or between K and μ. In Ng et al. (2007) we formulate additional conditions that lead to such a specialization.

6 Open problems

6.1 An entropy modified version of RDU

As we saw under segregation with branching and upper gamble decomposition, we obtain a specialized modification of RDU, (53), when the weighting function is not finitely additive. The question is whether there exists a property, or properties, that can be substituted for branching and/or upper gains decomposition that will permit a case of RDU with a more complex utility of gambling term, such as (49), than a constant over partitions of the maximal event.

6.2 The p-additive form of elements of chance

An important class of utility models in Luce (2000) leads naturally to the utility function U being separable, (23), and the following representation of $\oplus$, called *p-additivity*, holding for gains and losses separately:

$$U(f \oplus g) = U(f) + U(g) + \delta U(f)U(g). \tag{66}$$

When the p-additive form holds for all gambles, then the decomposition into kernel equivalents and elements of chance takes the interesting form

$$U(g_{[n]}) = U(KE(g_{[n]})) + U(e, C_1; \ldots; e, C_n)\left[1 + \delta U(KE(g_{[n]}))\right].$$

This means that the impact of $U(e, C_1; \ldots; e, C_n)$, i.e., of the elements of chance, depends on the importance of the gamble as measured by $1+\delta U(KE(g_{[n]}))$. Although working with the logarithm of this term yields an additive representation, that contrasts sharply with the additive representation with $\delta = 0$ with which we have worked in this article—in particular, $1 + \delta U(KE(x, C; e, D))$ does not have a separable representation. In the general case of gambles involving both gains and losses, there are issues about the sign of δ that have not yet been thought through.

If we relax the property of simple JR-decomposition as in Luce (2000), Sect. 4.4.7, we have that there exists such a U that is both p-additive and separable.

6.3 Additivity without the choice property

We have not addressed the question of solving either UGD, (34), or branching, (36), under duplex decomposition with finite additivity but without assuming the choice property, (29). Together UGD and branching imply the choice property (seen in the proof of Theorem 16), so any such solution will satisfy one but not the other. Note that we do not have an example of this case, which must not satisfy all of segregation, UGD, and branching (see Proposition 11).

6.4 Sign dependent representations

In a number of theories, particularly of the rank dependent type, the free functions and constants depend upon whether the gamble is seen as a gain or a loss. The results we have reported assume no such distinction, and it may prove tricky to incorporate it although we suspect that it will appear in terms of $KE(g_{[n]}) \succ$ or $\prec e$. Although the elements of chance depend only on the events, the recursions governing their utility depend on the weights and thus if they are sign dependent, then so too will be H. This somewhat tricky problem needs to be carefully formulated and developed in detail.

7 Conclusions

The purpose of this article was to provide an axiomatic formulation of the utility of gambling within the context of an ordering of uncertain alternatives. Our axioms are behavioral, but divide into two theories according to whether we assume segregation, Sect. 4, or duplex decomposition, Sect. 5. These representations are summarized in Table 1.

Table 1 Summary of representations for uncertain gambles

		U(KE) Seg	U(KE) DD		UofG Seg	UofG DD
S_Ω	FA	SEU	SEU	+	H	H
	Not FA	RDU	LWU		0 or A	H

where

$$H := \frac{1}{\mu(\Omega)}\left[h(\Omega) - \sum_{i=1}^{n} h(C_i)\right],$$

and the notation 0 or A means H is 0 when Ω is not maximal and a constant A when it is maximal. Codes: DD = Duplex Decomposition, FA = Finitely Additive, KE = Kernel Equivalent, LWU = Linear Weighted Utility, RDU = Rank-Dependent Utility, Seg = Segregation, SEU = Subjective Expected Utility, UofG = Utility of Gambling

Our arguments rested heavily on having joint receipts as well as gambles.

