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Comment

A behavioral condition for Prelec's weighting function on the positive line without assuming $W(1) = 1$

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Abstract

Prelec axiomatized a flexible 2-parameter weighting function W over probabilities in a utility context, and Luce reported a simpler axiomatic condition for the case $W(1) = 1$. This article modifies the latter to yield also the generalized Prelec functions with $W(1) \neq 1$ and deals also with the cases where the variables go through $[1, \infty[$ or through the whole positive line. This has arisen naturally in a psychophysical application.

Keywords: Double reduction invariance; Functional equation; Generalized Prelec function; Reduction invariance; Utility weighting function

1. Motivation

Let $(x, p; y, 1-p)$ denote the binary gamble where $x \in X$ is received with probability p and $y \in X$ with probability $1-p$. Many, if not most, versions of utility theory on binary gambles give rise to the well known subjective utility representation

$$U(x, p; y, 1-p) = U(x)W(p) + U(y)[1 - W(p)] \quad (p \in [0, 1], x, y \in X, x \succsim y), \tag{1}$$

This form was first suggested by Edwards (1955) and Savage (1954). Moreover, U preserves a weak preference order $\succsim$ over the binary gambles. Possible mathematical forms for the functions U and W have been of considerable interest. Here we focus only on W. For that it is enough to consider gambles yielding x with probability p and nothing otherwise. If $e \in X$ is the pure consequence that separates gains from losses, then various models provide arguments leading to $U(e) = 0$, and so we abbreviate $(x, p; e, 1-p)$ to (x, p), then we get the (multiplicative) *separability* condition[1]

$$U(x, p) := U(x, p; e, 1-p) = U(x)W(p). \tag{2}$$

In a different area, in psychology, Luce (2002, 2004) developed a global psychophysical theory of intensity involving a psychophysical function ψ over intensities[2] less the threshold intensity and a weighting function W that, like that in the utility area, represents a subjective distortion of numbers. Suppose that the respondent is presented signals $x, y\,(x > y)$ and a number $p > 0$, and is instructed to select the intensity $z = f(x, y, p)$ such that the "interval" from y to z is perceived to be p times the "interval" from y to x. Then the theory establishes, among other things, that

$$W(p) = \frac{\psi(z) - \psi(y)}{\psi(x) - \psi(y)} \quad (x > y \geqslant 0) \tag{3}$$

is satisfied.

[1]We do not use distinct symbols for utility of gambles and utility of pure consequences because the utility of a gamble is equal to that of its certainty equivalent, which is a pure consequence. And for simplicity we write $U(x, p)$ to replace $U[(x, p)]$, etc.

[2]Not decibels.

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Note that, if $p = 1$, which amounts to the instruction to choose z to match x, then the response is veridical, $z = x$, iff $W(1) = 1$. Some years ago, it came as a surprise to many psychophysists that empirical matches were, in general, not veridical. In the case of temporally successive presentations, this phenomenon is known as the *time order error*. The results are summarized by Hellström (1985, 2003). Steingrimsson and Luce are currently exploring the implications of (3) for the time order error.

Note that, if we formally identify U with ψ, and $U(x,p;y,1-p)$ with $U(f(x,y,p))$, then (1) and (3) are equivalent, but with one major difference. In the utility case the mapping W is from $[0,1]$ onto $[0,1]$, thereby forcing $W(0) = 0$, $W(1) = 1$, whereas in the psychophysical case W maps $]0,\infty[$ to $]0,\infty[$ and does not necessarily force $W(1) = 1$, although there has been considerable tendency in the theoretical literature to assume that without comment.

