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Ranked Additive Utility Representations of Gambles: Old and New Axiomatizations

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Abstract

A number of classical as well as quite new utility representations for gains are explored with the aim of understanding the behavioral conditions that are necessary and sufficient for various subfamilies of successively stronger representations to hold. Among the utility representations are: ranked additive, weighted, rank-dependent (which includes cumulative prospect theory as a special case), gains decomposition, subjective expected, and independent increments*, where * denotes something new in this article. Among the key behavioral conditions are: idempotence, general event commutativity*, coalescing, gains decomposition, and component summing*. The structure of relations is sufficiently simple that certain key experiments are able to exclude entire classes of representations. For example, the class of rank-dependent utility models is very likely excluded because of empirical results about the failure of coalescing. Figures 1–3 summarize some of the primary results.

Keywords: coalescing, component summing, event commutativity, gains decomposition, ranked additive utility, ranked weighted utility, utility representations

JEL Classification: D46, D81

This article explores the relations among a variety of utility representations. Some of the representations and results are fairly classical, others involve new axiomatizations. This undertaking involves partially summarizing and sharpening some existing results and proving six new ones.

The general strategy of the paper is as follows. We begin by assuming what we call a ranked-additive representation of gambles, where a ranked gamble consists of a finite number of consequence-event branches ordered by preferences among the consequences. The utility representation involves a sum of functions, one for each branch, that depends on two things: the utility of the consequence and the entire ordered vector of the event partition. This representation is a form of ranked additive conjoint measurement that was axiomatized by Wakker (1991). Starting there we formulate in Section 1 four specializations of the ranked additive form, which are among the representations that have arisen in the literature. Included are the class of ranked weighted utility, rank-dependent utility (also called Choquet utility and cumulative prospect theory), gains-decomposition utility

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(Marley and Luce, 2001), simple utility with weights that depend only on the relevant event, as used in the original prospect theory of Kahneman and Tversky, 1979), and subjective expected utility.

The thrust of Section 2 is to add qualitative conditions that are necessary and sufficient to go from the ranked-additive representation to each of these specializations, except that we are missing the conditions for simple utility. These conditions are all of a type that Luce (1990) has called accounting equivalences or, as we do here, indifferences. An accounting indifference asserts that two distinct gamble formulations that have the same bottom line are seen as indifferent by the decision maker. The section summarizes two old results, with some improvements, as well as three new ones about the interrelations among the representations. These results are summarized as Figures 1 and 2 in Section 2.7. One interesting conclusion is that the state space formulation of gambles, first used by Savage (1954) and frequently postulated in later attempts at generalization of subjective expected utility, in fact forces all of the accounting indifferences and so that once a ranked-additive utility representation is assumed in that state space context, subjective expected utility necessarily follows (Section 2.6).

The remainder of the article explores what happens when one adds to the framework of preferences among gambles the concept of joint receipts of consequences and of gambles. Section 3 defines, using joint receipt, a number of additional properties. The distinction between the joint receipt of subjectively independent gambles and the joint receipt of totally dependent ones is explored. The results for the independent case (Section 3.1.2) are referenced. The important totally dependent case, not previously studied, assumes that the ranked-additive representation holds and invokes the apparently innocent accounting indifference called component summing. With the additional assumption that the representation of joint receipts is decomposable with respect to the representation of the components, joint receipt is forced to have a generalized additive representation. Moreover, when the unranked case obtains, this actually becomes an additive representation and the representation of gambles reduces to a weighted utility one. These results are summarized in Figure 3 in Section 3.6.

Section 4 introduces the general cases of a relatively new class of additive representations, called ranked increasing utility increments, in which the component utilities of the additive form depend on functions of utility differences of consequences and on the entire ordered vector of the event partition. We study what happens when a gambling structure has both this representation and a ranked-additive one, and we find that this leads to the ranked-weighted representation. Next the closely related concepts of segregation and distribution are introduced. These amount to the joint receipt of a gamble and a consequence being seen as indifferent to the same gamble with each of its consequences replaced by the joint receipt of that consequence and the added one. Segregation concerns the case where the smallest consequence is no-change from the status quo and distribution excludes the status quo consequence. The latter case may be important given some empirical results (Birnbaum, 1997) that show that the status quo consequence affects preferences in special ways. We relate these concepts to ranked increasing utility increments in two theorems, and summarize these results in Section 4.5.