Appendix: Proofs

Proof of Proposition 4 (The following proof is related to the proof of Theorem 4.4.4 of Luce (2000)). Because U and U^* both preserve the order $\succsim$ and because $U(e) = U^*(e) = 0$, there is a strictly increasing function ψ such that $U(x) = \psi(U^*(x))$ and $\psi(0) = 0$. For the proof, let $t := W_{C\cup D}(C)$, then

$$\begin{aligned}
U(KE(x \oplus y, C; e, D)) &= U(KE(x, C; e, D) \oplus KE(y, C; e, D)) \\
&\quad \text{(using (25))} \\
&= U(KE(x, C; e, D)) \\
&\quad + U(KE(y, C; e, D)) \quad \text{(using (9))} \\
\Leftrightarrow \psi\left(U^*(KE(x \oplus y, C; e, D))\right) &= \psi\left(U^*(KE(x, C; e, D))\right) \\
&\quad + \psi\left(U^*(KE(y, C; e, D))\right) \quad \text{(using } U = \psi(U^*)) \\
\Leftrightarrow \psi\left(U^*(x \oplus y)t\right) &= \psi\left(U^*(x)t\right) + \psi\left(U^*(y)t\right) \quad \text{(using (23))} \\
\Leftrightarrow \psi\left(\psi^{-1}(U(x \oplus y))t\right) &= \psi\left(\psi^{-1}(U(x))t\right) \\
&\quad + \psi\left(\psi^{-1}(U(y))t\right) \text{(using } U^* = \psi^{-1}(U)).
\end{aligned}$$

If we set $u_1 = U(x)$, $u_2 = U(y)$ then, using (9) on the left hand side, this becomes

$$\psi(\psi^{-1}(u_1 + u_2)t) = \psi(\psi^{-1}(u_1)t) + \psi(\psi^{-1}(u_2)t) \quad (u_1, u_2 \in \mathbb{R}, t \in [0, 1]).$$

Thus, ψ being an increasing function for each fixed t, the function $\phi_t(u) := \psi(t\psi^{-1}(u))$ is additive, so there exists a constant c which depends on t such that

$$\psi(\psi^{-1}(u)t) = c(t)u \quad (u \in \mathbb{R}, t \in [0, 1]).$$

That is

$$\psi(vt) = c(t)\psi(v) \quad (v \in \text{range } U^*, t \in [0, 1]).$$

It follows from this Pexider functional equation that $c(t) = t^\kappa$ for some constant $\kappa > 0$. So the conversion function ψ from U^* to U satisfies

$$\psi(vt) = t^\kappa \psi(v) \quad (v \in \text{range } U^*, t \in [0, 1]). \tag{67}$$

Transforming both sides of (23) by ψ and using (67) we get (26). □

Proof of Theorem 6 We recall from (28) that $S_\Omega(C) = 0$ if and only if $C = \varnothing$. For the special case where the ring is an algebra, i.e., there exists a master event Ω_0 which contains all other events, we define

$$\mu(C) := S_{\Omega_0}(C),$$

and (30) is immediate, see Luce (1959/2005).

We now consider the case where the ring is not an algebra. The choice property (29) gives

$$S_{C_1 \cup C_2 \cup C_3}(C_1) = S_{C_1 \cup C_2}(C_1) S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2) \tag{68}$$

for disjoint $C_i, C_i \in \mathfrak{B}^*$. With

$$J(C_1, C_2) := \log_2 S_{C_1 \cup C_2}(C_1), \tag{69}$$

(68) becomes

$$J(C_1, C_2 \cup C_3) = J(C_1, C_2) + J(C_1 \cup C_2, C_3). \tag{70}$$

By the symmetry in C_2 and C_3 on the left we get

$$J(C_1, C_2) + J(C_1 \cup C_2, C_3) = J(C_1, C_3) + J(C_1 \cup C_3, C_2). \tag{71}$$

By Theorem 3.3 of Ebanks et al. (1988), there exist functions ϕ, ψ such that

$$J(E, F) = \phi(E) + \psi(F) - \phi(E \cup F) \tag{72}$$

for all disjoint E, F in $\mathfrak{B}^*$.

Putting (72) into (70) yields the additivity of ψ:

$$\psi(E \cup F) = \psi(E) + \psi(F) \tag{73}$$

for all disjoint E, F in $\mathfrak{B}^*$.

