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Evaluating a model of global psychophysical judgments—II: Behavioral properties linking summations and productions

Ragnar Steingrimsson[a,*,1], R. Duncan Luce[b]

[a]Department of Psychology, New York University, 6 Washington Place, New York, NY 10003, USA
[b]Department of Cognitive Science, University of California Irvine, Social Science Plaza, Irvine, CA 92697-5100, USA

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Abstract

Steingrimsson and Luce [Journal of Mathematical Psychology, in press] outlined the second author's proposed psychophysical theory [Luce (2002), Psychological Review, 109, 520–532; Luce (2004a) Psychological Review, 111, 446–454] and tested behavioral attributes that, separately, gave rise to two psychophysical functions, $\Psi_{\oplus}$ and $\Psi_{\circ_p}$. The function $\Psi_{\oplus}$ maps pairs of physical intensities onto the positive real numbers and represents subjective summation, and the function $\Psi_{\circ_p}$ represents a form of ratio production. This article evaluates properties linking summation and production in such a way as to force $\Psi_{\circ_p} = \Psi_{\oplus} = \Psi$. These properties, which are a form of distributivity, are subjected to an empirical evaluation in three experiments. The testing strategy is carried out in the auditory domain and concerns the subjective perception of loudness. Considerable support is provided for the existence of a single function Ψ for both summation and ratio production.

Keywords: Magnitude estimation; Auditory summation; Ratio production; Magnitude production; Psychophysics; Matching; Production commutativity; Bisymmetry; Joint-presentation decomposition; Segregation

This article, the second of two exploring the axioms underlying a psychophysical representation of joint presentations and ratio productions, is focused on the two distribution conditions that link the two structures. The motivating background and the underlying theory are outlined in detail in Steingrimsson and Luce (2005). Here we provide a brief summary of both.

1. Theory

1.1. Primitives

Consider, for example, loudness judgments of stimuli that involve, in general, different intensities of pure tones of the same frequency and phase presented to the two ears. Let x denote the physical intensity to the left ear, measured as the intensity of the signal presented less (not relative to) the respondent's threshold level for that ear, and let u denote the comparable intensity measure for the right ear. So, stimuli are of the form (x, u) where, as an idealization, we assume that x and u can be any non-negative real number.

Judgments of relative loudness are assumed to be described by a binary relation $\succsim$ such that $(x, u) \succsim (y, v)$ means (x, u) is judged to be at least as loud as (y, v). This idealization does not admit possible probabilistic aspects in the data. At present, we do not know how one might formulate a probabilistic theory for the linking relations of this article. We assume that $\succsim$ agrees with the intensity ordering $\geqslant$ in the sense that if the intensity is held fixed in one ear, it is assumed that the loudness ordering agrees with the intensities presented to the other ear. With $\sim$ denoting a judgment of equal loudness,[2] we assume that $\sim$ forms an

*Corresponding author.
E-mail addresses: ragnar@nyu.edu (R. Steingrimsson), rdluce@uci.edu (R.D. Luce).

[1]This article is based, in part, on the first author's Ph.D. dissertation (Steingrimsson, 2002).

[2]As usual, $(x, u) \sim (y, v)$ is defined to mean that both $(x, u) \succsim (y, v)$ and $(y, v) \succsim (x, u)$ hold.

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indifference relation. This is an idealization. We also assume that the subject can always establish matches of three types to each stimulus:

$$(x,u) \sim (z_l, 0), \quad (x,u) \sim (0, z_r), \quad (x,u) \sim (z_s, z_s). \tag{1}$$

The left and right matches z_l and z_r are called *asymmetric* and z_s is called a *symmetric* match.

In addition to judgments of loudness, we ask each respondent to provide ratio productions in which pairs (x,x) and (y,y), where $x>y$, are presented to the respondent who is to establish the intensity z such that the subjective "interval" from (y,y) to (z,z) is perceived to be p times the "interval" from (y,y) to (x,x). It is convenient to define the notation that makes explicit the dependence of z on x,y,p, namely,

$$(x,x) \circ_p (y,y) := (z,z). \tag{2}$$

Expression (2) appears to be, but in reality is not, more restrictive than

$$(x,u) \circ_p (y,v) = (z,w), \tag{3}$$

as was noted in Luce (2004a).

1.2. Representations

The theory developed by Luce (2002, 2004a) is of the following form. Given reasonable assumptions about $\succsim$, such as those above, and presuming the continuous nature of physical intensity and of the psychophysical functions involved, the goal is to state necessary and sufficient behavioral properties corresponding to the following numerical representation: There exist constants $\delta \geqslant 0$, $\gamma > 0$, a psychophysical function Ψ that is a strictly monotonic mapping of intensity pairs to the non-negative real numbers, and a strictly increasing, numerical distortion of each non-negative number p to a non-negative number $W(p)$ such that

$$(x,u) \succsim (y,v) \quad \text{iff } \Psi(x,u) \geqslant \Psi(y,v), \tag{4}$$

$$\Psi(x,u) = \Psi(x,0) + \Psi(0,u) + \delta \Psi(x,0)\Psi(0,u) \quad (\delta \geqslant 0), \tag{5}$$

$$W(p) = \frac{\Psi[(x,x)\circ_p(y,y)] - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} \quad (x \geqslant y \geqslant 0), \tag{6}$$

$$\Psi(x,0) = \gamma \Psi(0,x) \quad (\gamma > 0). \tag{7}$$

These representations hold in the theory when using asymmetric matching, z_l or z_r in (1). If symmetric matching, z_s of (1), is used instead, one can prove that the summation property of (5) with $\delta = 0$ and the ratio production property of (6) both hold. However, the constant bias property (7) need not hold under symmetric matching theory.

The results of Steingrimsson and Luce (2005) provided evidence that (5) and (6) hold separately. But for all we know from those results, there are two distinct psychophysical functions, $\Psi_\oplus$ for summations and bias and $\Psi_{\circ_p}$ for productions. The best we could say at that point was that for some strictly increasing function φ, $\Psi_{\circ_p}(x,u) = \varphi(\Psi_\oplus(x,u))$. This article focuses on the conditions linking the two structures so that $\Psi_\oplus = \Psi_{\circ_p} = \Psi$, i.e., φ is the identity function.[3]

Our research on global psychophysics also includes efforts to discover both the mathematical form of $\Psi(x,u)$ as an explicit function of x and u (with a few free parameters) and the form of $W(p)$ as a function of p. In both cases, we have testable behavioral conditions equivalent to certain functional forms. Most of the experimental work is complete and two resulting articles are nearing completion.

