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Empirical evaluation of a model of global psychophysical judgments: III. A form for the psychophysical function and intensity filtering

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Abstract

In part I, a concept of ratio estimation is defined and it is shown that if such estimates depend only upon the physical ratio of the signal to the reference signal, the psychophysical function must be a power function. Assuming the same exponents for each component, an invariance condition, equivalent to a sum of power functions, is studied empirically for binaural loudness. It is fully or partially sustained for 19 of 22 respondents. Since failures may be attributable to different exponents in the two ears, the ratio of the two exponents is estimated but that fails to explain the failures. Other possible explanations are suggested. In part II, an intensity filtering model is presented, accounting for the phenomenon where monaural loudness matches show a bias depending on the matching ear. We show (a) that the existence of such a bias does not alter the prior experimental results; and (b) assuming the power function, that five respondents attenuate the opposite ear and two enhance it.

Keywords: Auditory summation; Ratio production; Ratio estimation; Magnitude production; Psychophysics; Matching; Production commutativity; Multiplicative invariance; Intensity filtering; Power function

A recent global psychophysical theory of intensity perception is based on two primitives and several behavioral properties in the form of parameter-free axioms (Luce, 2002, 2004). The primitives are, in the case of auditory intensity, the loudness ordering over binaural presentations and what amounts to ratio productions that are closely related to Stevens' (1975) magnitude productions. The behavioral axioms giving rise to the representations were tested in the first two articles of this series (Steingrimsson & Luce, 2005a,b), and they were sufficiently well sustained that, for the purposes of this article, we accept the resulting representation as correct. That representation is stated explicitly in Section 1, see the following Eqs. (4)–(7).

The representation has three free functions: left and right ear psychophysical measures ψ_l and ψ_r of intensity, respectively, and a subjective distortion W of numbers, the respondents behavioral interpretation of numbers, possibly the one evidenced in magnitude estimation tasks. Having found considerable support for the representations, we turn here to the question of the exact functional form for the psychophysical functions.[2]

The article has the following structure: in Section 1 we summarize the primitives and resulting representations of the theory. In Section 2 (Part I), we present both the theoretical results of the article that relate to the form of the psychophysical function and its experimental evaluation in Sections 2.3–2.5. Section 3 (Part II) develops what may be called an intensity-filtering modification of the matching primitive that accounts for an anomaly discovered in the first experiment of this series, Steingrimsson and Luce (2005a). Although potentially important to the theory, it is shown not to affect any of the experiments we have designed to evaluate the theory. Finally, the article's results are summarized in Section 4.

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[1]This article is based, in part, on the first author's Ph.D. dissertation (Steingrimsson, 2002).

[2]Work is underway in an effort to determine the functional form of W.

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1. The primitives and representation

Let the notation (x, u) denote the simultaneous presentation of the stimuli x in the left ear and u in the right one. More precisely, x denotes the physical intensity presented to the left ear less the threshold intensity for that ear, and u is the corresponding quantity for the right ear. Further, x and u are of the same frequency and are presented in phase.[3] We assume that the respondent has a loudness ordering $\succsim$ over such intensity pairs and that it is compatible with physical intensity in the following sense: when the intensity is held constant in one ear, the loudness ordering agrees with the intensity ordering in the other ear.

1.1. Matching

The first primitive can be evaluated by any of three forms of loudness matching—left (l), right (r), and symmetric (s)—namely

$$(x,u)\sim(z_l,0), \quad (x,u)\sim(0,z_r), \quad (x,u)\sim(z_s,z_s), \tag{1}$$

where $\sim := \succsim \cap \precsim$ means equally loud. That is, respondents establish a z_i, $i = l, r, s$, such that it is perceived equally loud to (x, u). For obvious reasons, the left and right matches are called asymmetric.

Each of the z_i's is a function of both x and u, which we make explicit using the operator notation

$$x \oplus_i u := z_i \quad (i = l, r, s), \tag{2}$$

which are called *summation operators*—given our assumptions below, they can be proved to be operators. In this notation, the three matches are defined by the indifferences,

$$(x,u)\sim(x\oplus_l u, 0)\sim(0, x\oplus_r u)\sim(x\oplus_s u, x\oplus_s u). \tag{3}$$

1.2. Ratio productions

The other major primitive stems from the idea that one can present two stimuli (x, x) and (y, y), $x > y$, and a positive number p, and then ask the respondent to select the signal intensity z such that the subjective "interval" from (y, y) to (z, z) is deemed to stand in the ratio p to the subjective "interval" from (y, y) to (x, x). Note that z is a function of x, y, and p which have been provided in the experiment. We denote that function by $x \circ_p y := z$. Under the axioms of the theory, for each p the function $\circ_p$ is an operator.

1.3. The general representation

A set of necessary and sufficient behavioral axioms were given (Luce, 2002, 2004) that allowed us to construct a numerical mapping Ψ over the stimulus pairs that preserves the order $\succsim$, i.e.,

$$\Psi(x,u) \geqslant \Psi(y,v) \quad \text{iff } (x,u)\succsim(y,v), \tag{4}$$

and for which there exists a constant $\delta \geqslant 0$ such that

$$\Psi(x,u) = \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u), \tag{5}$$

and there is a strictly increasing numerical function W from the positive real numbers onto itself such that

$$W(p) = \frac{\Psi[(x,x)\circ_p(y,y)] - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} \quad (x > y \geqslant 0). \tag{6}$$

For the theories based on asymmetric matches, it also follows that

$$\Psi(x,0) = \gamma\Psi(0,x) \quad (\gamma > 0). \tag{7}$$

Although the property of constant bias, (7), is predicted by the theories based on asymmetric matches, for symmetric matches with $\delta = 0$, (7) may or may not hold. We provided empirical evidence in Steingrimsson and Luce (2005b) supporting $\delta = 0$, and one of our tasks (Section 2.2.3) is to see if constant bias seems to hold. We know from Steingrimsson and Luce (2005a) that the ears of most respondents are not symmetric in the sense that

$$(x,u)\sim(u,x) \tag{8}$$

generally fails to hold. Thus, if (7) is satisfied, then for most respondents $\gamma \neq 1$

It is useful to define three types of one-dimensional psychophysical functions:

$$\psi_l(x) := \Psi(x,0), \tag{9}$$

$$\psi_r(u) := \Psi(0,u), \tag{10}$$

$$\psi_s(x) := \Psi(x,x). \tag{11}$$

Whenever possible we state results for the generic ψ_i, $i = l, r$, and sometimes s.

Nothing in the theory so far dictates the explicit mathematical forms for ψ_i as a function of the physical intensity x nor of W as a function of p. Note that by (5), it is sufficient to understand what function $\psi_l(x)$ is of x and what function $\psi_r(u)$ is of u. This article begins to address the issues of the forms of these unspecified psychophysical functions. The issue of the form of W is currently under investigation.

2. Part I: ratio production, ratio estimation, and the psychophysical function

2.1. Ratio estimation

Ratio production is a fundamental primitive of our theory. But the dual process of ratio estimations for which the respondent is asked to state numerically the perceived ratio relation $t = z/x$ between two experimenter-presented signals x and z is not part of the axiomatization. We remedy that theoretical lacuna.

[3] Presumably the theory would also work for noise signals that are statistically identical. This experiment should be carried out. The theory has not been extended to deal explicitly with frequency and/or sum of a few pure tones.

Related to ratio estimation is the method of magnitude estimation, which is largely due to the influential contributions of Stevens, which are summarized in the posthumous book Stevens (1975). In some versions of magnitude estimation, the experimenter chooses a signal s, called a *standard*, and assigns it a number m, called the *modulus*. Usually the modulus is selected to be a number well above one so that fractions can be avoided because many people are not at ease with them. This procedure is sometimes called *magnitude estimation with a standard*. The unmodified term *magnitude estimation* is used when no standard is specified and the respondent is allowed to select his or her own standard on each trial so long as ratios are, in some sense, preserved (forthcoming work explores some possibilities for respondent-based standards).

Within the framework of the current theory, a natural interpretation of ratio estimation can be given in terms of (6) with $y = 0$. Instead of producing $t_i(x,p) = x \circ_{p,i} 0$, $i = l, r, s$, such that the intensity $t_i(x,p)$ stands in the ratio p to x, the respondent is asked to state the value of p_i that corresponds to the subjective ratio of z to x, i.e., that of $t = z/x$. It is convenient to substitute $z = tx$ in the equations. Thus, the experiment provides information about the ratio estimation function $p_i = p_i(t,x)$. This value, generated by the respondent, may be called the *perceived ratio* of z to x. According to (6) and using the definitions of ψ_i, $i = l, r, s$, (9)–(11),

$$W(p_i(t,x)) = \frac{\psi_i(tx)}{\psi_i(x)}. \tag{12}$$

This relation among the three unknown functions, ψ_i, p_i, W, plays a fundamental role in this article.

Fagot (1981) proposed the relationship of (12) under the assumptions that ψ_i is a power function and that $W(p)$ is proportional to p, but with different slopes depending on whether $t \geqslant 1$ or <1, which he called a bias—a usage different from ours. An argument is given below for the power function form for ψ_i, but on-going research work certainly suggests that W is more complex than Fagot's proposal.