In the representation (72), the additive ψ can be absorbed into ϕ:

$$J(C_1, C_2) = \tilde{\phi}(C_1) - \tilde{\phi}(C_1 \cup C_2) \tag{74}$$

where $\tilde{\phi} := \phi - \psi$. Letting $\mu := 2^{\tilde{\phi}}$ and returning to (68) and (69) we obtain the representation (30) for $C \neq \varnothing$. With $\mu(\varnothing) := 0$, it also holds for $C = \varnothing$. The assumption $S_D(C) \leq 1$ translates into $\mu(C) \leq \mu(D)$ for all $C \subseteq D$. So μ is increasing. □

Proof of Proposition 10 Using segregation applied to KEs, (38), followed by the additivity of joint receipts and then (26) and (27), we have

$$\begin{aligned} U\,(KE(x, C; y, D)) &= U(KE\,(x \ominus y, C; e, D)) + U(y) \\ &= [U(x) - U(y)]\, S_\Omega(C) + U(y) \\ &= U(x) S_\Omega(C) + U(y)\,[1 - S_\Omega(C)]\,, \end{aligned}$$

which is (39). And so by (16) we get (40). □

Proof of Proposition 11 1. The following arguments use UGD and branching, which are unranked, and (40), which is ranked. Thus care has to be taken that the relevant rank conditions are satisfied. Those conditions require that, given non-null events $C_i, i = 1, 2, 3$, we can select consequences x_1, x_2, x_3 independent of one another with $x_1 \succsim x_2 \succsim x_3$ and such that $x_1 \succsim (x_2, C_2; x_3, C_3)$ and $(x_1, C_1; x_2, C_2) \succsim x_3$. Since U is unbounded, this follows easily from the general representation of the binary gambles, (40).

Now, by UGD, (34), and (40), we have

$$\begin{aligned}U(g_{[3]}) &= U(x_1, C_1; (x_2, C_2; x_3, C_3), C_2 \cup C_3)\\ &= U(x_1)S_{C_1\cup C_2\cup C_3}(C_1) + U(x_2, C_2; x_3, C_3)[1 - S_{C_1\cup C_2\cup C_3}(C_1)]\\ &\quad + H_2(C_1, C_2 \cup C_3)\\ &= U(x_1)S_{C_1\cup C_2\cup C_3}(C_1) + H_2(C_1, C_2 \cup C_3)\\ &\quad + U(x_2)S_{C_2\cup C_3}(C_2)[1 - S_{C_1\cup C_2\cup C_3}(C_1)]\\ &\quad + U(x_3)[1 - S_{C_2\cup C_3}(C_2)][1 - S_{C_1\cup C_2\cup C_3}(C_1)]\\ &\quad + H_2(C_2, C_3)[1 - S_{C_1\cup C_2\cup C_3}(C_1)].\end{aligned}$$

And by branching, (36), and (40),

$$\begin{aligned}U(g_{[3]}) &= U((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3)\\ &= U(x_1, C_1; x_2, C_2)S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2) + U(x_3)[1 - S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)]\\ &\quad + H_2(C_1 \cup C_2, C_3)\\ &= U(x_1)S_{C_1\cup C_2}(C_1)S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)\\ &\quad + U(x_2)[1 - S_{C_1\cup C_2}(C_1)]S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)\\ &\quad + H_2(C_1, C_2)S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)\\ &\quad + U(x_3)[1 - S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)] + H_2(C_1 \cup C_2, C_3).\end{aligned}$$

Equating these and keeping in mind that the x_i can be varied independently subject only to the ordering constraints, their coefficients must be equal. That for x_1 yields

$$S_{C_1\cup C_2\cup C_3}(C_1) = S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2)S_{C_1\cup C_2}(C_1),$$

which is the choice property. That of x_2 together with the choice property, yields

$$S_{C_2\cup C_3}(C_2) = \frac{S_{C_1\cup C_2\cup C_3}(C_1 \cup C_2) - S_{C_1\cup C_2\cup C_3}(C_1)}{1 - S_{C_1\cup C_2\cup C_3}(C_1)}. \tag{75}$$

The one for x_3 is exactly equivalent to (75).