Finally, Section 5 provides a brief recapitulation and states seven open problems.

1. Classes of representations of gambles

1.1. Notation

Let X denote the set of pure consequences for which chance or uncertainty plays no role. A distinguished element $e \in X$ is interpreted to mean *no change from the status quo*. We assume a *preference order* $\succsim$ exists over X and that it is a weak order. Let $\sim$ denote the corresponding indifference relation. A typical *first-order gamble of gains* g with n consequences is of the form

$$g = (x_1, C_1; \ldots ; x_i, C_i; \ldots ; x_n, C_n) \equiv (\ldots ; x_i, C_i; \ldots),$$

where $x_i \succsim e$ are consequences (of gains) and $\mathbf{C}_n = (C_1, \ldots , C_i, \ldots , C_n)$ is a partition of some "universal event" $C(n) = \bigcup_{i=1}^{n} C_i$. The underlying event $C(n)$ is only "universal" for the purpose of this gamble. A (consequence, event) pair (x_i, C_i) is called a *branch*. Such a gamble is a $2n$-tuple composed of n branches. We deal mostly with cases where each $C_i \neq \emptyset$.

We intentionally do not follow the Savage (1954) route of having a common "state space" underlying all gambles, and for that reason some economists have criticized our approach. We contend that there is nothing holy in Savage's formulation and that ours has some distinct advantages if one wishes to account for observed behavior. First, the reader should attempt to formalize explicitly the state space corresponding to the several decisions made during an ordinary day of his or her life. Better yet, consider a year. It is unmercifully complex and hardly anyone approaches decisions in this fashion. Second, and closely related, when running experimental studies, the gambles are presented to the respondent in our form, not Savage's. Third, and probably decisively, the state space approach builds in some properties that are quite restrictive. This point is discussed more fully in Section 2.6 where we argue that if one is interested in potentially descriptive theories, one should avoid Savage's formulation.

This article is confined to all gains, i.e., where each $x_i \succsim e$ (or, equally, to all losses, i.e., where each $x_i \precsim e$).

We assume X is so rich that for any first-order gamble g, there exists $CE(g) \in X$, called the *certainty equivalent* of g, such that $CE(g) \sim g$. Thus, the preference order $\succsim$ can be extended to the domain of gains $\mathcal{D}_+$ that consists of X and all first-order gambles. For some results we need to expand $\mathcal{D}_+$ to include *second-order gambles* in which some of the x_i are replaced by first-order gambles. Under the usual co-monotonicity assumption, i.e., monotonicity is preserved so long as the consequence ranking is preserved, indifferent alternatives may be substituted without altering the appraisal of the gamble. So, in particular, if f is a second-order gamble with as one of its consequences, say the ith, a gamble g_i, then g_i can be replaced by the indifferent $CE(g_i) \sim g_i$ to obtain a first-order gamble $f' \sim f$. Thus, any second-order gamble can always be reduced to a first-order one by using certainty equivalents and preserving preferences. We do not bother to write things in that fashion.

We also assume that if ρ is a permutation of the indices $\{1, 2, \ldots, n\}$, then

$$\begin{aligned}&\big(x_{\rho(1)}, C_{\rho(1)}; \ldots; x_{\rho(i)}, C_{\rho(i)}; \ldots; x_{\rho(n)}, C_{\rho(n)}\big)\\ &\quad\sim (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n).\end{aligned} \tag{1}$$

Property (1) is an example of what has been called an *accounting indifference* (Luce, 2000). The basic feature of accounting indifferences is that the bottom lines on the two sides of $\sim$ are identical, and so normatively they should not be distinguished. We will introduce a number of additional accounting indifferences in Section 2. One can view accounting indifferences as invariance of preference under reformulations of the gamble.