1.3. Induced structures

Steingrimsson and Luce (2005) formulated the conditions corresponding to (5) and (6) separately and directly in terms of the ordering $\succsim$ over intensity pairs. Although we would like to be able to state conditions about how the two representation interlock in the same manner, we have not yet seen how to do that. Rather, the approach that we use entails mapping both the two-dimensional summation and the production structures onto a single intensity dimension by using one or the other of the solutions assumed to exist in (1). To that end we need to define exactly how these mappings are carried out.

1.3.1. Induced summation operations

First, it is useful to rename the three solutions of (1) using an operation notation that makes clear their dependence on x and u, namely,

$$x \oplus_i u := z_i \quad (i = l, r, s). \tag{8}$$

Thus, (1) becomes

$$(x,u) \sim (x \oplus_l u, 0), \quad (x,u) \sim (0, x \oplus_r u),$$
$$(x,u) \sim (x \oplus_s u, x \oplus_s u).$$

It is important to recognize that $x \oplus_i u$ is not a mere renaming of (x,u) but is the result of mapping (x,u) and its associated properties under $\succsim$ onto a single dimension of intensity in either the left ear, the right ear, or symmetrically in both.

It is not difficult to show that each of the $\oplus_i$ is, indeed, a binary operation that is defined for each pair (x,u) of intensities. Each operation encodes in one intensity dimension all of the information contained in the ordering of pairs of signals, i.e., in the conjoint structure of joint presentations. Luce (2002) studied $\oplus_l, \oplus_r$, and

[3] This is not strictly correct. One only shows that $\Psi_{\circ_p} = k\Psi_\oplus$, where $k>0$ is a constant. But one can replace $\Psi_{\circ_p}$ by $\Psi_{\circ_p}/k$ and the relevant expression (6) is not altered.

Luce (2004a) improved those results and studied $\oplus_s$ as well.[4]

These mappings permit, among other things, the possibility of such compounding as $(x \oplus_i u) \oplus_i v$, which is not meaningful for the joint presentation notation because $((x,u),v)$ has no direct meaning (see Appendix A). But, of course, $(x \oplus_i u, v)$ is meaningful because $x \oplus_i u$ is an intensity.

By Proposition 1 of Luce (2004a), (x,u) is strictly increasing in each argument, and therefore, so are the $\oplus_i$. Note that joint presentation symmetry, i.e., $(x,u) \sim (u,x)$, is equivalent to $\oplus_l \equiv \oplus_r$ and $\oplus_s$ all being commutative operations. Steingrimsson and Luce (2005) found that joint-presentation symmetry fails for most respondents, which rules out its holding as a general rule.

1.3.2. Induced subjective-production operations

We also define induced production operations $\circ_{p,i}$, $i = \mathrm{l,r,s}$, by the following special cases of the general operation $\circ_p$ defined by (3):

$$(x \circ_{p,l} y, 0) \sim (x,0) \circ_p (y,0), \tag{9}$$

$$(0, u \circ_{p,r} v) \sim (0,u) \circ_p (0,v), \tag{10}$$

$$(x \circ_{p,s} y, x \circ_{p,s} y) \sim (x,x) \circ_p (y,y). \tag{11}$$

1.3.3. Induced representations

For these induced operations, we are interested in representations ψ_i, $i = \mathrm{l,r,s}$, that map intensities onto the non-negative real numbers. This is done simply by relating them to the general representation Ψ by

$$\psi_l(x) := \Psi(x,0), \tag{12}$$

$$\psi_r(u) := \Psi(0,u), \tag{13}$$

$$\psi_s(x \oplus_s u) := \Psi(x,u). \tag{14}$$

These are the natural projections of Ψ.

In terms of these definitions, the p-additive form, (5), the constant bias form, (7), and the subjective proportion form, (6), are equivalent, respectively, to

$$\Psi(x,u) = \psi_s(x \oplus_s u) = \psi_l(x) + \psi_r(u) + \delta\psi_l(x)\psi_r(u) \quad (\delta \geqslant 0), \tag{15}$$

$$W(p) = \frac{\psi_i(x \circ_{p,i} y) - \psi_i(y)}{\psi_i(x) - \psi_i(y)} \quad (x > y \geqslant 0, i = \mathrm{l,r,s}), \tag{16}$$

$$\psi_l(x) = \gamma\psi_r(x) \quad (\gamma > 0). \tag{17}$$

Because $\psi_i(0) = 0$, we also have the following separable form for ordinary ratio production:

$$\psi_i(x \circ_{p,i} 0) = \psi_i(x) W(p), \tag{18}$$

which is what Stevens (1975) presumed in his classic studies of magnitude production methods. Of course, he identified ψ with his empirical psychophysical function. This is studied more thoroughly in articles under preparation.

1.4. Behavioral links between summations and productions

As just noted, the commutativity of $\oplus_i$ has effectively been ruled out for most respondents, so we may assume $\gamma \neq 1$. We turn first to what underlies $\delta = 0$.

1.4.1. Bisymmetry and $\delta = 0$

Luce (2004a) showed that, under the assumptions of the theory, $\gamma \neq 1$ and $\delta = 0$ are equivalent to the following property of *bisymmetry*:[5]

$$(x \oplus_i y) \oplus_i (u \oplus_i v) = (x \oplus_i u) \oplus_i (y \oplus_i v) \quad (i = \mathrm{l,r,s}). \tag{19}$$

Note that the two sides of this property simply involve interchanging y and u. However, bisymmetry fails to hold when $\delta \neq 0$, except for the unbiased case, i.e., $\gamma = 1$. Since we have considerable evidence that $\gamma \neq 1$, then whether bisymmetry holds tells us whether $\delta = 0$. When bisymmetry does hold, we are able to use symmetric matching to test several of the properties above (see Appendices Appendix B and Appendix C).

1.4.2. Simple joint-presentation decomposition

If we can assume $\delta = 0$ in (5), which is well sustained by the test of bisymmetry (Experiment 1, below), then one of the two key linking properties is *simple joint-presentation decomposition*: For all intensity pairs (x,u) and all positive real numbers p

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{p,i} 0) \quad (i = \mathrm{l,r,s}). \tag{20}$$

This is one form of "distributivity" (an analogue of the set theory property $(x \cap u) \cup \varnothing = (x \cup \varnothing) \cap (u \cup \varnothing)$ with $\oplus_i$ playing the role of $\cap$, $\circ_{p,i}$ the role of $\cup$, and 0 that of $\varnothing$).

The corresponding property in the case $\delta > 0$, the unbiased case, is vastly more complex to test because the right-most p has to be replaced by a function $q = q(x,p)$ that does not depend on u.