By the choice property, we know that the representation μ of (30) exists. Substituting that into (75) and letting $\Omega = C_1 \cup C_2 \cup C_3$, we obtain

$$\mu(\Omega) = \frac{\mu(C_2 \cup C_3)}{\mu(C_2)}[\mu(C_1 \cup C_2) - \mu(C_1)] + \mu(C_1). \tag{76}$$

If we redo this with $x_3 \succ x_2$, it amounts to interchanging the roles of C_2 and C_3 in (76). Equating the two resulting right hand sides, we obtain

$$\frac{\mu(C_2 \cup C_3)}{\mu(C_2)} [\mu(C_1 \cup C_2) - \mu(C_1)] + \mu(C_1) \\ = \frac{\mu(C_2 \cup C_3)}{\mu(C_3)} [\mu(C_1 \cup C_3) - \mu(C_1)] + \mu(C_1),$$

which is equivalent to

$$\frac{\mu(C_1 \cup C_2) - \mu(C_1)}{\mu(C_2)} = \frac{\mu(C_1 \cup C_3) - \mu(C_1)}{\mu(C_3)}.$$

Thus the expressions are independent of C_2 (and C_3) so there is a function φ such that

$$\frac{\mu(C_1 \cup C_2) - \mu(C_1)}{\mu(C_2)} = \varphi(C_1),$$

i.e.,

$$\mu(C_1 \cup C_2) = \mu(C_2)\varphi(C_1) + \mu(C_1). \tag{77}$$

Then, by the symmetry of C_1 and C_2, we also have

$$\mu(C_1 \cup C_2) = \mu(C_1)\varphi(C_2) + \mu(C_2) \tag{78}$$

and equating (77) and (78)

$$\frac{\varphi(C_1) - 1}{\mu(C_1)} = \frac{\varphi(C_2) - 1}{\mu(C_2)} = \Delta.$$

Thus,

$$\frac{\varphi(C) - 1}{\mu(C)} = \Delta \Leftrightarrow \varphi(C) = \Delta\mu(C) + 1,$$

which substituted back in (78) yields

$$\mu(C_1 \cup C_2) = \mu(C_1) + \mu(C_2) + \Delta\mu(C_1)\mu(C_2).$$

Note that the unit of Δ must be that of $1/\mu$. Dividing by $\mu(\Omega)$ and using (30) yields (45).

2. Now, consider general $n \geq 3$. Again, select x_1 such that $x_1 \succsim g_{[n],-1}$. By (16), we see that

$$U(g_{[n]}) = U(KE(g_{[n]})) + H_n(C_1, \ldots, C_n).$$

By UGD and (40), we have also that

$$\begin{aligned} U(g_{[n]}) &= U(x_1, C_1; g_{[n],-1}, \Omega\backslash C_1) \\ &= U(x_1)S_\Omega(C_1) + U(g_{[n],-1})\,[1 - S_\Omega(C_1)] + H_2(C_1, \Omega\backslash C_1) \\ &= U(x_1)S_\Omega(C_1) + U\left(KE(g_{[n],-1})\right)[1 - S_\Omega(C_1)] \\ &\quad + H_{n-1}(C_2, \ldots, C_n)\,[1 - S_\Omega(C_1)] + H_2(C_1, \Omega\backslash C_1). \end{aligned}$$

Therefore, using the fact that the consequences x_i and events C_i are independent, we have

$$U(KE(g_{[n]})) = U(x_1)S_\Omega(C_1) + U\left(KE(g_{[n],-1})\right)[1 - S_\Omega(C_1)], \tag{79}$$

and

$$H_n(C_1, \ldots, C_n) = H_{n-1}(C_2, \ldots, C_n)\,[1 - S_\Omega(C_1)] + H_2(C_1, \Omega\backslash C_1),$$

which is (46), and is not affected by restrictions on x_1. We deal with (79) in the proof of Theorem 13.

Next, consider consequences x_1, x_2, x_3 independent of one another with $x_1 \succsim x_2 \succsim x_3$ such that $x_1 \succsim (x_2, C_2; x_3, C_3)$ and $(x_1, C_1; x_2, C_2) \succsim g_{[n],-1,-2}$. As in the proof of Part 1, since U is order preserving and unbounded, the existence of such consequences follows easily from the general representation of the binary gambles, (40). Then using branching, UGD and (40), we have

$$\begin{aligned} U(g_{[n]}) &= U(KE(g_{[n]})) + H_n(C_1, \ldots, C_n) \\ &= U\left((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; \ldots; x_n, C_n\right) \\ &= U(KE\left((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; \ldots; x_n, C_n\right)) \\ &\quad + H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) \\ &= U(KE\left((x_1, C_1; x_2, C_2), C_1 \cup C_2; g_{[n],-1,-2}, \Omega\backslash(C_1 \cup C_2)\right)) \\ &\quad + H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) \\ &= U(x_1, C_1; x_2, C_2)S_\Omega(C_1 \cup C_2) + U(g_{[n],-1,-2})[1 - S_\Omega(C_1 \cup C_2)] \\ &\quad + H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) \\ &= \left[U(x_1)S_{C_1\cup C_2}(C_1) + U(x_2)(1 - S_{C_1\cup C_2}(C_1))\right] S_\Omega(C_1 \cup C_2) \\ &\quad + U(g_{[n],-1,-2})[1 - S_\Omega(C_1 \cup C_2)] + H_2(C_1, C_2)S_\Omega(C_1 \cup C_2) \\ &\quad + H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n). \end{aligned}$$