Prelec (1998) gave a behavioral condition for weighting functions to be of the form

$$W(p) = \exp[-\beta(-\ln p)^{\alpha}] \quad (p \in [0,1]; \alpha > 0, \beta > 0), \tag{4}$$

and Luce (2001) provided a simpler behavioral condition and named the function after Prelec. Under the assumption that W is strictly increasing over $[0,1]$, the condition arrived at, called *reduction invariance*, is

$$((x,p),q) \sim (x,r) \Leftrightarrow ((x,p^{\lambda}),q^{\lambda}) \sim (x,r^{\lambda}) \quad (p,q,r \in [0,1]; \lambda > 0), \tag{5}$$

which using (2) is readily seen to be equivalent to

$$W(p)W(q) = W(r) \;\Leftrightarrow\; W(p^{\lambda})W(q^{\lambda}) = W(r^{\lambda}). \tag{6}$$

It suffices that (5) be satisfied for just two values of λ, such as 2 and 3, because a simple proof, using induction and continuity, extends those two cases to all $\lambda > 0$. Notice that, in the form (4), $W(0) = 0$ and $W(1) = 1$, provided that the conventions $0^{\alpha} = 0$ $(\alpha > 0)$ and $\exp(-\infty) = 0$ are accepted. Also, if W is positive and continuous at 1, then (6) implies $W(1) = 1$ directly.

Observe that a power function p^{β} is the special case of Prelec for which $\alpha = 1$. That form was arrived at independently by Narens (1996) in an axiomatic treatment of psychophysical intensity, and he showed that this form is equivalent to the behavioral condition

$$((x,p),q) \sim (x,pq). \tag{7}$$

Several psychophysical empirical articles—Ellermeier and Faulhammer (2000), Steingrimsson and Luce (in press), Zimmer (2005)—tested this property quite thoroughly and found it grossly unsatisfactory. Initially all of these authors concluded that this meant that W is not a power function until, finally, Steingrimsson and Luce acknowledged that the trouble might lie in the implicit assumption $W(1) = 1$. Thus, they explored the possibility that $W(p) = W(1)p^{\beta}$ which they showed is equivalent to replacing (7) by

$$((x,p),q) \sim (x,kpq). \tag{8}$$

Their data sustained this property for six respondents run with $p > 1, q > 1$ and for three of five respondents with $p < 1, q < 1$, but the other two clearly failed that condition. (These two plus one other ran in both conditions.) This led them to consider the more general Prelec family where $W(1)$ need not be 1, i.e.,

$$W(p) = \begin{cases} W(1)\exp[-\beta(-\ln p)^{\alpha}] & (p \in]0,1[; \alpha > 0, \beta > 0), \\ W(1) & (p = 1), \\ W(1)\exp[\beta'(\ln p)^{\alpha'}] & (p \in]1,\infty[; \alpha' > 0, \beta' > 0) \end{cases} \tag{9}$$

whose behavioral equivalent was not immediately obvious to them. Finding such an equivalence is the reason for this note. (They found that the two respondents who failed (8) did, indeed, satisfy the (10) below).

Prelec (1998) also axiomatized this 3-parameter version, but Luce (2001) did not because in the utility situation, where $p \in]0,1]$, the condition $W(1) = 1$ seemed appropriate. When p can be greater or less than 1, it is not forced. In contrast to (6), we will obtain (see Proposition):

$$W(p)W(q) = W(r)^2 \;\Leftrightarrow\; W(p^{\lambda})W(q^{\lambda}) = W(r^{\lambda})^2$$

as the defining condition. So this article is an additional recasting of Prelec (1998).

2. Results

We state our result on $]0,1]$ (later on $]0,\infty[$) but it can be extended to 0 by continuity.

Proposition. *Assuming separability* $U(x,p) = U(x)W(p)$, (2), *excluding* $U(x) = 0$ *for all* $x \in X$, *assuming that the bijection* $W :]0,1] \to]0,\omega]$ *is strictly increasing and that* $0 < \omega = W(1) = \lim_{p\to 1} W(p) < \infty$, *then the following three statements are equivalent*:

$$((x,p),q) \sim ((x,r),r) \Leftrightarrow ((x,p^{\lambda}),q^{\lambda}) \sim ((x,r^{\lambda}),r^{\lambda}) \quad (p,q,r \in]0,1]; \lambda > 0), \tag{10}$$

$$W(p)W(q) = W(r)^2 \Leftrightarrow W(p^{\lambda})W(q^{\lambda}) = W(r^{\lambda})^2 \quad (p,q,r \in]0,1]; \lambda > 0), \tag{11}$$

$$W(p) = \omega\, \exp(-\beta(-\ln p)^{\alpha}) \quad (p \in]0,1]; \alpha > 0, \beta > 0; \omega = W(1)). \tag{12}$$

We may call (10) *double reduction invariance*. It is easy to see that double reduction invariance, in the form (11), plus $W(1) = 1$ implies reduction invariance, (6), but the latter does not imply the former, in general.