2.1.1. Independence of the reference signal

A small part of the literature has focused on the question of whether or not the ratio estimation function $p_i(t,x)$ is actually independent of the standard reference signal x:

$$p_i(t,x) = p_i(t) \quad (i = l, r, s). \tag{13}$$

This certainly seems to have been an implicit assumption throughout Stevens' (1975) work. As we show below in Proposition 1 this assumption yields a power function representation.

The two most directly relevant empirical publications of which we are aware are Hellman and Zwislocki (1961) and Beck and Shaw (1965), both of which explore the effect of changing the standard on the magnitude estimates of loudness. Beck and Shaw (1965) also show the impact of changes of the modulus. These data are graphed in Fig. 1(a), (b) and their slope relationships are summarized in Table 1.

Hellman and Zwislocki (1961) used symmetric stimuli (x,x) and five different standards with each separately

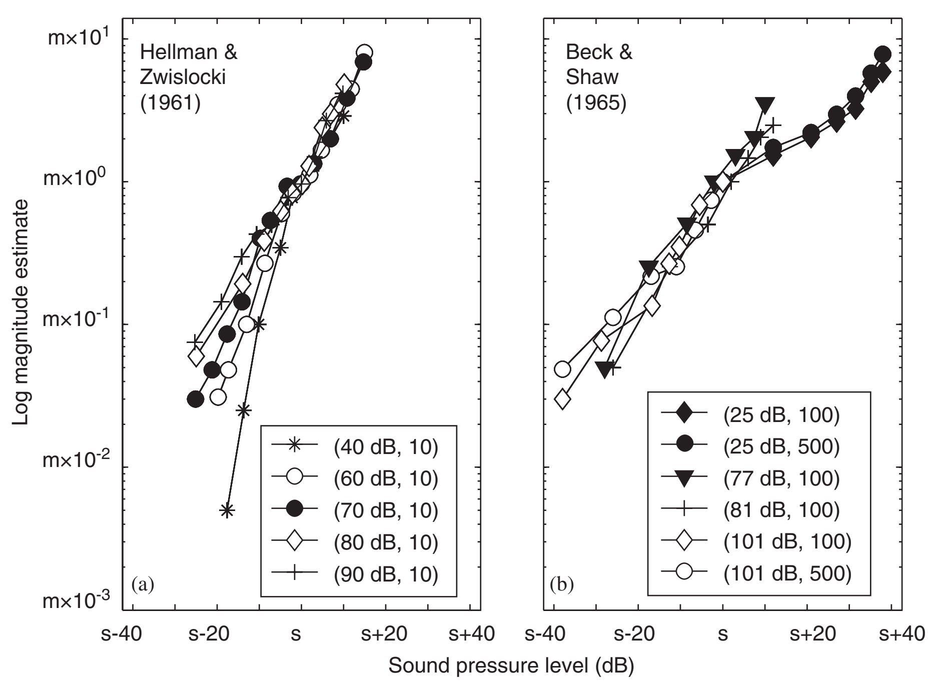

Fig. 1. Panel (a) contains auditory data adapted from Fig. 6 of Hellman and Zwislocki (1961). Plotted in panel (b) are data adapted from the data in Fig. 1 of Beck and Shaw (1961). Each graph shows results of magnitude estimates as a function of stimuli in dB SPL and with respect to a common standard (s) and modulus (m), indicated as (s, m).

Table 1
Summary of the relationship between the standard/modulus values and the slope of magnitude estimates below and above standards, respectively. Each pair of numbers indicates the standard and modulus (*s*,*m*), where the standards are recoded as $1, 2, \ldots$, going from low to high intensity

Study	Below standard
H & Z	$(1,10) > (2,10) > (3,10) > (4,10) > (5,10)$
B & S	$(2,100) > (3,100) > (3,500)$
HZ & BS	$1 > 2 > 3 > 4 > 5 > 6 > 7$
	Above standard
H & Z	No discernible effect
B & S	$(2,100) > (1,100) \geqslant (1,500)$
HZ & BS	$6 \approx 5 \approx 4 \approx 3 \approx 2 > (1,100) > (1,500)$

assigned the modulus value of 10. They had nine respondents provide ratio estimates, using a 1000 Hz pure tone stimuli, to five different standard pairs $(x_0, 10)$ where $x_0 = 40, 60, 70, 80, 90$ dB SPL. The geometric-mean results for the respondents are shown in their Fig. 6. If one shifts the intensity scale (in dB) so that all the standard pairs are at the same point of the graph, their plot is transformed to Fig. 1(a).

Several aspects of these data are notable. First, for values above the standard, there does not seem to be any difference in the curves in Fig. 1(a), in agreement with (13). But things are not so favorable for values below the standard. Second, the slopes well below the standards for the three larger standards are fairly similar and somewhat less steep than those for the two lower standards. This belies the assumption of independence of the standard, at least below the standard. Third, the slopes below the standard differ from those above it.

We suspect, but have no supporting evidence, that part of the reason behind these variable slopes below the standard may have to do with the respondents feeling bounded from below and being reluctant to use fractions less than 1. It therefore seems reasonable to do the study with moduli of, say, 100 or larger.

This is exactly what Beck and Shaw (1965) did: they collected magnitude estimates of loudness as a function of four standards, 25, 77, 81, and 101 dB SPL, and two moduli, 100 and 500 (incomplete factorial design) using a binaural 1000 Hz pure tone and 25–27 respondents in each condition and reported their median estimates. They collected data for both even and irregularly spaced stimuli, but concluded the results were the same. Hence, we have averaged over the stimulus spacing conditions. In our Fig. 1(b), we have replotted their data by shifting them to a common standard (s) and modulus (m) and on the same scale as those of Hellman and Zwislocki (1961) (our Fig. 1(a)). The legend indicates the standard and modulus corresponding to each graph, e.g. (25 dB, 100) means a standard of 25 dB and a modulus of 100. Note, only the 77 and 81 dB conditions extend both below and above the standard. Here we find, contrary to the data of Hellman and Zwislocki (1961), for values below the standard, there does not seem to be much, if any, difference in the slope of the curves, which agrees with (13). However, for at least two of the four graphs, the slope is shallower for values above as compared to below the standard. The shallower slopes above the standard are both for graphs generated by the lowest standard (25 dB), where as for the higher standards (77 and 81 dB), the slopes appear unchanged on either side of the standard. It is as if the respondents had established an upper bound to the response scale, and so exhibited response attenuation to achieve that.

In our theory, one should treat the abscissa as the intensity less than the threshold intensity, which neither Hellman and Zwislocki (1961) nor Beck and Shaw (1965) had any reason to do. This has the potential of changing the slopes closest to threshold, i.e., for Hellman and Zwislocki's (1961) data below the standard ($p < 1$) but not for intensities well-above threshold ($p > 1$). They reported an average threshold of 6 dB SPL, which is clearly too small to alter the results in a material way. Nevertheless, were these experiments to be repeated, it would be important, from our perspective, to collect data on individual thresholds and plot the data in terms of the intensity less the threshold intensity for individual respondents.

In an effort to glean a trend from these data, we summarize in Table 1 the relation between the slopes of the magnitude estimate curves and standards/moduli. We did this both separately for the two studies (Beck & Shaw, 1965; Hellman & Zwislocki, 1961) as well as by collapsing the data across them. In the table, we order, by visual inspection, the slopes of the curves above and below the standard from the highest to the lowest, where each curve is indicated as in the graph legends, i.e., (standard, modulus). For the data collapsed across the two studies, only intensities are compared. We found it helpful to recode the standards as $1, 2, \ldots$, going from low to high intensity, so we did that. For the Beck and Shaw (1965), the 77, 81 dB standard conditions are collapsed because we could not see much difference between them; in the inter-study comparison, we collapsed over the standard conditions of 77, 80, 81 dB and ignored the modulus values.

Immediately striking is the fact that for both studies the data are consistent with slopes below the standard decreasing with increasing standards and above it increasing with decreasing standards. This also holds true for the inter-study comparison of the two experiments.

Although the evidence for the effects of moduli on slopes is less rich than for the intensities, a clear trend emerges when the data of the two studies are combined. Namely, the data are consistent with the slopes, both below and above the standard, increasing with decreasing moduli.

Poulton (1968), who examined both the data of Hellman and Zwislocki (1961) and those of Beck and Shaw (1965) seems to come to a conclusion similar to ours, when he writes: "When the physical magnitude of the standard is near the lower end of the range, variables smaller than the standard give steeper slopes than variables larger than the standard. Conversely, when the standard is near the upper end of the range, variables larger than the standard give the steeper slopes.'' (p. 6). In his view, these slope differences can be considered experimental artifacts of standards at the extreme end of the audible intensity spectrum as well as moduli that do not accord well with the way most people use numbers, e.g., their presumed discomfort with using fractions compared to integers. He models these effects in his Fig. 1C, according to which there is a range of standards and moduli for which $p(z,t) = p(t)$ is true in magnitude estimation. Although we do not test this hypothesis, the assertion that pairs of standards and moduli can be chosen such that magnitude estimates above and below the standard are the same, does accord with the available data. That is ratio independence, (13), is satisfied in at least some cases.

If indeed Poulton's (1965) assertion is correct, namely that certain choices of standards and moduli in effect introduce experimental artifacts, it suggests that relying on ratio productions rather than estimations, as do Steingrimsson and Luce (2005a,b), is to be preferred experimentally.

2.2. Power functions

2.2.1. Derivation of power functions from invariance of the standard

In any event, we explore further the consequences of assuming that ratio independence, (13), is satisfied.