Thus, we have proved that, if $x_1 \succsim (x_2, C_2; x_3, C_3)$ and $(x_1, C_1; x_2, C_2) \succsim g_{[n],-1,-2}$, then

$$\begin{aligned} U(KE(g_{[n]})) &= U(x_1)S_{C_1\cup C_2}(C_1)S_\Omega(C_1 \cup C_2) \\ &\quad + U(x_2)(1 - S_{C_1\cup C_2}(C_1))S_\Omega(C_1 \cup C_2) \\ &\quad + U(g_{[n],-1,-2})[1 - S_\Omega(C_1 \cup C_2)] \end{aligned} \tag{80}$$

and also that

$$H_n(C_1, \ldots, C_n) = H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) + H_2(C_1, C_2) S_\Omega(C_1 \cup C_2),$$

which is (47). □

Proof of Lemma 12 In view of the expansibility of H_n, (43), we focus on partitions $(C_1, \ldots, C_n)$ with $C_i \in \mathfrak{B}^*$.

1. Suppose that (48) holds, i.e.,

$$\begin{aligned} &H_n(C_1, C_2, C_3, \ldots, C_n) \\ &\quad = H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) + H_2(C_1, C_2) \frac{\mu(C_1 \cup C_2)}{\mu(\Omega)}. \end{aligned} \tag{81}$$

Defining L_n by

$$L_n(C_1, C_2, \ldots, C_n) := \mu(\Omega) H_n(C_1, C_2, \ldots, C_n), \tag{82}$$

equation (81) becomes

$$L_n(C_1, C_2, C_3, \ldots, C_n) = L_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n) + L_2(C_1, C_2) \tag{83}$$

where L_n is symmetric because H_n is. We first observe that $L_3(C_1, C_2, C_3) = L_2(C_1 \cup C_2, C_3) + L_2(C_1, C_2)$ and the symmetry of the left side yields the cocycle equation (71):

$$L_2(C_1 \cup C_2, C_3) + L_2(C_1, C_2) = L_2(C_2 \cup C_3, C_1) + L_2(C_2, C_3).$$

With the symmetry of L_2, this equation becomes

$$L_2(C_2, C_3) + L_2(C_1, C_2 \cup C_3) = L_2(C_1, C_2) + L_2(C_1 \cup C_2, C_3). \tag{84}$$

By Davidson and Ng (1981) when the events form an algebra, and by Ebanks (1982) when they do not, L_2 has the representation

$$L_2(C_1, C_2) = h(C_1 \cup C_2) - h(C_1) - h(C_2) \tag{85}$$

for some function h. Computing L_n using (83) and (85) we arrive at

$$L_n(C_1, C_2, \ldots, C_n) = h(\Omega) - \sum_{i=1}^{n} h(C_i). \tag{86}$$

Returning to (82), we have the representation (49) for $C_i \in \mathfrak{B}^*$. In view of the expansibility of H_n, we let $h(\varnothing) = 0$. With that, (49) holds for $C_i \in \mathfrak{B}$.

2. Now consider (50), i.e.,

$$H_n\,(C_1, C_2, \ldots, C_n) \\ = H_{n-1}(C_2, \ldots, C_n)\left(1 - \frac{\mu\,(C_1)}{\mu(\Omega)}\right) + H_2\,(C_1, \Omega\backslash C_1)\,. \tag{87}$$