1.4.3. Segregation

The second linking property can also be viewed as a version of "distributivity" (an analogue to the set-theoretical condition $u \cap (x \cup z) = (u \cap x) \cup (u \cap z)$). *Left segregation* of type i, for $i = \mathrm{l,r,s}$, is satisfied if for intensities x, u and positive real number p,

$$u \oplus_i (x \circ_{p,i} 0) = (u \oplus_i x) \circ_{p,i} (u \oplus_i 0). \tag{21}$$

[4]At the time that many of the reported experiments were carried out, the later results were not available. But, as we shall see, this does not greatly matter because we conclude that most likely $\delta = 0$ (Experiment 1).

[5]Because we are working on a single dimension of intensity and because in this case $\succsim$ agrees with the intensity ordering, we write the several expressions below in terms of $=$ rather than $\sim_i$.

Right segregation of type i is satisfied if

$$(x\circ_{p,i}0)\oplus_i u = (x\oplus_i u)\circ_{p,i}(0\oplus_i u). \quad (22)$$

In the unbiased case, right and left segregation are equivalent, but not in the biased case, which we have seen is empirically correct. Because it is shown in the theory that 0 is a right identity for $i = \mathrm{l}$, one can replace $u\oplus_{\mathrm{l}}0$ by u, and because 0 is a left identity for $i = \mathrm{r}$, $0\oplus_{\mathrm{r}}u$ can also be replaced by u.

Luce (2004a) established that another key to getting the representations of (5), (6), and (7) is that left segregation holds for $i = \mathrm{l}$ and right segregation holds for $i = \mathrm{r}$, leading to (5). Additionally, both left and right segregation hold for $i = \mathrm{s}$, leading to

$$\psi_i(x\oplus_i u) = \mu_{\mathrm{l}}(i)\psi_i(x) + \mu_{\mathrm{r}}(i)\psi_i(u) \quad (i = \mathrm{l}, \mathrm{r}, \mathrm{s},\ \mu_j(i)\geqslant 0), \quad (23)$$

which is a general linear weighting with the identifications

i	$\mu_{\mathrm{l}}(i)$	$\mu_{\mathrm{r}}(i)$
l	1	$1/\gamma$
r	γ	1
s	$\gamma/(1+\gamma)$	$1/(1+\gamma)$

where γ is defined by (7), i.e., $\gamma = \Psi(x,0)/\Psi(0,x)$.

1.4.4. Experimental hypotheses

The evidence supporting each of (5) and (6) motivated us to evaluate, in three additional experiments, the linking properties mentioned earlier. Experiment 1 tests (19), which if sustained establishes that $\delta = 0$ in (15), making it sufficient to test (20), rather than its more complicated counterpart (see Section 1.4.2) and to do so using a two-ear testing procedure (See Appendix C). Bisymmetry also makes easier the testing of (21) and (22). These tests are carried out in Experiments 2 and 3.

2. Tests of linked summations and productions

2.1. Experimental methods common to all experiments

Most of the methods employed here are identical to those used in Steingrimsson and Luce (2005), and so we give only an abbreviated account of the methods in this article.

2.1.1. Signal presentations and notation

The experiments were carried out in the auditory domain using a 1000 Hz sinusoidal tone presented for 100 ms, which included 10 ms on and off ramps. A safety limit of 85 dB was imposed.

The theory is cast in terms of intensities less threshold, i.e., for the left ear $x = x' - x_\tau$. The right is similar. In methods, we report x' in dB SPL rather than $x_{\mathrm{dB}} = (x' - x_\tau)_{\mathrm{dB}}$. Well above threshold, the error is negligible.

2.1.2. Respondents

Data were obtained from 16 students—graduate and undergraduate—from the University of California, Irvine, with normal hearing thresholds assessed by an audiometric test (Micro Audiometric EarScan ES-AM). The first author was one of them (R22).[6] Of the 16, two stopped for personal reasons before sufficient data had been collected and two participated in piloting sessions only. Hence, data are reported from 12 (two male and 10 female) respondents.

Respondents (except the author) received compensation of $10 per session, provided written consent, and were treated in accordance with the ‘‘Ethical Principles of Psychologists and Code of Conduct’’ (American Psychological Association, 1992). Consent forms and procedures were approved by UC Irvine's Institutional Review Board.

2.1.3. Estimating one- and two-ear matches

There are three types of matches (1). To precisely indicate form of a trial, let $\langle A, B\rangle$ denote a presentation of A followed 450 ms later by presentation of B. Then, the three trial types used are:

$$\langle (x,u),(z_{\mathrm{l}},0)\rangle, \quad (24)$$

$$\langle (x,u),(0,z_{\mathrm{r}})\rangle, \quad (25)$$

$$\langle (x,u),(z_{\mathrm{s}},z_{\mathrm{s}})\rangle. \quad (26)$$

That is, respondents heard a tone followed 450 ms later by another tone in the left, right, or both ears and used key presses to either adjust the sound pressure level of z_i, $i = \mathrm{l}, \mathrm{r}, \mathrm{s}$, to repeat the previous trial, or to indicate satisfaction with the loudness match. Adjustments of 0.5, 1, 2 and 4 dB were possible. This process was repeated until respondents were satisfied with the match.

The task was explained as that of making the second stimulus equal in loudness to the first and to pay attention only to loudness.

2.1.4. Estimating the $\circ_p$ operation

The basic trial form is $\langle\langle A, B\rangle, \langle A, C\rangle\rangle$ where $\langle A, B\rangle$ and $\langle A, C\rangle$ represent the first and the second intensity interval, respectively. The temporal delay between $\langle A, B\rangle$ and $\langle A, C\rangle$ was 750 ms and between A and B (and A and C), the delay was 450 ms. The longer delay created a subjective sense of two distinct intervals.

[6]We judged this acceptable because the behavioral measures of matching and ratio production are not determined by the experimental design.

An estimate of $x \circ_{p,i} y = v_i$, in the case of $i = \text{s}$, was obtained using the trial type

$$\langle\langle(y, y), (x, x)\rangle, \langle(y, y), (v_\text{s}, v_\text{s})\rangle\rangle, \quad (27)$$

where the value of v_s was under the respondents' control.

In practice, respondents heard two tones separated by 450 ms (the first loudness interval). After 750 ms, another set of two tones (the second loudness interval) was played. The first tone in both intervals was the same and quieter than the second tone. Respondents controlled the loudness of the second tone in the second interval. The value of the proportion p was displayed on the monitor. Other aspects of this process were identical to that for joint presentations, see Section 2.1.3.