Proposition 1. *Suppose that* (12) *holds. Then ratio independence* (13) *is equivalent to, for* $i = l, r$,

$$\psi_i(x) = \alpha_i x^{\beta_i} \quad (x \geqslant 0), \tag{14}$$

$$W(p_i(t)) = t^{\beta_i} \quad (t \geqslant 0), \tag{15}$$

where $\alpha_i > 0, \beta_i > 0$ *are constants.*

A proof is given in the Appendix.

Assuming that ψ_i is a power function, (14), then the following inverse relationships hold between ratio productions $t_i(p)$ and estimates $p_i(t)$ that are defined by:

$$t_i(p) := W(p)^{1/\beta_i} \quad (p \text{ given}), \tag{16}$$

$$p_i(t) := W^{-1}(t^{\beta_i}) \quad (t \text{ given}). \tag{17}$$

These expressions have two important implications.

First, although ratio estimation, (12), leads to the psychophysical function $\psi_i(x)$ being a power function, neither the empirical ratio production function nor the estimation function is itself predicted to be a power function unless W is also a power function. It is trivial from (15) that $p_i(t)$ is a power function iff W is a power function, but not necessarily with $W(1) = 1$. Ample data reject that W is a power function if $W(1) = 1$ is assumed: Ellermeier and Faulhammer (2000) for $p, q > 1$; Zimmer (2005) for $p, q < 1$. In current work, we are exploring how well a power function works with $W(1) \neq 1$.

Second, within this theory and on the assumption that ratio estimates are independent of the standard, we conclude that ratio production and ratio estimation are simply inverse functions, but in general they are not linearly related. This prediction seems, at first, not to accord very precisely with the data of Stevens (1975, Fig. 12, p. 31), on the so-called regression effect. There, both sets of data are plotted with numbers corresponding to p in logarithmic units on the ordinate and t, the ratio of the signal produced to the reference signal, in dB on the abscissa. The empirical finding is that the production slope is somewhat greater than the estimation one. However, depending on the exact form for W, these relations may not really be linear as Stevens assumed—although they are if W is a power function—and so his observation that the functions are not inverse is a, more or less, crude approximation. Until the form of W is known, we cannot be sure exactly what is predicted.

2.2.2. A testable property of power ψ_i: multiplicative invariance

Consider the property:

σ-Multiplicative Invariance (σ-MI)[4] : For all signals x, u and for all factors $\lambda > 0$, there is some constant $\sigma > 0$ such that

$$\lambda x \oplus_i \lambda^\sigma u = \lambda_i (x \oplus_i u) \quad \left(\lambda_i := \begin{cases} \lambda, & i = l \\ \lambda^\sigma, & i = r \end{cases} \right), \tag{18}$$

where $\oplus_i$, $i = l, r$ is given by (3).

Proposition 2. *The following statements are equivalent*:

(i) *The representation* (5) *holds with*[5] $\delta = 0$ *and* σ*-MI is satisfied.*
(ii) *The form of the psychophysical function is*

$$\Psi(x,u) = \alpha_l x^{\beta_l} + \alpha_r u^{\beta_r}, \tag{19}$$

where $\alpha_i > 0, \beta_i > 0$ $(i = l, r)$ *are constants and* $\sigma = \beta_l / \beta_r$.

The proof is given in the Appendix.

Note that

$$\frac{\Psi(x,0)}{\Psi(0,x)} = \frac{\alpha_l}{\alpha_r} x^{\beta_l - \beta_r}.$$

This ratio is constant (7) iff $\sigma = 1 \Leftrightarrow \beta_l = \beta_r$.

For the case where $\delta > 0$ and assuming that the ψ_i are power functions, the overall function becomes

$$\Psi(x,u) = \alpha_l x^{\beta_l} + \alpha_r u^{\beta_r} + \delta \alpha_l \alpha_r x^{\beta_l} u^{\beta_r}. \tag{20}$$

[4]In mathematics, the property 1-MI is usually described as $\oplus_i$ being a first-order homogeneous operator.

[5]Steingrimsson and Luce (2005b) gave supporting data.

This representation falls within our theoretical framework only if (7) holds, i.e., $\beta_l = \beta_r$. We do not know of a behavioral condition comparable to (18) that characterizes the form (20). Finding one is an interesting, if apparently difficult, open problem.

2.2.3. Estimating σ

The purpose of the following result is to provide means to estimate the important parameter σ.

Proposition 3. *Assume that the sum of power functions representation* (19) *holds. Then*:

$$(0 \oplus_l x)_{\mathrm{dB}} = c_1 + \frac{1}{\sigma} x_{\mathrm{dB}}, \tag{21}$$

$$(x \oplus_r 0)_{\mathrm{dB}} = c_2 + \sigma x_{\mathrm{dB}}, \tag{22}$$

where with $\log := \log_{10}$

$$c_1 = -\frac{10}{\beta_l} \log \gamma,$$

$$c_2 = \frac{10}{\beta_r} \log \gamma,$$

and $(x \oplus_i y)_{\mathrm{dB}} := 10 \log (x \oplus_i y)$.

A simple proof of this proposition is in the Appendix.

Corollary 4. *The following expressions also hold*:

$$(x \oplus_r 0)_{\mathrm{dB}} = c_3 + \sigma^2 (0 \oplus_l x)_{\mathrm{dB}}, \tag{23}$$

and

$$\log \gamma = \frac{\beta_r c_3}{10(1+\sigma)}, \quad (c_3 = c_2 - \sigma^2 c_1). \tag{24}$$

The proposition provides two ways to estimate σ if the power function representation obtains, which is combined into a single way in (23) of the corollary. Probably the latter is the most useful one. The expressions for c_1, c_2, and (24) each state a relation between γ and β_r but these are not sufficient, alone or together, to estimate either parameter.

2.3. Experimental methods

The first experiment described below was aimed at testing 1-MI, which is (18) with $\sigma = 1$ (Experiment 1). This empirical study was carried out before we knew Proposition 2 for $\sigma \neq 1$. As a consequence of those results, we turned to the question of whether or not the 1-MI results were likely to change by doing the σ-MI experiment (Experiment 2). The two experiments have a number of testing strategies in common that are now outlined. Other aspects are described later as relevant. Because most of the methods employed here are identical to those used in Steingrimsson and Luce (2005a), we give here only an abbreviated account of the methods.

2.3.1. Signal presentations and notation

The experiments were carried out in the auditory domain using a 1000 Hz sinusoidal tone presented for 100 ms, which included 10 ms on and off ramps.

The theory is cast in terms of intensities less the threshold intensity, i.e., for the left ear $x = x' - x_\tau$. Similarly, for the right ear we have $u = u' - u_\tau$. In experimental descriptions we report x' in dB SPL instead of $x_{\mathrm{dB}} = (x' - x_\tau)_{\mathrm{dB}}$, and likewise for u'. All signals used are well-above threshold using respondents with normal hearing, for which they were selected, and so the resulting errors are negligible.

2.3.2. Respondents

A total of 22 students—graduate and undergraduate, seven males and 15 females—from the University of California, Irvine (UCI), and New York University (NYU) participated in the experiments reported in this article. The first author was one of them (R22).[6] All respondents were within 20 dB of normal hearing thresholds (ANSI, 1996) in the range 250–8000 Hz, assessed by an audiometric test (Micro Audiometric EarScan ES-AM).

All respondents, except the first author, were compensated \$10 per session. Each person provided written consent and was treated in accordance with the "Ethical Principles of Psychologists and Code of Conduct" (American Psychological Association, 1992). Consent forms and procedures were approved by the UCI's and NYU's respective Institutional Review Boards.

2.3.3. Equipment

At UCI, stimuli were generated digitally using a personal computer and played through a 16-bit digital-to-analog converter (Quikki; Tucker-Davis Technology), at conversion rate of 40μ's per sample. Presentation level was controlled by manual and programmable attenuators, and stimuli were presented over Sennheiser HD265L headphones to the listener seated in an individual, single-walled, IAC sound booth. At NYU, stimuli were generated digitally using a personal computer and played through a 24-bit digital-to-analog converter (RP2.1 Real-time processor, Tucker-Davis Technology). Presentation level was controlled by build-in features of the RP2.1 and stimuli were presented over Sennheiser HD265L headphones to the listener seated in an individual, double-walled, IAC sound booth.

At both universities, a 85 dB SPL safety limit was imposed in all experiments.

2.3.4. Procedure

Experiments were conducted in sessions lasting no more than 1 h. All respondents completed one training session

[6]Because the behavioral estimates of matching and ratio production that enter into our tests do not seem to be affected one way or the other by knowledge of the experimental design, we judged it acceptable to use a knowledgeable respondent.

with loudness matching. Because some observers participated in multiple experiments, the total practice that individuals had prior to any one experiment varied substantially. Depending on the experiment, practiced respondents typically completed 60–64 estimates per session.

2.3.5. Estimating one-ear and two-ear matches

The three types of matches used are listed in (3). Let $\langle A, B\rangle$ denote a presentation of A followed by a temporally displaced presentation of B. We used a temporal delay of 450 ms between A and B. Three trial types, corresponding to (3), were used

$$\langle (x,u),(z_l,0)\rangle, \tag{25}$$

$$\langle (x,u),(0,z_r)\rangle, \tag{26}$$

$$\langle (x,u),(z_s,z_s)\rangle. \tag{27}$$

That is, respondents heard a tone followed 450 ms later by another tone in the left, right, or both ears depending upon the experimental condition. Following the tone presentation, respondents used key presses to either adjust the intensity of z_i, $i = l, r, s$, to repeat the previous trial, or to indicate satisfaction with the loudness match. Adjustments were freely done in steps of 0.5, 1, 2 or 4 dB.