Under the form (49) just derived using the choice property, this is equivalent to

$$\frac{1}{\mu(\Omega)}\left[h(\Omega) - \sum_{i=1}^{n} h(C_i)\right] \\ = \frac{1}{\mu(\Omega\backslash C_1)}\left[h(\Omega\backslash C_1) - \sum_{i=2}^{n} h(C_i)\right]\left[1 - \frac{\mu(C_1)}{\mu(\Omega)}\right] \\ + \frac{1}{\mu(\Omega)}\left[h(\Omega) - h(C_1) - h(\Omega\backslash C_1)\right], \tag{88}$$

which simplified is equivalent to

$$h(\Omega) - \sum_{i=1}^{n} h(C_i) = \left[\frac{\mu(\Omega) - \mu(C_1)}{\mu(\Omega\backslash C_1)}\right]\left[h(\Omega\backslash C_1) - \sum_{i=2}^{n} h(C_i)\right] \\ + h(\Omega) - h(C_1) - h(\Omega\backslash C_1),$$

which in turn is equivalent to

$$\left[h(\Omega\backslash C_1) - \sum_{i=2}^{n} h(C_i)\right]\left[\frac{\mu(\Omega\backslash C_1) - \mu(\Omega) + \mu(C_1)}{\mu(\Omega\backslash C_1)}\right] = 0.$$

So, for each partition $(C_1, \ldots, C_n)$, either

$$\mu(\Omega) = \mu(\Omega\backslash C_1) + \mu(C_1), \tag{89}$$

or

$$h(\Omega\backslash C_1) - \sum_{i=2}^{n} h(C_i) = 0. \tag{90}$$

First, suppose that (89) holds for some partition $(D_1, \ldots, D_n)$ ($n \geq 3$, $D_i \in \mathfrak{B}^*$ understood). In conjunction with (44) which gives

$$\mu(\Omega) = \mu(\Omega\backslash D_1) + \mu(D_1) + \Delta\mu(\Omega\backslash D_1)\mu(D_1),$$

it follows that $\Delta = 0$. So, we arrive at 2(a).

Otherwise, we have (90) holding for all partitions with $n \geq 3$. Restated, it means that, for all $n \geq 2$,

$$H_n(C_1, C_2, \ldots, C_n) = 0 \quad (\forall \text{ non} - \text{maximal } \Omega). \tag{91}$$

If $\mathfrak{B}$ has no maximal event, we have arrived at 2(b).

Finally, assume the case that $\mathfrak{B}$ has a maximal event Ω_0 (which is unique) and we consider partitions $(C_1, \ldots, C_n)$ of Ω_0.

By (87) and (90), $H_n(C_1, C_2, \ldots, C_n) = H_2(\Omega_0 \backslash C_1, C_1)$. Using the symmetry of H_n we get

$$H_n(C_1, C_2, \ldots, C_n) = H_2(\Omega_0 \backslash C_i, C_i) \quad (\forall i \leq n). \tag{92}$$

To come to the conclusion of 2(b), it is sufficient to show that H_2 is a constant, A, over all partitions of Ω_0 by members of $\mathfrak{B}^*$. For that purpose, let (C_1, C_2) and (D_1, D_2) be given partitions of Ω_0 (with non-empty C_i, D_i) and we proceed to show that $H_2(C_1, C_2) = H_2(D_1, D_2)$. If $\{C_1, C_2\} = \{D_1, D_2\}$ we are done because H_2 is symmetric. Otherwise $\{C_i \cap D_j \mid i, j = 1, 2\}$ is a partition with at least three non-empty events. By symmetry of H_2 again, we may assume that $C_1 \cap D_1$, $C_1 \cap D_2$ and $C_2 \cap D_1$ are non-empty. There are two cases to consider.

Case (i). Suppose that $C_2 \cap D_2 = \varnothing$. Then

$$\begin{aligned}
&H_4(C_1 \cap D_1, C_1 \cap D_2, C_2 \cap D_1, C_2 \cap D_2)\\
&\quad = H_3(C_1 \cap D_1, D_2, C_2) \quad \text{by expansibility}\\
&\quad = H_2(\Omega_0 \backslash C_2, C_2) \quad \text{by (92)}\\
&\quad = H_2(C_1, C_2).
\end{aligned}$$

Using parallel arguments we also get $H_4(C_1 \cap D_1, C_1 \cap D_2, C_2 \cap D_1, C_2 \cap D_2) = H_2(D_1, D_2)$. This proves $H_2(C_1, C_2) = H_2(D_1, D_2)$ for case (i).