Proof. It is easily shown by use of separability $U(x,p) = U(x)W(p)$ that (10) and (11) are equivalent. So we suppose that (11) holds. To show (12), define

$$P = -\ln p, \quad Q = -\ln q, \tag{13}$$

$$G(P) := -\ln W(e^{-P}), \tag{14}$$

$$H_\lambda(P) := G(\lambda P). \tag{15}$$

(Note that, since W is a strictly increasing bijection, so is G). Then (11) translates into

$$G^{-1}\left(\frac{G(P)+G(Q)}{2}\right) = H_\lambda^{-1}\left(\frac{H_\lambda(P)+H_\lambda(Q)}{2}\right)$$
$$(\lambda > 0, P \geqslant 0, Q \geqslant 0).$$

This is equivalent to

$$G(\lambda P) = H_\lambda(P) = A(\lambda)G(P) + B(\lambda) \quad (\lambda > 0, P \geqslant 0) \tag{16}$$

(see e.g. Aczél, 1966/2006, Section 3.1.3). Setting $P = 1$ yields $G(\lambda) = A(\lambda)G(1) + B(\lambda)$, whence, with

$$K(P) := G(P) - G(1) \tag{17}$$

(since G is strictly increasing, so is K), (16) is equivalent to

$$K(\lambda P) = A(\lambda)K(P) + K(\lambda) \tag{18}$$

$$= A(P)K(\lambda) + K(P). \tag{19}$$

This holds because of symmetry; but then only for $\lambda > 0, P > 0$. So

$$[A(\lambda) - 1]K(P) = [A(P) - 1]K(\lambda) \quad (\lambda > 0, P > 0). \tag{20}$$

If A is identically 1, then (18) becomes

$$K(\lambda P) = K(P) + K(\lambda) \quad (\lambda > 0, P > 0),$$

whose strictly increasing solution is $K(P) = \beta \ln P$ $(P > 0)$. Thus,

$$-\ln W(e^{-P}) + \ln W(e^{-1}) = G(P) - G(1) = K(P) = \beta \ln P,$$

that is,

$$W(p) = \frac{\Gamma}{(-\ln p)^\beta} \quad (p \in]0,1[).$$

($\Gamma > 0, \beta > 0$, because W is positive and strictly increasing), which implies that $\lim_{p\to 1} W(p) = \infty$, contrary to assumption.

So, assuming that there exists a $\lambda_0 > 0$ with $A(\lambda_0) \neq 1$, Eq. (20) yields

$$K(P) = \beta[A(P) - 1], \tag{21}$$

($\beta = \frac{K(\lambda_0)}{A(\lambda_0)-1} \neq 0$) which inserted into (18) gives

$$A(\lambda P) = A(\lambda)A(P) \quad (\lambda > 0, P > 0),$$

with $A(P) = P^\alpha$ $(\alpha \neq 0)$ as the strictly monotonic solution. Thus, using (21) and the definitions (14), (17), and (13) of G, K, and P, respectively, we obtain

$$W(p) = \exp(-G(-\ln p)) = \omega \exp(-\beta(-\ln p)^\alpha)$$
$$(p \in]0,1[; \omega > 0).$$

This is strictly increasing iff $\alpha > 0, \beta > 0$ or $\alpha < 0, \beta < 0$. In the latter case $\lim_{p\to 1} W(p) = \infty$, contrary to assumption. Thus $\alpha > 0, \beta > 0, W(1) = \lim_{p\to 1} W(p) = \omega$ and we get (12).

It is routine to show that (12) implies (11). □

A similar result holds for $W : [1,\infty[\to [\omega,\infty[$ $(\omega > 0)$, namely

$$W(q) = \omega \exp(\beta'(\ln q)^{\alpha'})$$
$$(q \in [1,\infty[; \alpha' > 0, \beta' > 0; \omega = W(1)) \tag{22}$$

(then $p, q, r \in [1,\infty[$ in (10), (11)). The proof is analogous to that of (12).