After an intensity adjustment, the altered tone sequence was played. This process was repeated until the respondent was satisfied with the match.

In verbal instructions to respondents, the task was explained as that of making the second stimulus equal in loudness to the first. The instructions stressed the importance of paying attention solely to the loudness of the stimuli and ignoring the subjective sense of tone location.

2.3.6. Statistical method

As in the previous articles, this one also examines a number of parameter-free null hypotheses of the form $L_{\text{side}} = R_{\text{side}}$. This reflects the nature of the empirical axioms being tested. If the hypothesis $L_{\text{side}} = R_{\text{side}}$ is correct, we are asserting that both L_{side} and R_{side} are drawn from the same distribution. Because we do not have a theory that predicts the distributions of our estimates, we therefore chose the nonparametric Mann–Whitney U test for our statistical evaluation, with a significance level of .05.

Although medians would be preferable to means if we could accurately estimate them, the discrete nature of the signal values makes the mean a better estimate provided that their distribution is approximately Gaussian, which they appear to be. So we report means. To indicate variability in adjustments, we report the standard deviations.

A particular concern is whether the sample sizes for L_{side} and for R_{side} are sufficiently large so that a true failure of the null hypothesis can be distinguished within the power of the statistical method employed. To address this issue all statistical results were verified using Monte Carlo simulations based on the bootstrap technique (Efron & Tibshirani, 1993, see Steingrimsson & Luce, 2005a, for details). We asked whether L_{side} and R_{side} could, at the .05 level, be argued to come from the same underlying distribution. This was our criterion for accepting the null hypothesis as supporting the behavioral property.

2.4. Experiment 1: 1-multiplicative invariance

The empirical study of this section was carried out before we knew Proposition 2 for $\sigma \neq 1$. The experiment was designed on the assumption that $\sigma = 1$ and so 1-MI was studied, i.e.,

$$\lambda x \oplus_i \lambda u = \lambda(x \oplus_i u) \quad (\lambda > 0, i = l, r). \tag{28}$$

Failures of (28) can arise for either of two reasons: because the power representation (19) is incorrect or because the respondent did not exhibit constant bias (7) which in the presence of (19) is equivalent to $\sigma \neq 1$.

2.4.1. Method

The test is carried out in two steps: the first, an experimental one, consists of two respondent estimates:

$$t_i = (\lambda x) \oplus_i (\lambda u) \quad \text{and} \quad z_i = x \oplus_i u \quad (i = l, r).$$

This is followed by a purely "arithmetic" step in which the experimenter calculates $t_l' = \lambda \times z_i$ where $i = l$ or $= r$. The property 1-MI is said to hold if t_i and t_i' are found to be statistically equivalent.

That property was tested using one-ear matching with the respondents split between left and right ear matching. We did not understand at the time that we could have used symmetric matching.

For the present experiment we used $x = 64$ dB and $u = 70$ dB and two values for λ_{dB}, 4 and -4 dB ($\lambda = 2.5$ and 0.4, respectively). Of course, in dB notation, the impact of the multiplicative factor λ is additive.[7] The trial forms used to estimate t_i and z_i, with $i = l, r$, are given by expressions (25) and (26), respectively.

2.4.2. Results

Data for 10 respondents who completed the experiment at UCI and the 12 at NYU are reported in Tables 2 and 3, respectively. In the tables, T_1 and T_2 stand for the mean of t and t', respectively—standard deviations are in parentheses. The number of observations for each of T_1 and T_2 is reported as n. Left and right ear matching is marked with l and r, respectively.

Of the 22 respondents who participated in the experiment, three (R2, R20, R22) failed both tests, seven (R13, R16, R24, R41, R43, R44, R46) failed one of the two, and

[7] The behavioral properties are formulated in terms of physical intensity, but the testing is specified using decibel measure. In most cases the distinction does not cause confusion, but obviously the distinction is crucial in the current case.

Table 2
Experiment 1: 1-multiplicative invariance, UCI data

Resp.	i	λ_{dB}	Mean (s.d)		p_{stat}	n	Stat. trend
			T_1	T_2			
R2	l	4	76.15 (1.92)	78.30 (2.42)	<.001	40	$T_1 \neq T_2$
		−4	71.36 (3.59)	70.30 (2.42)	.015		$T_1 \neq T_2$
R5	l	4	76.53 (3.02)	76.74 (3.03)	.409	80	$T_1 = T_2$
		−4	69.18 (3.69)	68.74 (3.03)	.169		$T_1 = T_2$
R7	r	4	75.47 (3.90)	76.73 (4.53)	.143	60	$T_1 = T_2$
		−4	69.78 (3.92)	68.73 (4.53)	.146		$T_1 = T_2$
R8	r	4	75.38 (0.80)	75.07 (1.32)	.212	60	$T_1 = T_2$
		−4	67.42 (1.68)	67.07 (1.32)	.163		$T_1 = T_2$
R11	r	4	75.99 (1.72)	76.36 (2.46)	.281	38	$T_1 = T_2$
		−4	67.79 (3.31)	68.36 (2.46)	.610		$T_1 = T_2$
R12	l	4	76.24 (1.37)	75.68 (2.01)	.129	60	$T_1 = T_2$
		−4	67.02 (2.55)	67.68 (2.01)	.113		$T_1 = T_2$
R13	r	4	75.37 (1.58)	75.57 (1.43)	.726	41	$T_1 = T_2$
		−4	68.33 (2.06)	67.57 (1.43)	.048		$T_1 \neq T_2$
R16	r	4	74.63 (1.61)	74.79 (1.43)	.646	80	$T_1 = T_2$
		−4	65.96 (2.03)	66.79 (1.43)	.003		$T_1 \neq T_2$
R20	r	4	75.30 (3.48)	77.54 (2.99)	<.001	80	$T_1 \neq T_2$
		−4	71.14 (3.63)	69.54 (2.99)	.001		$T_1 \neq T_2$
R24	l	4	73.56 (1.51)	74.45 (2.04)	.010	40	$T_1 \neq T_2$
		−4	66.56 (2.06)	66.45 (2.04)	.860		$T_1 = T_2$

Table 3
Experiment 1: 1-multiplicative invariance, NYU data

Resp.	i	λ_{dB}	Mean (s.d)		p_{stat}	n	Stat. trend
			T_1	T_2			
R10	l	4	74.07 (0.91)	74.17 (1.02)	.697	30	$T_1 = T_2$
		−4	66.73 (1.54)	66.17 (1.02)	.120		$T_1 = T_2$
R22	l	4	71.45 (0.88)	72.07 (0.94)	.003	40	$T_1 \neq T_2$
		−4	64.81(1.52)	64.07 (0.94)	.036		$T_1 \neq T_2$
R37	l	4	74.55 (1.58)	74.65 (2.25)	.864	30	$T_1 = T_2$
		−4	65.37 (2.97)	66.65 (2.25)	.099		$T_1 = T_2$
R38	r	4	75.13 (1.82)	75.47 (1.76)	.288	40	$T_1 = T_2$
		−4	67.64 (1.67)	67.47 (1.76)	.872		$T_1 = T_2$
R39	r	4	75.60 (0.98)	75.47 (1.44)	.611	30	$T_1 = T_2$
		−4	67.03 (2.32)	67.47 (1.44)	.364		$T_1 = T_2$
R40	r	4	74.95 (1.28)	74.4 (1.91)	.291	30	$T_1 = T_2$
		−4	65.63 (2.19)	66.43 (1.91)	.140		$T_1 = T_2$
R41	l	4	73.68 (2.79)	71.50 (3.56)	.016	30	$T_1 \neq T_2$
		−4	64.12 (5.09)	63.50 (3.56)	.882		$T_1 = T_2$
R42	r	4	73.70 (1.01)	73.95 (1.51)	.438	40	$T_1 = T_2$
		−4	66.42 (1.55)	65.95 (1.51)	.085		$T_1 = T_2$
R43	l	4	73.77 (2.38)	75.17 (2.36)	.029	30	$T_1 \neq T_2$
		−4	67.38 (3.55)	67.17 (2.36)	.553		$T_1 = T_2$
R44	l	4	76.65 (1.09)	75.83 (1.07)	.005	30	$T_1 \neq T_2$
		−4	67.45 (1.44)	67.83 (1.07)	.144		$T_1 = T_2$
R45	r	4	76.03 (1.83)	75.93 (2.53)	.858	30	$T_1 = T_2$
		−4	68.07 (2.28)	67.93 (2.53)	.688		$T_1 = T_2$
R46	l	4	72.41 (1.56)	72.74 (1.75)	.406	40	$T_1 = T_2$
		−4	66.06 (2.87)	64.74 (1.75)	.014		$T_1 \neq T_2$

12 (R5, R7, R8, R10, R11, R12, R37, R38, R39, R40, R42, R45) did not reject the invariance condition. Although the test was accepted slightly more often in the NYU sample than in the UCI one, the difference is far too small to suggest any systematic differences between the samples.