Case (ii). Suppose that $C_2 \cap D_2 \neq \varnothing$. Then

$$\begin{aligned}
&H_4(C_1 \cap D_1, C_1 \cap D_2, C_2 \cap D_1, C_2 \cap D_2)\\
&\quad = H_2(\Omega_0 \backslash (C_1 \cap D_1), C_1 \cap D_1) \quad \text{by (92)}\\
&\quad = H_3(C_1 \cap D_1, C_1 \cap D_2, C_2) \quad \text{by (92)}\\
&\quad = H_2(\Omega_0 \backslash C_2, C_2) \quad \text{by (92)}\\
&\quad = H_2(C_1, C_2).
\end{aligned}$$

Again, parallel arguments lead to $H_4(C_1 \cap D_1, C_1 \cap D_2, C_2 \cap D_1, C_2 \cap D_2) = H_2(D_1, D_2)$. This proves $H_2(C_1, C_2) = H_2(D_1, D_2)$ for case (ii).

The deduction of 2(b) is complete. □

Proof of Theorem 13 First, consider Part (ii), i.e., where finite additivity fails, which corresponds to the case of Part 2(b) of Lemma 12. This reduces the problem to one already dealt with in the literature Luce (2000) but we provide a simple proof. Consider as the induction hypothesis the rank-dependent form

$$U(KE(g_{[n]})) = \sum_{i=1}^{n} U(x_i)\left[S_\Omega(C(i)) - S_\Omega(C(i-1))\right]. \tag{93}$$

It follows immediately from (40) and (28) that (93) is true for $n = 2$. Suppose that it is true for $n-1$, $n > 2$, then using the choice property and (80),

$$\begin{aligned} U(KE(g_{[n]})) &= U(x_1)S_{C_1\cup C_2}(C_1)S_\Omega(C_1 \cup C_2) \\ &\quad + U(x_2)(1 - S_{C_1\cup C_2}(C_1))S_\Omega(C_1 \cup C_2) \\ &\quad + \sum_{i=3}^{n} U(x_i)\left[S_\Omega(C(i)) - S_\Omega(C(i-1))\right] \\ &= \sum_{i=1}^{n} U(x_i)\left[S_\Omega(C(i)) - S_\Omega(C(i-1))\right], \end{aligned} \tag{94}$$

which form is known as general rank-dependent utility, denoted $RDU\big(g_{[n]}\big)$.

In any RDU representation,

$$\sum_{i=1}^{n} \left[S_\Omega(C(i)) - S_\Omega(C(i-1))\right] = 1.$$

Then using (30) and (45), we have

$$\begin{aligned} S_\Omega(C(i)) - S_\Omega(C(i-1)) &= S_\Omega(C_i \cup C(i-1)) - S_\Omega(C(i-1)) \\ &= S_\Omega(C_i) + \Delta\mu(\Omega)S_\Omega(C_i)S_\Omega(C(i-1)), \end{aligned}$$

which yields the RDU representation of (53).

We turn now to Part (i), i.e., where finite additivity holds, which corresponds to the case of Part 2(a) of Lemma 12. As we pointed out after the statement of Proposition 10, in this case (39) becomes

$$U(KE(f, C; g, D)) = U(f)S_{C\cup D}(C) + U(g)S_{C\cup D}(D),$$

which is clearly independent of the ordering between f and g. It is then easily checked that in this case the proof that leads to (79) goes through without the ordering constraint, leading to

$$U(KE(g_{[n]})) = U(x_1)S_\Omega(C_1) + U(KE(g_{[n],-1}))S_\Omega\left(\Omega\backslash C_1\right).$$

Note that the above symmetry of the roles of $S_\Omega(C_1)$ and $S_\Omega\left(\Omega\backslash C_1\right)$ does not exist in the general case of segregation, but it does once we have finite additivity. Thus, with SEU, (51), as the induction hypothesis for $n-1$, it follows using the choice property that SEU holds for n. □

Proof of Theorem 16 By Lemma 15, (60) is satisfied, and, in particular, there is no ordering constraint between f and g. For $n = 3$, branching and (61) give

$$\begin{aligned} U(KE(x, C; y, D; z, E)) &= U(KE((x, C; y, D), C \cup D; z, E)) \\ &= U(x, C; y, D)S_\Omega(C \cup D) + U(z)S_\Omega(E) + H_2(C \cup D, E) \\ &= U(x)S_{C\cup D}(C)S_\Omega(C \cup D) + U(y)S_{C\cup D}(D)S_\Omega(C \cup D) + U(z)S_\Omega(E) \\ &\quad + H_2(C, D)S_\Omega(C \cup D) + H_2(C \cup D, E). \end{aligned}$$