Instructions consisted of a description of the task coupled with graphical examples in which, e.g., $p = 2$ and $p = \frac{2}{3}$ were represented as bars that were 2 and $\frac{2}{3}$ times a reference bar, respectively.

The special case of $y = 0$ with $i = \text{s}$ was estimated using the trial type

$$\langle(x, x), (v_\text{s}, v_\text{s})\rangle. \quad (28)$$

In practice, respondents heard two tones, separated by 450 ms, and they adjusted the second tone to be a proportion p of the first tone.

Trial forms in the case of $i = \text{r}, \text{l}$ are constructed in a manner analogous to (27) and (28).

2.1.5. Equipment and procedure

Stimuli were generated digitally using a personal computer and played through a 16-bit digital-to-analog converter (Quikki; Tucker-Davis Technology). Presentation level was controlled by programmable attenuators, and stimuli were presented over Sennheiser HD265L headphones to listeners seated in individual, single-walled, IAC sound booths.

Experiments were conducted in sessions lasting no more than 1 h. All respondents completed at least one training session with matching. Respondents typically completed 18–19 sessions, with 60–64 estimates per session.

2.1.6. Statistical method and result presentation

We used the nonparametric Mann–Whitney U-test for the statistical analysis and a significance level of 0.05.

Data are reported as means, rather than medians, of repeated results because the signal values were discrete, making the mean a better estimate provided that their distributions are reasonably Gaussian, which they appear to be. Likewise, to indicate variability, standard deviations are reported.

This article examines parameter-free null hypotheses of the form $A = B$. A particular concern is whether the sample sizes for A and B are large enough for a failure of the null hypothesis to be distinguished within the power of the statistical test. To address this issue all statistical results were verified using Monte Carlo simulations based on the bootstrap technique (Efron & Tibshirani, 1993). As long as the actual test-statistics fell within for $p > 0.05$ or outside for $p \leqslant 0.05$ of a 95% confidence interval for the statistic, the test statistic was accepted.

2.1.7. Multi-step testing

The experiments required that an estimate made in one step be used in a subsequent step. Initially, we used the average of the first step as the input into the subsequent step, which has the drawback of not allowing the variance to propagate through the testing process. We realized that using individual estimates as input for subsequent estimates was a statistical improvement. This method was partially incorporated into Experiment 3.

2.2. Experiment 1: bisymmetry

Bisymmetry was stated in (19), namely

$$(x \oplus_i y) \oplus_i (u \oplus_i v) = (x \oplus_i u) \oplus_i (y \oplus_i v) \quad (i = \text{l}, \text{r}, \text{s}).$$

We ran the bisymmetry experiment first because if it is sustained, we then know that $\delta = 0$, which greatly simplifies running the experiments on simple joint-presentation decomposition, (20), and segregation, (21) and (22). The property of bisymmetry can be tested using either one- or two-ear matching (Appendix B). For generality, both matching approaches were employed, with the single ear matching carried out in the left ear (there is no obvious reason to suspect that the right ear matching results should be qualitatively different from the left ear ones, so we did not do that).[7]

2.2.1. Method

To study (19) requires one to obtain six matches made in two steps. The first consists of

$$w_i = x \oplus_i y \text{ and } w_i' = u \oplus_i v, \quad \text{[right-hand side of (19)]}$$

and

$$z_i = x \oplus_i u \text{ and } z_i' = y \oplus_i v \quad \text{[left-hand side of (19)]}.$$

Using the averages of the individual estimates of w_i, w_i', z_i, and z_i', the second step consists of the two matches

$$t_i = w_i \oplus_i w_i' \quad \text{and} \quad t_i' = z_i \oplus_i z_i'.$$

The property is said to hold if t_i and t_i' are found to be statistically equivalent.

[7] Steingrimsson and Luce (2005) found that in single-ear matching, the nature of inter-aural bias depended on the matching ear. This result is not methodologically problematic here as the direction of bias plays no role in this experiment and the matching ear remains constant throughout.

For the left ear match ($i = \text{l}$) and the two-ear matches ($i = \text{s}$), the six matches were obtained using the trial forms given by (24) and (26), respectively.

One instantiation of signals was used: $x = 58\,\text{dB}$, $y = 64\,\text{dB}$, $u = 70\,\text{dB}$, and $v = 76\,\text{dB}$. Trials were grouped into two blocks. The first block contained two instances of each of the trial types in step one, eight trials in all; the second block contained three of each of the two trial types in step two, six trials in all. The two blocks were run in separate sessions.

2.2.2. Results

The data for six respondents are presented in Table 1, where T_1 and T_2 stand for the means of t_i and t_i', respectively; standard deviations are given in parentheses. Sound pressure levels are given in dB SPL. The number of observations in each mean is indicated by n (n for step one is given in parentheses). Left ear and two-ear matching are marked with l and s, respectively. Results of statistical tests are indicated by corresponding p-values.

The bisymmetry property was not rejected for any of the six respondents.

2.2.3. Discussion

These results provide good initial support for the bisymmetric property. Within the context of the theory, this means that either $\gamma = 1$ or, when $\gamma \neq 1$, $\delta = 0$. Because we have considerable evidence that mostly $\gamma \neq 1$ (Steingrimsson & Luce, 2005), we are justified in assuming that $\delta = 0$ in designing Experiments 2 and 3.

2.3. Experiment 2: simple joint-presentation decomposition

Simple joint-presentation decomposition was stated in (20), namely,

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{p,i} 0) \quad (i = \text{l}, \text{r}, \text{s}).$$

The production judgments required are of the general form $v_\text{l} = x \circ_{p,i} 0$, which is a special case of general production judgments where $y = 0$ (see Section 2.1.4). On the assumption that $\delta = 0$, which was sustained in Experiment 1, the simple joint-presentation decomposition property can be tested using either one- or two-ear matching (see Appendix C).

Estimates such as $t_i = (x \oplus_i u) \circ_{p,i} 0$ can be carried out in one or two steps. In terms of expression (28), a one-step process for $i = \text{l}$ is performed using the trial $\langle (x,u),(t,0) \rangle$, and respondents are instructed to make the second tone $(t,0)$ stand in proportion p to the first tone (x,u), i.e., the $x \oplus_i u$ estimate is realized implicitly (the $i = \text{r}, \text{s}$ cases are analogous). A two-step process involves first making the estimate $z_i = x \oplus_i u$ and then using that to make a second one, $t_i = z_i \circ_{p,i} 0$. We found in pilot studies that, for one-ear productions ($i = \text{l}, \text{r}$), results from a two-step process were more satisfactory than those from a one-step process—the difference is likely related to tone-localization issues, which are easier to ignore in matching than in productions. This makes a two-ear ($i = \text{s}$) procedure experimentally more efficient, so we used it.