2.4.3. Discussion

With the test failing in 14 out of 44 tests, the experiment does not provide strong evidence for 1-MI holding for more than about 50% of the young people sampled. As was noted earlier, the rejections might come about in several ways: a failure of $\delta = 0$, or a failure of 1-MI, in which case either $\sigma \neq 1$, or a failure of the functions ψ_i, $i = l, r$, to be exactly power functions. Of course, as discussed in Section 2.1.1, we know the ratio independence assumption, which underlies power functions, is not necessarily always true. Also, as noted, these apparent failures are plausibly related to the choices of standards and moduli in magnitude estimation studies. Here, these potential problems are absent. Particularly noteworthy is that with 19 out of 24 respondents passing one or both of the conditions, the data seem to suggest that the sum of power functions closely approximates behavior but that for some respondents the ears differ sufficiently so that $\sigma = 1$ will fail. Certainly, our results favoring power functions accords with numerous experiments done over the years. However, those were mostly group data and in the few cases where data from individual respondents have been reported, deviations from power functions were seen.[8]

2.5. Experiment 2: estimates of σ

In Experiment 1, we arrived at the approximate estimate of about half of the respondents satisfying multiplicative invariance with $\sigma = 1$. This provided motivation to explore whether or not the 1-MI results were likely to change by doing the σ-MI experiment, which first involves estimating σ. These results led us to test σ-MI for just one respondent. Although, strictly, it is a different experiment, we fold the discussion of its execution and results into the discourse of the current experiment. The discussions of the σ estimates and σ-MI testing are provided in separate subsections.

2.5.1. Method

We estimated σ based on the estimates of $z_l = 0 \oplus_l x$ and $z_r = x \oplus_r 0$ using (23), i.e., matching a stimulus presented in one ear to another in the other ear. These were obtained using the trial forms given by (25) and (26), respectively, with three instantiations of x, namely 58, 66, and 74 dB SPL. With two trial forms and three instantiations of each, a total of six estimates needed to be made. Initially, the estimates of z_l and z_r were obtained in separate sessions,

[8] A reviewer pointed out that in several cases where equality of medians is statistically rejected, the magnitude of the mean differences in sound pressure level are close to the average discrimination threshold, suggesting that perhaps our statistical analysis is overly conservative. We agree, but in the absence of a demonstratively better objective criterion, reinterpretation of the results seems difficult. This is especially delicate since we, in essence, attempt to accept the null hypothesis (see Section 2.3.6). It would be desirable in the future to improve our understanding of these issues.

but fearing there might be inter-session effects, we switched to collecting both estimates within blocks of trials.

Different coefficients are not unexpected depending on whether one regresses on $(0 \oplus_l x)_{\text{dB}}$ or on $(x \oplus_r 0)_{\text{dB}}$ in (23), hence we do both regressions and report the geometric mean of σ estimated both ways. Using the estimate $\widehat{\sigma}$ from the linear regression, we statistically tested whether or not $\widehat{\sigma} = 1$.

Because c_1 and c_2 are estimated using (21) and (22), we also explored whether a σ estimated using those two equation differed from that of using (23). We calculate $\widehat{\sigma}_l$ and $\widehat{\sigma}_r$ from (21) and (22), respectively, and report their geometric mean estimate, i.e., $\widehat{\sigma}^* = (\widehat{\sigma}_l \widehat{\sigma}_r)^{1/2}$. We found $\widehat{\sigma} < 1$ for five out of seven cases when regressing on $(0 \oplus_l x)_{\text{dB}}$ but in only in one of seven cases when regressing on $(x \oplus_r 0)_{\text{dB}}$. We do not understand why that is the case. These tendencies were not mirrored in the estimates $\widehat{\sigma}_r$ and $\widehat{\sigma}_l$, which were both less than one for one out of seven cases.

2.5.2. Results

Results for seven respondents are given in Table 4. For five of the seven (all but R39 and R42), sufficient number of observations of z_l and z_r were collected in separate sessions to estimate σ. When these estimates were compared to those obtained from data collected in the same session, the two estimates were not found to differ materially. It is of some interest that the matching data appear to be relatively immune to inter-session effects, something that is not at all the case for the productions. Hence, for those five respondents, all the collected data were pooled together to provide the reported estimates. In Table 4, estimates for $\widehat{\sigma}^*$ and $\widehat{\sigma}$ are both reported. However, only the p-value for the test $\widehat{\sigma} = 1$ is reported and is indicated along with the statistical trend indicated by that test. As an indicator of variability of the estimate $\widehat{\sigma}$, its associated standard error is reported.

The largest difference between $\widehat{\sigma}^*$ and $\widehat{\sigma}$ was 0.080, which is sufficiently small to be immaterial. For three of seven respondents, R10, R39, R46, σ was found to be statistically different from one.

Table 4
Experiment 2.5: estimating σ

Respondent	$\widehat{\sigma}^*$	$\widehat{\sigma}$	$\text{SE}_{\widehat{\sigma}}$	p_{stat}	n	Stat. trend
R10	0.907	0.916	0.035	.017	60	$\sigma \neq 1$
R22	1.060	1.034	0.033	.304	70	$\sigma = 1$
R37	1.046	1.047	0.035	.215	60	$\sigma = 1$
R38	1.000	1.004	0.038	.917	30	$\sigma = 1$
R39	1.208	1.204	0.035	<.001	60	$\sigma \neq 1$
R42	1.059	1.060	0.051	.242	40	$\sigma = 1$
R46	1.349	1.269	0.050	<.001	65	$\sigma \neq 1$

2.5.3. Discussion

Our aim was to explore whether or not the 1-MI results from Experiment 1, were likely to change by doing the σ-MI experiment, which first involves estimating σ. Previously (see Tables 2 and 3) we did not reject 1-MI for R10, R37, R38, R39, R42, which strongly suggests that for these respondents $\sigma \approx 1$. Of these five respondents, R10 and R39 are estimated to have $\sigma \neq 1$ but for the rest $\sigma = 1$ is acceptable. Neither R22 nor R46 passed both conditions of 1-MI in the prior test but statistically, only R46 had $\sigma \neq 1$.

The important questions are how good do the estimates of σ appear to be and what do these results say about the value of carrying out the σ-MI experiment?

To address these questions we first asked: in which direction would σ have to deviate from one in order to alter the previous data testing 1-MI (Experiment 1) toward equality of the property's two sides? With reference to Experiment 1, we observe that a value of $\sigma < 1$ is equivalent to the prediction that the value named T_1 should be decreased in case of $\lambda_{\text{dB}} = 4\,\text{dB}$ and increased for $\lambda_{\text{dB}} = -4\,\text{dB}$, and vice versa for $\sigma > 1$.

The result of this investigation is summarized in Table 5. The first column indicates the group (failed/passed 1-MI), the second and third columns, labeled $\sigma < 1$ and $\sigma > 1$, indicate the numerical direction for σ needed to make T_1 closer to T_2. The fourth column indicates the number of cases where the two conditions of 1-MI provide contradictory indication of the direction needed for σ. To facilitate comparison with the results in the current experiment, the last three columns summarize the results of the σ estimations.

First, the contradictory results are in all cases for results where T_1 and T_2 were very close numerically, hence these cases are likely just an issue of variance.

Looking first at the trend for what the numerical direction of σ would need to be in order to bring the

Table 5
Summary of numerical direction of σ needed to fit data and obtained estimates

1-MI	Total	Based on Experiment 1		Contradictory	Total	Based on Experiment 2	
		Needed				Numerical	
		$\sigma < 1$	$\sigma > 1$			$\widehat{\sigma} < 1$	$\widehat{\sigma} > 1$
Passed	12	2	7	3	5	1	4
Failed	10	2	7	1	2	0	2
Total	22	4	14	4	7	1	6

1-MI results closer to accept, then, excluding the contradictory cases, 14 of the 18 consistent ones needed a value of $\sigma > 1$ and four needed < 1. In the subset of seven respondents for which we actually obtained estimates of σ, six had estimates of $\sigma > 1$. Thus the trend between the needed values and estimated ones is similar. In this respect, the estimation results appear satisfactory.

Looking more closely, we see that for R10 and R39 the numerical direction of the estimation results are opposite to what is needed to improve the 1-MI fit. For R42, the estimated and needed direction for σ were contradictory which, however, is not too surprising because the estimate of σ is close to one. For the remaining respondents, the needed and estimated numerical directions of σ agreed. This means the pattern of results appears reasonable for five and inappropriate for two of the seven respondents.

Only R46 both failed 1-MI and showed an estimated σ that matched the direction of the needed one. From (18) with a $\lambda_{\mathrm{dB}} = 4\,\mathrm{dB}$ and $\sigma = 1.269$, we calculate a correction factor of 5.1 dB, which, while not large, may suffice for the respondent to pass σ-MI.

Similarly, the correction factors for R10 and R39 are 3.7 and 4.8 dB, respectively. At the sound pressure levels used, the change for R10 is smaller than a JND and unlikely to affect the final results. Given that R39 passed 1-MI, the correction factor seems larger than would be needed. In the case of R22, who failed 1-MI, σ is smaller than needed. Both of these cases suggest that while the σ estimations appear reasonably in line with expectations in at least five out of seven cases, these estimates do show some variability. Such variability is expected in empirical studies and does not seem to be too bad overall.

2.5.4. σ-MI test for R46

We tested σ-MI for R46 using the sigma estimate obtained here. The method for testing was identical to that for 1-MI. With reference to the terminology used to report result in Experiment 1, the results are reported in Table 6.