Similarly, upper gamble decomposition with (61) yields

$$\begin{aligned} U(KE(x, C; y, D; z, E)) = U(x)S_\Omega(C) &+ U(y)S_{D\cup E}(D)S_\Omega(D \cup E) \\ &+ U(z)S_{D\cup E}(E)S_\Omega(D \cup E) \\ &+ H_2(D, E)S_\Omega(C \cup D) + H_2(C, D \cup E). \end{aligned}$$

Equating these, collecting terms, and noting that x, y, z are independent immediately forces the choice property. By Theorem 6, the representation (62) holds.

Using upper gamble decomposition with (61) we get

$$\begin{aligned} U(g_{[n]}) &= U\left(x_1, C_1; g_{[n],-1}, \Omega\backslash C_1\right) \\ &= U(x_1)S_\Omega(C_1) + U\left(g_{[n],-1}\right) S_\Omega\left(\Omega\backslash C_1\right) + H_2\left(C_1, \Omega\backslash C_1\right) \\ &= U(x_1)S_\Omega(C_1) + H_2\left(C_1, \Omega\backslash C_1\right) \\ &\quad + \left[U(KE(g_{[n],-1})) + H_{n-1}\left(C_2, \ldots, C_n\right)\right] S_\Omega\left(\Omega\backslash C_1\right) \\ &= U(x_1)S_\Omega(C_1) + U(KE(g_{[n],-1}))S_\Omega\left(\Omega\backslash C_1\right) \\ &\quad + H_{n-1}\left(C_2, \ldots, C_n\right) S_\Omega\left(\Omega\backslash C_1\right) + H_2\left(C_1, \Omega\backslash C_1\right). \end{aligned} \tag{95}$$

Comparing it with (16):

$$U(g_{[n]}) = U(KE(g_{[n]})) + H_n(C_1, C_2, \ldots, C_n), \tag{96}$$

we get the recursions

$$U(KE(g_{[n]})) = U(x_1)S_\Omega(C_1) + U(KE(g_{[n],-1}))S_\Omega\left(\Omega\backslash C_1\right) \tag{97}$$

and

$$\begin{aligned} H_n\left(C_1, C_2, \ldots, C_n\right) = H_{n-1}(C_2, \ldots, C_n)S_\Omega\left(\Omega\backslash C_1\right) \\ + H_2\left(C_1, \Omega\backslash C_1\right). \end{aligned} \tag{98}$$

By simple induction, from the first recursion (97) we get

$$U(KE(g_{[n]})) = \sum_{i=1}^{n} U(x_i)S_\Omega(C_i). \tag{99}$$

The choice property and (98) give

$$H_n(C_1, C_2, \ldots, C_n) = H_{n-1}(C_2, \ldots, C_n)\,\frac{\mu(\Omega\backslash C_1)}{\mu(\Omega)} + H_2(C_1, \Omega\backslash C_1)\,. \tag{100}$$

Similar to (82), we define

$$L_n(C_1, C_2, \ldots, C_n) := \mu(\Omega)\, H_n(C_1, C_2, \ldots, C_n) \tag{101}$$

and equation (100) becomes

$$L_n(C_1, C_2, \ldots, C_n) = L_{n-1}(C_2, \ldots, C_n) + L_2(C_1, \Omega\backslash C_1)\,. \tag{102}$$

Notice that L_n is symmetric because H_n is. The relation

$$L_3(C_1, C_2, C_3) = L_2(C_2, C_3) + L_2(C_1, C_2 \cup C_3)$$

and the symmetry of the left side yields the cocycle equation (71), equivalently (84), i.e.,

$$L_2(C_2, C_3) + L_2(C_1, C_2 \cup C_3) = L_2(C_1, C_2) + L_2(C_1 \cup C_2, C_3),$$

with symmetric L_2. So, as shown in the proof of Lemma 12, L_2 has the representation (85) for some function h. Computing L_n using (102) and (85) we also arrive at (86). Returning to (101), we have the representation (49). With (49) and (99), (96) reduces to (63).

The verification of the converse statement is straight forward. □

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