2.3.1. Method

The property has two levels of estimation which were carried out in two steps. First, the estimates

$$t_\text{s} = (x \oplus_\text{s} u) \circ_{p,\text{s}} 0,$$
$$w_\text{s} = x \circ_{p,\text{s}} 0,$$
$$s_\text{s} = u \circ_{p,\text{s}} 0,$$

were made. The averages of w_s and s_s were used in the second step, which consisted of the match

$$t_\text{s}' = w_\text{s} \oplus_\text{s} s_\text{s}.$$

The property is considered to hold if t_s and t_s' are found to be statistically equivalent.

We used one pair of sound pressure levels, $x = 64$ and $u = 70\,\text{dB}$, resulting in three trial types in the first step. An estimate for t_s was obtained using the trial type $\langle (x,u),(t_\text{s},t_\text{s}) \rangle$, estimates for w_s and s_s used trial types given by expression (27), and in the second step, t' was estimated using the trial type given by expression (26).

A theoretical prediction is that the property holds for both $p < 1$ and for $p \geqslant 1$. Two production conditions meeting these constraints were chosen, namely $p = \frac{2}{3}$ and $p = 2$.

Table 1
Experiment 1: bisymmetry

Resp.	i	Mean (s.d.)		p_stat	n (Step 1)	Stat. concl.
		T_1	T_2			
R2	s	68.90 (2.24)	69.15 (1.48)	0.464	40 (46)	$T_1 = T_2$
R3	s	62.36 (1.14)	61.90 (1.39)	0.144	45 (60)	$T_1 = T_2$
R11	s	61.56 (3.26)	62.20 (3.41)	0.261	32 (34)	$T_1 = T_2$
R14	s	68.31 (3.90)	68.34 (2.20)	0.208	45 (60)	$T_1 = T_2$
R16	l	77.24 (1.27)	76.83 (1.25)	0.106	60 (60)	$T_1 = T_2$
R17	l	76.04 (1.16)	75.73 (0.60)	0.064	60 (60)	$T_1 = T_2$

Trials were blocked by steps. The first block contained the estimates completed in step one (six estimates); the second block contained estimates completed in step two (three of each of two types of estimates). The blocks were run in separate sessions.

2.3.2. Results

Four respondents completed this experiment and their data are presented in Table 2.

In the table, T_1 and T_2 stand for the means of t_s and t'_s, respectively; standard deviations are given in parentheses. All sound pressure levels are given in dB SPL. The number of observations t_s and t'_s are indicated by n_{T_1} and n_{T_2}, respectively. By design, n_{T_1} also indicates the number of individual observations used for estimating w_s and s_s. Statistical results are indicated by corresponding p-values labeled p_{stat}.

The property held in both conditions for two of four respondents with the other two failing one of the two conditions, $p = \frac{2}{3}$, 2 respectively. That is, the property was accepted in six out of eight tests.

On a few trials of the $p = 2$ condition, marked by ∗, respondent R5 sought to produce a t_s and a t'_s with sound pressure level above the procedural safety limit. Hence, the possibility cannot be excluded that, without this limit, the two values would have been found statistically different.

A fifth person, R15, completed the first step of the experiment but was observed to violate monotonicity in step two. Specifically, we obtained t'_s by the matching $(w_s, s_s) \sim (t'_s, t'_s)$. We found $w_s < s_s < t'_s$, which violates the monotonicity of ψ that dictates $w_s < s_s$ and $t'_s < s_s$, a result that was obtained for all other respondents for both values of p. Note that R15 participated in an earlier matching experiment (Steingrimsson & Luce, 2005) and in that case the data do not show any violations of monotonicity. We do not have any firm hypothesis to account for R15's violation of monotonicity in step two. The fact that monotonicity was repeatedly observed in the respondent's responses prior to the current task leads us to suspect that performing production judgments in the way actualized here has led to some time-order effect in the matching task. Yet, absent further information, this case remains a mystery.

2.3.3. Discussion

Productions are well known to exhibit greater variability than matching, and variability increases with a decrease in intensity. Because when the present experiment was carried out we had not yet realized that the multi-step procedure explained in Section 2.1.7 would have been better, the variance in t'_s is not accumulated over the two estimation steps. Given this, it is not surprising that the average variance for T_1 (2.46) is greater than for T_2 (1.95), i.e., productions are more variable than matching, and that the variability is smaller for $p = 2$ (1.60) than for $p = 2/3$ (2.77).

We know that non-parametric ranking statistics (such as the Mann–Whitney) show bias towards rejecting the null hypothesis (Type I) in face of heterogeneous variance (Zimmerman, 2000). Because of the lost variance information in the two-step estimates, there are some cases with substantial heterogeneity in the variance between T_1 and T_2, which makes rejecting the null harder, not easier. In a ranking process it is, of course, equality of medians that we are testing, hence the most troublesome cases are those where the difference between the medians ($|T_1 - T_2|$) appears to be relatively large, the variability is relatively high in both samples (T_1 and T_2), and the test accepts the equality of the medians ($T_1 = T_2$). Upon close inspection, there is only one such case, namely for R5 in the $p = \frac{2}{3}$ condition. Even though this, and the other test results, have been checked using the bootstrapping process (see 2.1.6), this particular case should be accepted only with some reservation. (The estimated variances stated above are considerably reduced if this one case is excluded.)

With reservations about the one test result, we find simple joint-presentation decomposition holding statistically in six out of eight tests, which provides reasonable initial support of this property.

Table 2
Experiment 2: simple joint-presentation decomposition

Resp.	p	Mean (s.d.)		p_{stat}	n_{T_1}, n_{T_2}	Stat. concl.
		T_1	T_2			
R4	2	74.0 (1.34)	72.63 (1.79)	<0.001	60,60	$T_1 \neq T_2$
	2/3	64.53 (1.94)	64.28 (2.95)	0.823		$T_1 = T_2$
R5	2*	82.58 (3.43)	84.98 (1.06)	0.257	60,60	$T_1 = T_2$
	2/3	48.64 (5.99)	50.25 (5.20)	0.096		$T_1 = T_2$
R10	2	78.95 (1.13)	78.97 (0.71)	0.480	30,28	$T_1 = T_2$
	2/3	62.42 (1.71)	61.13 (1.38)	0.003		$T_1 \neq T_2$
R23	2	78.59 (2.58)	79.58 (0.80)	0.065	60,56	$T_1 = T_2$
	2/3	63.54 (1.54)	63.43 (1.47)	0.779		$T_1 = T_2$

2.4. Experiment 3: segregation

Recall that, for $i = \mathrm{l, r, s}$, left segregation is given by

$$u \oplus_i (x \circ_{p,i} 0) \sim (u \oplus_i x) \circ_{p,i} (u \oplus_i 0)$$

and right segregation by

$$(x \circ_{p,i} 0) \oplus_i u \sim (x \oplus_i u) \circ_{p,i} (0 \oplus_i u).$$

Although right and left segregation are equivalent in the case of symmetric joint presentations (i.e., $\oplus_i$ is commutative), we know, as mentioned earlier, that joint presentations are not symmetric for most respondents, i.e., $(x, u) \sim (u, x)$ does not hold in general. Hence, as explained in Section 1.4.3, right and left segregation are not equivalent.