Clearly, the results indicate that the estimated σ-correction was not sufficient for R46 to pass σ-MI. We are not sure why this is, but some possibilities are discussed in the following.

2.5.5. Conclusions

In conclusion, the σ estimates appear reasonably in line with expectations. These results coupled with those from Experiment 1 give us evidence that for about half of the respondents loudness perceptions are well described by the sum of power functions, but we do not know what forms fit the other half. For these other respondents, at least three sources of trouble may exist: (1) $\delta = 0$ may not be strictly correct for some respondents. The theory for $\delta > 0$ has not yet been worked out for power functions and it does not appear to be easy to do so. (2) The constant bias assumption $\sigma = 1$ is not satisfied by the respondents. Or (3) the representation of ψ_i as power functions is simply wrong. Of these, we believe that studying the first may be the most promising. The reason is that rough calculations of how large δ needs to be to have a fairly large effect show that it can be quite small, such as .01, which, of course, would be very difficult to test for using the bisymmetry property (studied by Steingrimsson & Luce, 2005b) that led us to accept the hypothesis $\delta = 0$. We do not yet know how to do this.

Table 6
Evaluate σ-multiplicative invariance for R46

Resp.	i	$\lambda^{\sigma}_{\mathrm{dB}}$	Mean (s.d)		p_{stat}	n	Stat. trend
			T_1	T_2			
R46	l	5.1	71.02 (1.27)	72.77 (1.38)	<.001	30	$T_1 \neq T_2$
		−5.1	66.17 (1.91)	64.77 (1.38)	.015		$T_1 \neq T_2$

3. Part II: intensity filtering

The topic of the present section is related to the earlier material primarily because power functions are assumed. The issue first arose in trying to understand a phenomenon that appeared in Experiment 1 of Steingrimsson and Luce (2005a). At that time we did not have a way to incorporate it into Luce's (2002, 2004) theoretical framework. We now believe we have such a description, which we call an intensity filtering model. In part I of the current paper, we explored the possibility of Ψ being a power function and, under that assumption, we were able to derive several consequences of the intensity filter model.

3.1. Asymmetric matching and intensity filtering

Experiment 1 of Steingrimsson and Luce (2005a) attempted, using asymmetric matches, to test whether joint presentation (jp-) symmetry, $(x,u) \sim (u,x)$, holds. An unexpected phenomenon emerged: for some respondents, the order between (x,u) and (u,x), established by the comparison $x \oplus_l u \lesseqgtr u \oplus_l x$, (left-ear matches) did not agree with the order established using right matches. And for the remaining respondents, although the order was preserved, the magnitude of the difference $x \oplus_i u - u \oplus_i x$, was not the same for $i = r$ and $i = l$. Despite efforts to find a methodological explanation, including alternating matching ear within a block, the phenomenon persisted. This places some limitations on experimental procedures, namely, we may not automatically assume that the ordering established by asymmetric matches is independent of matching ear. We propose an explanation.

The observation just described seems to suggest that the procedure of repeated adjustments of the sound pressure level in one ear to achieve a match to (x,u) has somehow biased the respondent to attend differentially to the matching ear and the opposite one. One can think of this as what amounts to a behavioral modification, enhancement or attenuation, of the intensity that reaches the hearing mechanism. It is not unlike the mechanical, inner ear protective attenuation of very loud sounds. We assume the simplest form of modification, namely, a constant factor[9] ζ on the intensity of the matching ear or a constant factor of η on intensity of the opposite ear. As we show below, an enhancement in one ear is equivalent to an attenuation in the other one. Enhancement corresponds to a factor >1, which is an additive factor in dB, and attenuation corresponds to a factor less <1, which is subtractive in dB. To be noncommittal, we speak of *intensity filtering*. In what follows we formulate matters in terms of modifications to the opposite ear.

For left matching, the experimental stimulus (x,u) becomes, effectively, $(x,\eta u)$. And for right matching, (x,u) effectively becomes $(\eta x,u)$. Thus, when we ask the respondents to solve the three indifferences of (3), what they in fact do, according to this filtering theory, is set them to

$$(x,u)\sim(x\oplus_s u, x\oplus_s u), \tag{29}$$

$$(x,\eta u)\sim(x\oplus_l \eta u, 0), \tag{30}$$

$$(\eta x,u)\sim(0,\eta x\oplus_r u). \tag{31}$$

Note that the intensity filter plays no role in the symmetric matches of (29). One must interpret the notation of (30) and (31) carefully in the following sense. When, for example, we present (x,u) and ask for a left ear match, the experimental notation will necessarily be (x,u) and $x\oplus_l u$ even though, under the theory, the respondents use (30) and report what amounts to $x\oplus_l \eta u$. So, in dB notation we will write $(x\oplus_l u)_{\mathrm{dB}}$ but use the filtered version of the representation.

To show the equivalence of matching ear and opposite ear modifications, note that for the same ear we have

$$(\zeta x\oplus_l u,0)\sim(x,u)\sim(0,x\oplus_r \zeta u).$$

If we make the following simple changes of variable:

$$y=\zeta x,\quad v=\zeta u,\quad \eta=1/\zeta,$$

those indifferences reduce to

$$y\oplus_l \eta v=\zeta x\oplus_l \eta\zeta u=\zeta x\oplus_l u,$$

which is a simple notational variant on opposite ear intensity modification. The right case is similar. Of course, enhancement and attenuation on the same ear are not at all equivalent; one or the other must occur.

We proceed in three steps. First show that the filter theory, indeed, accommodates the inconsistent results of jp-symmetry using left and right ear matches that motivated it. In Section 3.3 we explore the—what turns out to be a lack of—impact of intensity filtering on our experiments on the qualitative properties of the theory. And in Section 3.4 we give some estimates of η.

[9]A more general model would assume that the factor depends on the intensity level and/or the ear used in the matches.

3.2. *Inconsistent asymmetric matches for jp-symmetry*

In the left-match test of symmetry we are in effect asking for the sign of $\Psi(x,\eta u)-\Psi(u,\eta x)$, and in the right-match, the sign of $\Psi(\eta x,u)-\Psi(\eta u,x)$. For the case of pure additivity in Ψ, i.e., $\delta=0$ in (5) (a result empirically supported by Steingrimsson & Luce, 2005b, but see Section 2.5.5), and assuming the arguments above leading to power functions for the psychophysical functions ψ_i are correct, we know that (19) holds, which may be rewritten as

$$\frac{1}{\alpha_r}\Psi(x,u)=\gamma x^{\beta_l}+u^{\beta_r}\quad(\gamma=\alpha_l/\alpha_r). \tag{32}$$

Then,

Proposition 5. *On the assumption of* (32) *and intensity filtering, inconsistent right ear and left ear matches for* $x>(<)u$ *yield*

$$\begin{aligned}&x\oplus_l u\gtrless u\oplus_l x \quad\text{and}\quad x\oplus_r u\lessgtr u\oplus_r x\\ &\quad\Leftrightarrow \Psi(x,\eta u)-\Psi(u,\eta x)\gtrless 0\gtrless \Psi(\eta x,u)-\Psi(\eta u,x)\\ &\quad\Leftrightarrow \frac{1}{\gamma\eta^{\beta_l}}\gtrless(\lessgtr)\frac{x^{\beta_r}-u^{\beta_r}}{x^{\beta_l}-u^{\beta_l}}\gtrless(\lessgtr)\frac{\eta^{\beta_r}}{\gamma}\\ &\quad\Rightarrow \eta\lessgtr(\gtrless)1.\end{aligned}$$

A proof is in the Appendix.

In the following analysis of Experiment 2, we have $x>u=0$.

This result provides a way to decide whether one is dealing with attenuation or enhancement of the opposite ear. We do this in Section 3.4.

3.3. *Impact of intensity filtering on behavioral properties*

Although intensity filtering is consistent with the observations from the very first experiment of the series, a concern immediately arises: what impact does it have on the tests of various properties that used asymmetric matches? We take up first the issues of Experiment 1 and then that of earlier experiments.

3.3.1. *σ-multiplicative invariance*

For σ-MI, (18), the expressions under filtering becomes

$$\lambda x\oplus_l \eta\lambda^\sigma u=\lambda(x\oplus_l \eta u), \tag{33}$$

$$\eta\lambda x\oplus_r \lambda^\sigma u=\lambda^\sigma(\eta x\oplus_r u)\quad(\lambda>0). \tag{34}$$

Because $\eta\lambda^\sigma=\lambda^\sigma\eta$, then setting $v=\eta u$ in the first and $y=\eta x$ in the second yields in both cases (18) with a trivial change of notation. So the filter does not affect a σ-MI experiment despite using asymmetric matches.

3.3.2. *Bisymmetry, production commutativity, and segregation in Steingrimsson and Luce (2005b)*

The first paper of this series did not invoke asymmetric matching except in the first test of jp-symmetry, whose results led us to the current intensity filter theory. The second paper, Steingrimsson and Luce (2005b), made some use of asymmetric matches.[10] Specifically, the properties tested experimentally were three: bisymmetry using both asymmetric and symmetric matches, jp-decomposition using only symmetric matches, and segregation using only asymmetric matches.