To discuss the $p, q, r \in]0,\infty[$ case, we write the analogue of (11) in the form

$$[W^{-1}([W(p)W(q)]^{1/2})]^\lambda = W^{-1}([W(p^\lambda)W(q^\lambda)]^{1/2}) \tag{23}$$

$(\lambda > 0)$ for the case $p, q, r \in]0,\infty[$.

For $p, q \in]0,\infty[$ we have $W :]0,\infty[\to]0,\infty[$; that, being an increasing bijection, is continuous. If, in particular, $p, q \in [1,\infty[$ then (22) holds; if $p, q \in]0,1]$ then (12) holds.

If $0 < p \leqslant 1 \leqslant q$ (this is the remaining case, since p and q are interchangeable) then, W being a continuous bijection, there exists an $s \in]0,\infty[$ such that

$$W(s) = [W(p)W(q)]^{1/2} =: \tau.$$

Either $s \in]0,1]$ or $s \in [1,\infty[$. The result and proof being similar, let us take, say, $s \in [1,\infty[$. Then (12) holds for p and (22) for q and s. From (22),

$$s = W^{-1}(\tau) = \exp([(1/\beta')\ln(\tau/\omega)]^{1/\alpha'}). \tag{24}$$

In view of (12), (22), and (24), we have

$$W^{-1}([W(p)W(q)]^{1/2}) = \exp\left[\left(\frac{\beta'(\ln p)^{\alpha'} - \beta(-\ln q)^\alpha)}{2\beta'}\right)^{1/\alpha'}\right], \tag{25}$$

$$W^{-1}([W(p^\lambda)W(q^\lambda)]^{1/2})$$
$$= \exp\left[\lambda\left(\frac{\beta'(\ln p)^{\alpha'} - \beta\lambda^{\alpha-\alpha'}(-\ln q)^\alpha}{2\beta'}\right)^{1/\alpha'}\right]. \tag{26}$$

Clearly, the λth power of (25) equals (26), i.e., Eq. (23) is satisfied, iff $\alpha' = \alpha$ (β' may be different from β). Thus we have proved that the general solution of (23) with $\lambda > 0, p, q \in]0,\infty[$ or, equivalently, of

$$((x,p),q) \sim ((x,r),r) \iff ((x,p^\lambda),q^\lambda) \sim ((x,r^\lambda),r^\lambda)$$
$$(p,q,r \in]0,\infty]; \lambda > 0) \tag{27}$$

or of

$$W(p)W(q) = W(r)^2 \iff W(p^\lambda)W(q^\lambda) = W(r^\lambda)^2$$
$$(p,q,r \in]0,\infty]; \lambda > 0) \tag{28}$$

is given by

$$W(r) = \begin{cases} \omega \exp[-\beta(-\ln r)^\alpha] & \text{for } r \in]0,1], \\ \omega \exp[\beta'(\ln r)^\alpha] & \text{for } r \in]1,\infty[\end{cases}$$

$(\alpha > 0, \beta > 0, \beta' > 0;\ \omega = W(1))$.

However, empirical evidence shows also $\alpha' \neq \alpha$ cases (which indicates that people treat numbers below and above 1 differently). To accommodate them, one could consider $p, q \in]0,1]$ and $p, q \in [1,\infty[$ separately, using (12) and (22), respectively, (the appearance of $\alpha' \neq \alpha$ in the psychophysical context makes the exclusion of the mixed case necessary).

On the other end, for $\alpha' = \alpha$ and $\beta' = \beta$ and assuming (27) or (28), or (23) for $p, q, r \in]0, \infty[$, we get, by combining the case $\alpha' = \alpha, \beta' = \beta$ of (12) and (22),

$$W(p) = \omega \exp(\beta \operatorname{sign}(\ln p)|\ln p|^{\alpha}) \quad \text{for } p \in]0, \infty[\quad (\omega = W(1)).$$

As pointed out by a referee, this equation satisfies the property

$$W(p)W(1/p) = W(1)^2.$$

Perhaps this consequence could be used to test the theory. To our knowledge, that has not been attempted.

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