As explained in Section 1.3.3, because 0 is a right identity of $\oplus_i$, testing left segregation is easier for $i = \mathrm{l}$ and right segregation is easier for $i = \mathrm{r}$. For $i = \mathrm{s}$ both need to be tested.

The experiment was carried out before Luce (2004a) derived the case for $i = \mathrm{s}$ so it was not tested here, but the two other cases were tested. However, when we designed and ran the experiment we did not fully understand the theory and we believed there to be an interlock between the nature of the bias and the type of segregation needed for $i = \mathrm{l, r}$. For this reason, we determined for each respondent the nature of the bias, left or right, and studied the corresponding form of segregation, left or right. This meant two things: (i) there was an unnecessary step in determining the nature of the bias and (ii) for each respondent (except R22), we studied only one form of segregation, either left or right.

In the following, methodological issues are discussed for right segregation with $i = \mathrm{r}$—the other cases follow readily from that discussion.

The compound production $(x \oplus_\mathrm{r} u) \circ_{p,\mathrm{r}} u$ on the left-hand side of right segregation can be obtained with a trial of the form, a variation on (27),

$$\langle\langle(0, u), (x, u)\rangle, \langle(0, u), (0, v_\mathrm{r})\rangle\rangle,$$

thus eliminating the need to first estimate $x \oplus_\mathrm{r} u$. In practice, respondents would hear two tone-intervals, the first a monaural tone u followed by a binaural one (x, u), then the u is repeated and is followed by v_r, whose intensity respondents can adjust. Thus, at least as long as $|x - u|$ is not large, the presentation $(0, u)$ is subjectively experienced in the right ear whereas the subsequent joint-presentation (x, u) has a head-centered location. So, when $(0, u)$ is presented again the presentation can create the sensation of rapid movement of a sound from the right ear to a somewhat centered location and then back to the right ear. This is similar to the situation discussed in relation to estimating simple joint-presentation decomposition (Section 2.3). Here as there, this experience was reported to be distracting. For this reasons, in the case of $i = \mathrm{l, r}$ it is beneficial to estimate $x \oplus_\mathrm{r} u$ first.

2.4.1. Method

Because of the number and variety of estimated values, testing segregation was quite challenging with respect to experimental design and procedure. Four estimates must be made

$$w_\mathrm{r} = x \circ_{p,\mathrm{r}} 0, \quad \text{then using } w_\mathrm{r}, \tag{29}$$

$$t_\mathrm{r} = w_\mathrm{r} \oplus_\mathrm{r} u, \tag{30}$$

$$z_\mathrm{r} = x \oplus_\mathrm{r} u, \quad \text{then using } z_\mathrm{r}, \tag{31}$$

$$t'_\mathrm{r} = z_\mathrm{r} \circ_{p,\mathrm{r}} u. \tag{32}$$

The property is said to hold if t_r and t'_r are not found statistically different.

Both t_r (30) and t'_r (32) are estimated using the prior estimates of w_r (29) and z_r (31). Note, however, that while w_r and z_r were estimated in the right ear, in (30) and (32) they are presented in the left ear (the converse is the case for left segregation). (See Footnote 7.)

The property is predicted to hold for both $p < 1$ and $p \geqslant 1$, hence $p = \frac{2}{3}$ and $p = 2$ were used. One sound pressure level pair, $x = 72\,\mathrm{dB}$ and $u = 68\,\mathrm{dB}$, was used for all respondents except R23 for whom it was dropped to $x = 68\,\mathrm{dB}$ and $u = 64\,\mathrm{dB}$ to avoid attempted productions above the safety limit.

The experiment was carried out using three block types (additional comments follow the list).

(B1) The estimates for z_r were carried out in a block containing two dummy trials to ensure stimulus variety within a block, namely $x \oplus_\mathrm{r} u$ and $u \oplus_\mathrm{r} u$. Each block contained two instances of each trial type, given by expression (25), or six in all.

(B2) The estimates for w_r and t'_r were carried out together in a block containing sub-blocks for each estimate. Each sub-block contained three trials of each of the two trials forms, six in all. Respondents were given the textual alerts of "One interval" for a sub-block estimating of w_r and "Two intervals" for t'_r. The trial type for estimating w_r is given by

$$\langle(0, x), (0, v'_\mathrm{r})\rangle$$

and for estimating t'_r is given by

$$\langle\langle(0, u), (0, z_\mathrm{r})\rangle, \langle(0, u), (0, v_\mathrm{r})\rangle\rangle,$$

where an average is used for z_r.

(B3) Two instances, corresponding to the two proportion values, of the estimate for t_r were needed. These were performed in a block containing three trials of each of the two trials forms for a total of six trials. The trial types are given by expression (25) with an average of w_r used as input.

In (B2), all production judgment conditions were run together within a session, which prevents systematic effects of inter-session variability. Of course, when data for these conditions are pooled across sessions, inter-session effects do manifest themselves in increased variability. However, as the Mann–Whitney test is based on ranking, only intra-session variability is of statistical concern. These conditions were estimated using procedurally different trial types, and the respondents strongly preferred knowing in advance which type of trial they were about to experience. Additionally, respondents preferred having only one kind of trial form in a block. Hence, these two types were placed in two sub-blocks within a block and the respondents were alerted to the upcoming task type.

Sessions were blocked on these block types. Even though w_{r} and t'_{r} were both estimated within a block, pilot data from (B2) showed substantial inter-session variation in the relative relationship for the two production tasks. Extensive practice, consisting of at least three practice sessions with (B2), seemed to counter this problem.

For R22 with $i = \mathrm{r}$, an improved version of the experiment was employed (see Section 2.1.7). In this version, all trial conditions were run within a session and individual estimates, rather than averages, were propagated through the process. Within the session, 20 instances of each trial condition were collected before moving to the next step. Hence, a session consisted of 80 trials.