A summary of these properties is

Bisymmetry:

$$(x\oplus_i y)\oplus_i(u\oplus_i v) = (x\oplus_i u)\oplus_i(y\oplus_i v) \quad (i = l, r, s). \tag{35}$$

Simple jp-decomposition:

$$(x\oplus_i u)\circ_{p,i}0 = (x\circ_{p,i}0)\oplus_i(u\circ_{p,i}0) \quad (i = l, r, s), \tag{36}$$

where $\circ_{p,i}$ are defined by

$$(x\circ_{p,l}y, 0)\sim(x, 0)\circ_p(y, 0), \tag{37}$$

$$(0, u\circ_{p,r}v)\sim(0, u)\circ_p(0, v), \tag{38}$$

$$(x\circ_{p,s}y, x\circ_{p,s}y)\sim(x, x)\circ_p(y, y). \tag{39}$$

Left segregation:

$$u\oplus_i(x\circ_{p,i}0)\sim(u\oplus_i x)\circ_{p,i}(u\oplus_i 0) \quad (i = l, r, s). \tag{40}$$

Right segregation:

$$(x\circ_{p,i}0)\oplus_i u\sim(x\oplus_i u)\circ_{p,i}(0\oplus_i u) \quad (i = l, r, s). \tag{41}$$

We explore how each of these are affected, if at all, by the filtering assumption. Of course, for $i = s$, there is nothing to show.

Proposition 6. *Suppose that the power representation* (19) *holds, which under the general representation is equivalent to σ-MI holding. Then for $i = l, r$, bisymmetry and jp-decomposition are invariant under intensity filtering. If, in addition, the subjective proportion representation* (6) *holds, then multiplicative invariance for $\circ_{p,l}$ is satisfied in the sense of* (33) *and* (34),

$$\lambda(x\circ_{p,i}u) = \lambda x\circ_{p,i}\lambda u \quad (i = l, r, \lambda > 0), \tag{42}$$

and left and right segregation are invariant under intensity filtering.

A proof is given in the Appendix. Actually, as the proof shows, the assumption of σ-MI is not needed for simple jp-decomposition.

3.4. Estimating η

3.4.1. Theory

Recall that in Section 2.2.3 we arrived at the regressions (21)–(23) that were used to estimate σ. Intensity filtering played no role in those computations. We now carry out the comparable regressions under the assumption of filtering and that, it turns out, permits the independent estimation of σ and η. In the more general case of intensity filtering, these regression expressions are unchanged but the intercepts $c_i, i = 1, 2, 3$ are changed.

[10]At the time those experiments were run, we had not developed the theory for symmetric matches.

Proposition 7. *Assume that the sum of power functions representation* (32) *holds and that intensity filtering exists. Then Eqs.* (21)–(23) *are satisfied with*

$$c_1 = -\frac{10}{\beta_l}\log\gamma + \frac{10}{\sigma}\log\eta,$$

$$c_2 = \frac{10}{\beta_r}\log\gamma + \sigma 10\log\eta.$$

These yield the estimation equations

$$10\log\eta = \frac{\sigma c_1 + c_2}{1+\sigma}, \tag{43}$$

$$10\log\gamma = \frac{\beta_r c_3}{1+\sigma} = \frac{\beta_r(c_2 - \sigma^2 c_1)}{1+\sigma}. \tag{44}$$

A proof of this proposition is in the Appendix.

Note that (43) can actually be used to estimate η whereas (44) only gives an estimate of γ if we know β_r, which we do not know.

3.4.2. η-Estimation

We used the data developed in Experiment 2 to estimate both σ and η. We reported and discussed the σ estimates there.

Table 7 shows the estimates of η, $\widehat{\eta}$, obtained using (43). For five of seven respondents, $\widehat{\eta}<1$ whereas for R22 and R46 they were $\widehat{\eta}>1$. To the extent we may take these estimates seriously, it suggests that we have found both attenuation and enhancement among the respondents. We did not anticipate the latter would happen. Of course, the η-estimation method assumes the power-function representation (32), and both R22 and R46 failed 1-MI and thus their behavior is not well described by a sum of power functions. Hence, their seemingly enhancement estimates may be a result of an erroneous model. Assuming that the estimates can be taken seriously, for two people the effect of asymmetric matching is to enhance signals to the nonmatching ear or, equivalently, to attenuate the matching ear.

Table 7
Experiment 2: estimating η

Respondent	n	$\widehat{\eta}$
R10	60	0.544
R22	70	1.378
R37	60	0.660
R38	30	0.997
R39	60	0.343
R42	40	0.270
R46	65	4.978

4. Summary

Under the binaural loudness interpretation of the primitives, Steingrimsson and Luce (2005a,b) tested and found adequate support for a number of behavioral axioms that together established the representations in (5) and (6). Here, we turned to the question of possible functional forms for the psychophysical functions ψ_l and ψ_r.

Under the assumption that ratio estimates are independent of the standard, (13), we arrived at the power function form (14) for ψ_i. Under the same assumption, we concluded that ratio production and ratio estimation are simply inverse functions.

We then showed that the representation (5) holding with $\delta = 0$, which was sustained by Steingrimsson and Luce (2005b), is equivalent to the testable property of σ-MI (18) holding and Ψ being a sum of power functions, (20). We also developed a regression method for estimating $\sigma = \beta_l/\beta_r$, where β_l, β_r are the powers of the two power functions.

In Experiment 1 we studied 1-MI and found that it was sustained for 12 of 22 respondents, with seven partially satisfying it, and three not at all. In Experiment 2 we estimated σ for seven respondents who had previously participated in Experiment 1, and we found that for those who passed 1-MI, $\sigma \sim 1$ as might be expected, but those who failed it, $\widehat{\sigma}$ for one was clearly not adequate for the respondent to pass σ-MI, for the other respondent σ-MI was evaluated but did not hold. Thus we find the power function is a good description for those who pass 1-MI, but less so for the rest. One possible explanation is that, in reality, perhaps our empirical estimate of $\delta = 0$ is not sufficiently sensitive to detect people with a small, nonzero value. So far, no behavioral equivalent to the case of p-additive (5) power functions has been discovered.

Steingrimsson and Luce (2005a) first tried to use asymmetric matching summation to study (x, u) versus (u, x) and found that its magnitude and even direction appeared dependent on the matching ear. Here we developed, under the assumption of power function ψ_i, a behavioral account of this phenomenon in the form of an intensity filtering model with a constant parameter η.

We first studied the question whether any of the properties previously tested using asymmetric matches were compromised by such a filter. Steingrimsson and Luce (2005b) tested three such properties (bisymmetry, production commutativity, and segregation). Under the assumption of the power representation (19), the filter was shown not to affect the testing of these properties. Following this, we estimated the filtering constant η, and for five of seven respondents we found evidence of attenuation in the nonmatching ear, corresponding to $\eta < 1$, but for two we estimated $\eta > 1$, i.e., enhancement.

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Appendix: Proofs

Proof of Proposition 1. Using (12) and (14), it is trivial that

$$W(p_i(r,x)) = \frac{\psi_i(rx)}{\psi_i(x)} = r^{\beta_i},$$

whence (13). Conversely, under that assumption, (12) simplifies to the Pexider functional equation

$$\psi_i(rx) = W(p_i(r))\psi_i(x) \quad (r > 0, x > 0, i = l, r).$$

With all functions being strictly increasing, the solutions of this functional equation are well-known to be power functions, as asserted (Aczél, 1966, p. 144–145). Setting $r = 1$ in this equation, we see that $W(p_i(1)) = 1$, and so the multiplicative constant of the W power function must be 1, thereby yielding both (14) and (15). □

Proof of Proposition 2. (ii) implies (i). This is a routine calculation.

(i) implies (ii). Suppose $i = l$, (we give a proof for the case of left matches, the one for right is similar), and recall that

$$\psi_l(x \oplus_l u) = \Psi(x \oplus_l u, 0) = \Psi(x, u) = \psi_l(x) + \psi_r(u).$$

Therefore,

$$\begin{aligned}\psi_l(\lambda x \oplus_l \lambda^\sigma u) &= \Psi(\lambda x \oplus_l \lambda^\sigma u, 0) \quad (9)\\ &= \Psi(\lambda x, \lambda^\sigma u) \quad (3)\\ &= \Psi(\lambda x, 0) + \Psi(0, \lambda u) \quad (5)\\ &= \psi_l(\lambda x) + \psi_r(\lambda^\sigma u) \quad (9), (10).\end{aligned}$$

Multiplicative invariance, (18) is therefore equivalent to the functional equation

$$\psi_l^{-1}(\psi_l(\lambda x) + \psi_r(\lambda^\sigma u)) = \lambda \psi_l^{-1}(\psi_l(x) + \psi_r(u)). \tag{45}$$

For $\sigma = 1$, this equation is of the form of Eq. (2.20) of Aczél et al. (2000). János Aczél (personal communications July 23, 2004 and February 1, 2005) has modified their proof for any $\sigma > 0$. We give this proof with his permission.