2.4.2. Results

Four respondents completed this experiment. Their results are reported in Table 3. In that table, the form of segregation is given by l and r for left and right segregation, respectively (i has, by design, the same value).

Furthermore, T_1 and T_2 stand for the means of t_i and t'_i, respectively; standard deviations are given in parentheses. Sound pressure levels are given in dB SPL. The value n is the number of observations of t_i, t'_i, and w_i; n_z is the number of observations of z_i. Statistical results are indicated by corresponding p-values.

The property held in four out of five cases for both $p = \frac{2}{3}$ and $p = 2$ (although for R24 using $p = 2$ the statistic is close to rejecting equality of the means). That is, binary segregation was not rejected in eight (perhaps seven) out of 10 tests.

2.4.3. Discussion

An open question is whether people perform production judgments of $v_i = x \circ_{p,i} y$ in the same qualitative fashion for $y > 0$ as for $y = 0$. For a general discussion of this phenomenon see Luce (2004b). Substantial inter-session variation in the relationship between the two production types was observed. Although practice seemed to alleviate this problem, the observation supports the notion that respondents do not perform the two production judgments equivalently.

With one exception (R22, $i = \mathrm{r}$), the data were collected before we understood how to preserve variance information in multi-step estimations. Hence, the true variance of both t_i and t'_i is likely larger than indicated. Because t_i was estimated using matches and t'_i estimated using productions, the variance associated with the former is expected to be less than that of the latter. A similar situation arose in the testing of simple joint-presentation decomposition and we refer the reader to the discussion of it there (Section 2.3.3). Contrary to that experiment, we find no immediately suspect results here, except for perhaps R10, $p = 2$, where the difference between T_1 and T_2 is not overly large, but the variance appears unusually low—one might suspect that if the accumulated variance had been preserved, the equality of the medians might not have been rejected.

Given the challenges that the testing of this property generates, reasonable initial support has been established for segregation.

Table 3
Experiment 3: segregation

Resp.	Seg.	p	Mean (s.d.)		p_{stat}	n	Stat. concl.
			T_1	T_2		(n_z)	
R10	l	2	80.32 (0.61)	79.21 (0.55)	<0.001	30	$T_1 \neq T_2$
		2/3	73.60 (1.04)	73.86 (1.09)	0.656	(42)	$T_1 = T_2$
R22	l	2	78.95 (1.13)	78.27 (1.63)	0.197	30	$T_1 = T_2$
		2/3	71.98 (1.27)	72.22 (2.53)	0.573	(30)	$T_1 = T_2$
	r	2	69.18 (1.10)	69.73 (1.73)	0.121	40	$T_1 = T_2$
		2/3	62.30 (1.78)	64.33 (1.82)	<0.001	(40)	$T_1 \neq T_2$
R23	l	2	81.18 (1.14)	80.62 (1.44)	0.202	30	$T_1 = T_2$
		2/3	71.10 (2.26)	70.55 (1.69)	0.110	(30)	$T_1 = T_2$
R24	r	2	81.30 (1.22)	82.01 (1.42)	0.052	30	$T_1 = T_2$
		2/3	73.10 (1.74)	73.37 (1.98)	0.594	(30)	$T_1 = T_2$

3. General discussion

3.1. Summary

The topic has been a theory of global psychophysical judgments leading to the two representation classes. For asymmetric matches, the theory leads to the following three properties:

$$\Psi(x,u) = \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u) \quad (\delta \geqslant 0), \quad (5)$$

$$W(p) = \frac{\Psi[(x,u)\circ_p(y,v)] - \Psi(y,v)}{\Psi(x,u) - \Psi(y,v)} \quad [(x,u) \succ (y,v) \succsim (0,0)], \quad (6)$$

$$\Psi(x,0) = \gamma\Psi(0,x) \quad (\gamma > 0). \quad (7)$$

For symmetric matches, (5) with $\delta = 0$ and (6) both hold, whereas the constant bias property of (7) need not hold.

These representations have a number of necessary consequences (behavioral properties) that in turn are sufficient under certain structural conditions to give rise to the representations. Those that underlie (5) and (6) separately were examined and sustained in Steingrimsson and Luce (2005). Those that link the two representations and force a common psychophysical function were examined here. Results of the tests of bisymmetry and the two linking properties are summarized in Table 4.

Within the framework of the theory, in particular representation (5), the statement that $\delta = 0$ is equivalent to bisymmetry holding for each of the three induced operations. That prediction was strongly supported (Experiment 1). Accepting this made it possible to assume $\delta = 0$ in designing tests for simple joint-presentation decomposition (Experiment 2) and segregation (Experiment 3). Both properties were partially sustained with 2 failures each in 8 and 10 tests, respectively.

3.2. Further work

Sufficient support for the theory has been found in the two articles to suggest that further theoretical and experimental work be undertaken. This includes the possibility of testing the theory in other domains such as vision as well as some additional directions suggested by the present experiments, by some informal observations, and by parallel evidence from utility theory.

Table 4
Summary of experimental results

Exp. #	Name	#R	#Tests	#Fail
1	Bisymmetry	6	6	0
2	JP decomp.	4	8	2
3	Segregation	4	10	2

One observation is that respondents seem to process $(x,x)\circ_{p,i}(y,y)$, $y>0$, differently from $(x,x)\circ_{p,i}(0,0)$. One can imagine that this might take the form of the respondents distorting p differently in the two cases. So, for example, this might lead to replacing (6) by something of the form

$$\frac{\Psi[(x,x)\circ_p(y,y)] - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} = \begin{cases} W(p) & y>0, \\ W_0(p) & y=0. \end{cases}$$

This kind of representation and a generalization of it are the topics of Luce (2004b). A major feature of the case of $y>0$ is that the property of segregation is replaced by the following two forms of distribution:

Left distributivity:

$$z\oplus_i(x\circ_{p,i}u) = (z\oplus_i x)\circ_{p,i}(z\oplus_i u) \quad (u>0). \quad (33)$$

Right distributivity:

$$(x\circ_{p,i}u)\oplus_i z = (x\oplus_i z)\circ_{p,i}(u\oplus_i z) \quad (u>0). \quad (34)$$

Segregation is the special case of (33) and (34) with $u = 0$. Again, in the unbiased case no distinction exists between the left and right version. For $i = \text{l}, \text{r}$, one can deduce the two forms of distributivity from the representation. For $i = \text{s}$ and with both versions of segregation holding for $\oplus_\text{s}$, one can show that both left and right distributivity are satisfied.

These theoretical results need to be explored more carefully in the psychophysical context and additional empirical studies conducted, including of left distributivity for $\oplus_\text{l}$ and right distributivity for $\oplus_\text{r}$ and both for $\oplus_\text{s}$.

3.3. Conclusions

Our central conclusion from the empirical evaluations in this and the previous paper of Luce's (2002, 2004a) theory is that it has received reasonable support in the auditory domain with stimuli well-above threshold and with common frequency and phase.

An important message of these articles is that one can study the adequacy of a representation in which there are both free functions and free parameters without estimating either the unspecified functions or the parameters and carrying out some form of goodness-of-fit. The latter approach is common in cognitive modeling and encounters considerable difficulty. The behavioral properties we tested are all parameter free.

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Appendix A. Commutativity and associativity

In the unbiased case, one can easily show from (15) that the following two conditions are met:

Commutativity (*or symmetry*) *of* $\oplus_i$:

$$x \oplus_i u = u \oplus_i x. \tag{A.1}$$

Associativity of $\oplus_i$:

$$x \oplus_i (y \oplus_i z) = (x \oplus_i y) \oplus_i z. \tag{A.2}$$

Although Steingrimsson and Luce (2005), show that a substantial proportion of respondents exhibited bias as measured by commutativity, for three of them they did not reject the hypothesis of being unbiased and so commutativity appears to hold. However, as Zimmer, Luce, and Ellermeier (2001) showed in their work, associativity was rejected in those cases when commutativity was not rejected. Thus, associativity is an important further test to carry out when trying to decide if a person is actually unbiased. We ignored this fact when we tested symmetry.

Appendix B. Testing bisymmetry using two-ear matches

The object is to show that bisymmetry can be tested using two-ear matching. Bisymmetry is given as

$$(x \oplus_s y) \oplus_s (u \oplus_s v) = (x \oplus_s u) \oplus_s (y \oplus_s v).$$

We suppose that $\delta = 0$, which must be true if bisymmetry holds. Define

$$(x,y) \sim (w,w), \quad (u,v) \sim (w',w'), \quad (w,w') \sim (t,t),$$
$$(x,u) \sim (z,z), \quad (y,v) \sim (z',z'), \quad y(z,z') \sim (t',t').$$

We show that bisymmetry of $\oplus_s$ is equivalent to $t = t'$.

If we define $\zeta = \gamma/(1+\gamma)$ and look at the table just below (23), we see that

$$\psi_s(w) = \zeta\psi_s(x) + (1-\zeta)\psi_s(y),$$
$$\psi_s(w') = \zeta\psi_s(u) + (1-\zeta)\psi_s(v),$$
$$\psi_s(t) = \zeta\psi_s(w) + (1-\zeta)\psi_s(w').$$

Starting at the bottom and substituting

$$\psi_s(t) = \zeta^2\psi_s(x) + \zeta(1-\zeta)[\psi_s(y) + \psi_s(u)] + (1-\zeta)^2\psi_s(v).$$

The analogous expression for t' is

$$\psi_s(t') = \zeta^2\psi_s(x) + \zeta(1-\zeta)[\psi_s(u) + \psi_s(y)] + (1-\zeta)^2\psi_s(v).$$

Because of the commutativity of $+$ and the strict monotonicity of ψ_s, we conclude that bisymmetry of $\oplus_s$ is equivalent to $t = t'$.

Appendix C. Using two-ear matching to test simple joint-presentation decomposition when $\delta = 0$

The object is to show that when $\delta = 0$ we may test simple joint-presentation decomposition using two-ear matching and production judgments. For the sake of simplifying the exposition, the proof is demonstrated with the use of the $\oplus_l$ and $\circ_{p,l}$ operators only; the argument using $\oplus_r$ is similar.

The simple joint-presentation decomposition hypothesis is, for $i = l$,

$$(x \oplus_l u) \circ_{p,l} 0 = (x \circ_{p,l} 0) \oplus (u \circ_{p,l} 0). \tag{C.1}$$

Define z for the left-hand side of (C.1) as the solution to $(z,z) \sim (x,u)$ in which case

$$x \oplus_l u = z \oplus_l z.$$

Using this and $0 = 0 \oplus_l 0$ we let t be the solution to

$$(x \oplus_l u) \circ_{p,l} 0 \sim (z \oplus_l z) \circ_{p,l} (0 \oplus_l 0) \sim t \oplus_l t.$$

And for the right-hand side of (C.1), define v, w, t' as the solutions to

$$(x \oplus_l x) \circ_{p,l} (0 \oplus_l 0) = v \oplus_l v,$$
$$(u \oplus_l u) \circ_{p,l} (0 \oplus_l 0) = w \oplus_l w,$$
$$v \oplus_l w = t' \oplus_i t'.$$

Using (23) and the separable representation

$$\psi_i(x \circ_{p,l} 0) = W(p)\psi_l(x)$$

yields

$$\psi_l(x \oplus_l u) = \psi_l(z \oplus_l z) = [\mu_l(l) + \mu_r(l)]\psi_l(z) \tag{C.2}$$

and

$$\begin{aligned} \psi_l(t \oplus_l t) &= [\mu_l(l) + \mu_r(l)]\psi_l(t) \\ &= \psi_l((x \oplus_l u) \circ_{p,l} 0) \\ &= \psi_l((z \oplus_l z) \circ_{p,l} 0) \\ &= \psi_l(z \oplus_l z) W(p) \\ &= [\mu_l(l) + \mu_r(l)]\psi_l(z) W(p). \end{aligned}$$

Because of (C.2) with z replaced by t, it follows that

$$\psi_l(t) = \psi_l(z) W(p).$$

So, from (23),

$$\begin{aligned}\psi_1(t) &= \psi_1(z)W(p)\\ &= \frac{\psi_1(x\oplus_1 u)}{\mu_1(l)+\mu_r(l)}W(p)\\ &= \frac{\mu_1(l)\psi_1(x)+\mu_r(l)\psi_1(u)}{\mu_1(l)+\mu_r(l)}W(p)\\ &= \frac{\mu_1(l)\psi_1(x)W(p)+\mu_r(l)\psi_1(u)W(p)}{\mu_1(l)+\mu_r(l)}\\ &= \frac{\mu_1(l)\psi_1(v)+\mu_r(l)\psi_1(w)}{\mu_1(l)+\mu_r(l)}\\ &= \frac{\psi_1(v\oplus_1 w)}{\mu_1(l)+\mu_r(l)}\\ &= \psi_1(t'),\end{aligned}$$

whence $t = t'$, proving that a two-ear method may be employed.

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