Let

$$y{:=}\psi_l(x), \quad v{:=}\psi_r(u), \quad g_\lambda(y){:=}\psi_l[\lambda \psi_l^{-1}(y)],$$
$$h_\lambda(v){:=}\psi_r[\lambda^\sigma \psi_r^{-1}(v)], \tag{46}$$

then (45) becomes

$$g_\lambda(y+v) = g_\lambda(y) + h_\lambda(v),$$

which is the Pexider equation whose solutions under weak regularity (e.g., strict monotonicity) are well known (Aczél, 1966) to be

$$\psi_l[\lambda\psi_l^{-1}(y)] = g_\lambda(y) = m(\lambda)y + a(\lambda),$$

$$\psi_r[\lambda^\sigma\psi_r^{-1}(v)] = h_\lambda(v) = m(\lambda)v,$$

which with (46) and $\Lambda = \lambda^\sigma$ yields

$$\psi_l(\lambda x) = m(\lambda)\psi_l(x) + a(\lambda), \quad \psi_r(\Lambda u) = C(\Lambda)\psi_r(u),$$

where $C(\Lambda) := m(\Lambda^{1/\sigma}) = m(\lambda)$. Again under weak regularity (e.g., strict monotonicity)

$$\psi_l(x) = \alpha_l x^{\beta_l} + \lambda, \quad \psi_r(u) = \alpha_r u^{\beta_l/\sigma}.$$

Because $\psi_l(0) = 0$, $\lambda = 0$, and the proof is completed. □

Proof of Proposition 3. Assume the power representation, (32), then

$$(0\oplus_l x) = \frac{1}{\gamma^{1/\beta_l}} x^{1/\sigma},$$

where $\gamma = \alpha_l/\alpha_r$ and $\sigma = \beta_l/\beta_r$. Taking logarithms yields

$$(0\oplus_l x)_{\text{dB}} = c_1 + \frac{1}{\sigma} x_{\text{dB}},$$

where $c_1 = (-10/\beta_l)\log\gamma$. Similarly,

$$(x\oplus_r 0)_{\text{dB}} = c_2 + \sigma x_{\text{dB}},$$

where $c_2 = (10/\beta_r)\log\gamma$.

For the corollary, eliminating x_{dB} between these two expressions yields

$$(x\oplus_r 0)_{\text{dB}} = c_3 + \sigma^2(0\oplus_l x)_{\text{dB}},$$

where $c_3 = c_2 - \sigma^2 c_1 = (10/\beta_r)(1+\sigma)\log\gamma$.

It is easy to verify (43) and (24). □

Proof of Proposition 5. Assume (32) with $\delta = 0$ and that right and left matching disagree and $x > (<u)$. Assume $x > u$. Under left matching

$$\begin{aligned}
&\Psi(x,\eta u) \gtrless \Psi(u,\eta x)\\
&\quad\Leftrightarrow \Psi(x,0) + \Psi(0,\eta u) \gtrless \Psi(u,0) + \Psi(0,\eta x)\\
&\quad\Leftrightarrow \gamma x^{\beta_l} + \eta^{\beta_r} u^{\beta_r} \gtrless \gamma u^{\beta_l} + \eta^{\beta_r} x^{\beta_r}\\
&\quad\Leftrightarrow \gamma(x^{\beta_l} - u^{\beta_l}) \gtrless \eta^{\beta_r}(x^{\beta_r} - u^{\beta_r})\\
&\quad\Leftrightarrow \frac{x^{\beta_l} - u^{\beta_l}}{x^{\beta_r} - u^{\beta_r}} \gtrless \frac{\eta^{\beta_r}}{\gamma}.
\end{aligned}$$

And under right matching

$$\begin{aligned}
&\Psi(\eta x, u) \lesseqgtr \Psi(\eta u, x)\\
&\quad\Leftrightarrow \Psi(\eta x,0) + \Psi(0,u) \lesseqgtr \Psi(\eta u,0) + \Psi(0,x)\\
&\quad\Leftrightarrow \gamma\eta^{\beta_l} x^{\beta_l} + u^{\beta_r} \lesseqgtr \gamma\eta^{\beta_l} u^{\beta_l} + x^{\beta_r}\\
&\quad\Leftrightarrow \gamma\eta^{\beta_l}(x^{\beta_l} - u^{\beta_l}) \lesseqgtr (x^{\beta_r} - u^{\beta_r})\\
&\quad\Leftrightarrow \frac{1}{\gamma\eta^{\beta_l}} \gtrless \frac{x^{\beta_l} - u^{\beta_l}}{x^{\beta_r} - u^{\beta_r}}.
\end{aligned}$$

Note that these imply

$$\frac{1}{\gamma\eta^{\beta_l}} \gtrless \frac{\eta^{\beta_r}}{\gamma} \Leftrightarrow 1 \gtrless \eta^{\beta_l+\beta_r} \Leftrightarrow 1 \gtrless \eta,$$

whence the conclusion.

Under $x < u$, the inequalities in the last two lines for left and right matching and the ones of the last display are reversed. □

Proof of Proposition 6. It suffices to do the proof just for $i = l$ because $i = r$ is similar. Throughout the proof we assume that σ-MI, (18), holds.

Consider bisymmetry under filtering, (30):

$$\begin{aligned}
(x\oplus_l\eta y)\oplus_l\eta(u\oplus_l\eta v) &= (x\oplus_l\eta y)\oplus_l(\eta u\oplus_l\eta^{1+\sigma}v), \quad (18)\\
&= (x\oplus_l\eta u)\oplus_l(\eta y\oplus_l\eta^{1+\sigma}v), \quad (35)\\
&= (x\oplus_l\eta u)\oplus_l\eta(y\oplus_l\eta v), \quad (18),
\end{aligned}$$

which is filtered bisymmetry.

Next consider simple jp-decomposition under filtering: this requires no more than a change of notation because

$$(x\oplus_i\eta u)\circ_{p,i}0 = (x\circ_{p,i}0)\oplus_i(\eta u\circ_{p,i}0)$$

is the ordinary property with $y = \eta u$. Note that this does not depend on multiplicative invariance.

Next we show that $\circ_{p,l}$ is invariant in the sense of (42) under the assumption of subjective production representation, (6) and a power function for ψ_l. We have

$$\begin{aligned}
\psi_l[\lambda(x\circ_{p,l}u)] &= \alpha_l\lambda^{\beta_l}(x\circ_{p,l}u)^{\beta_l}\\
&= \lambda^{\beta_l}\psi_l(x\circ_{p,l}u)\\
&= \lambda^{\beta_l}([\psi_l(x) - \psi_l(u)]W(p) + \psi_l(u))\\
&= [\psi_l(\lambda x) - \psi_l(\lambda x)]W(p) + \psi_l(\lambda u)\\
&= \psi_l(\lambda x\circ_{p,l}\lambda u).
\end{aligned}$$

Taking ψ_l^{-1} yields the desired invariance property.

Finally, consider left segregation under filtering: Using the previous result with $\lambda = \eta$ and the underlying segregation property

$$\begin{aligned}
u\oplus_l\eta(x\circ_{p,l}0) &= u\oplus_l(\eta x\circ_{p,l}0) \quad (42)\\
&= (u\oplus_l\eta x)\circ_{p,l}(u\oplus_l 0), \quad (40)
\end{aligned}$$

which is left segregation with filtering. □

Proof of Proposition 7. Consider first left matching under filtering η of the opposite ear, (30)

$$(0,\eta x) \sim (0\oplus_l\eta x, 0).$$

Applying the sum of power functions form (19) and recalling that $\gamma = \alpha_l/\alpha_r$ and $\sigma = \beta_l/\beta_r$, we have

$$\begin{aligned}
&\alpha_r(\eta x)^{\beta_r} = \alpha_l(0\oplus_l \eta x)^{\beta_l}\\
&\quad \Leftrightarrow 0\oplus_l \eta x = \left(\frac{1}{\gamma}\right)^{1/\beta_l}(\eta x)^{1/\sigma}\\
&\quad \Leftrightarrow (0\oplus_l \eta x)_{\mathrm{dB}} = -\frac{10}{\beta_l}\log\gamma + \frac{10}{\sigma}\log\eta + \frac{1}{\sigma}x_{\mathrm{dB}},
\end{aligned}$$

whence

$$c_1 = -\frac{10}{\beta_l}\log\gamma + \frac{10}{\sigma}\log\eta.$$

Next, consider under right matching and filtering (31):

$$\begin{aligned}
&(\eta x, 0)\sim(0, \eta x\oplus_r 0)\\
&\quad \Leftrightarrow \alpha_l(\eta x)^{\beta_l} = \alpha_r(\eta x\oplus_r 0)^{\beta_r}\\
&\quad \Leftrightarrow (\eta x\oplus_r 0) = \gamma^{1/\beta_r}(\eta x)^{\sigma}\\
&\quad \Leftrightarrow (\eta x\oplus_r 0)_{\mathrm{dB}} = \frac{10}{\beta_r}\log\gamma + 10\sigma\log\eta + \sigma x_{\mathrm{dB}},
\end{aligned}$$

whence

$$c_2 = \frac{10}{\beta_r}\log\gamma + \sigma 10\log\eta.$$

Thus, adding c_2 to

$$\sigma c_1 = -\frac{10}{\beta_r}\log\gamma + 10\log\eta$$

yields

$$\begin{aligned}
&\sigma c_1 + c_2 = (1+\sigma)10\log\eta\\
&\quad \Leftrightarrow 10\log\eta = \frac{\sigma c_1 + c_2}{1+\sigma},
\end{aligned}$$

which is (43). And

$$\begin{aligned}
c_2 - \sigma^2 c_1 &= \frac{10}{\beta_r}\log\gamma + \sigma 10\log\eta + \frac{\sigma 10}{\beta_r}\log\gamma - \sigma 10\log\eta\\
&= (1+\sigma)\frac{10}{\beta_r}\log\gamma\\
&\quad \Leftrightarrow 10\log\gamma = \frac{\beta_r}{1+\sigma}(c_2 - \sigma^2 c_1),
\end{aligned}$$

as asserted. □

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