In stating numerous axioms and representations, it is convenient to assume that we have carried out a permutation of the indices such that the consequences are (rank) ordered from the most preferred to the least preferred, in which case we simply use the notation

$$x_1 \succsim \cdots \succsim x_i \succsim \cdots \succsim x_n \succsim e. \tag{2}$$

For the most part, this article is concerned with properties formulated for the latter ranked case. In the ordered case, there is an order induced on the underlying partition $\mathbf{C}_n$, which we emphasize by the vector notation $\vec{\mathbf{C}}_n$. We give the rank ordered representation the same name as the corresponding unordered one, but with the prefix "ranked" added, and the abbreviations also prefixed with R. We do not explicitly include the relevant definitions for the unranked cases. In general, when people test theories empirically involving ranked consequences, such as RDU below, they present the gambles in the ordered form. But always keep in mind that so long as (1) is satisfied, the ranked form is nothing but a convenience for writing the representation or for writing an axiom leading to a ranked representation.

We explore utility representations U onto real intervals of the form $I = [0, \kappa[$, where $\kappa \in]0, \infty]$, that meet various, increasingly stronger, restrictions. Two conditions that are common to all representations in this article are:

$$g \succsim h \quad \text{iff } U(g) \geq U(h), \tag{3}$$

$$U(e) = 0. \tag{4}$$

We refer to these as *order-preserving representations.* Note that because I is open on the right, there is no maximal element in the structure.

1.2. Rank additive utility representations

Definition 1. An order-preserving representation $U\colon \mathcal{D}_+ \xrightarrow{\text{onto}} I$ is a *rank additive utility (RAU)* one iff, for all $x_i \in X$ satisfying (2) and for every corresponding ordered partition $\vec{\mathbf{C}}_n$ of $C(n)$, there exist strictly increasing functions $L_i(\cdot, \vec{\mathbf{C}}_n)\colon I \xrightarrow{\text{onto}} I$, with the following properties:

$$\begin{aligned} U(x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) &= \sum_{i=1}^{n} L_i(U(x_i), \vec{\mathbf{C}}_n), \\ L_i(0, \vec{\mathbf{C}}_n) &= 0, \\ C_i = \emptyset \text{ implies } L_i(Z, \vec{\mathbf{C}}_n) &= 0 \quad (Z \in I). \end{aligned} \tag{5}$$

It is an *additive utility (AU)* representation iff the functions $L_i(\cdot, \vec{\mathbf{C}}_n)$ are the same for every ordering of the partition $\mathbf{C}_n$ of $C(n)$.

Note that in Definition 1, we assume that $C_i = \emptyset$ implies $L_i(Z, \vec{\mathbf{C}}_n) = 0$ $(Z \in I)$, whereas, for simplicity, we restrict the proofs of the relevant representation theorems to partitions with all non-null sets.

Given (5), one would like to know of specific behavioral constraints that limit it to the particular forms found in the literature including rank-dependent utility, subjective expected utility, and the several descriptive configural weighted models,[1] such as RAM and TAX, of M. H. Birnbaum and his colleagues (for summaries, see Birnbaum, 1997, 1999). We do not discuss the general TAX model in this article as it is known to be equivalent to the idempotent weighted utility representation of the next subsection (Marley and Luce, 2004).

1.3. Ranked weighted utility representations

A concept of a rank weighted utility, RWU, representation, is defined in Marley and Luce (2001). Karamarkar's (1978) earlier use of the term "subjectively weighted utility" is a special case of our unranked usage, whereas Chew's (1983) use of "weighted utility" is rather different. Included in the RWU class is the general class of rank-dependent utility representations and others as well, such as one based on the concept of gains decomposition first defined in Liu (1995). The form of the RWU representation is given by:

Definition 2. An order-preserving representation $U: \mathcal{D}_+ \xrightarrow{\text{onto}} I \subseteq \mathbb{R}_+$ is a *ranked weighted utility (RWU)* one iff there exists weights $S_i(\vec{\mathbf{C}}_n)$ assigned to each index $i = 1, \ldots, n$ and possibly dependent on the entire ordered partition $\vec{\mathbf{C}}_n$, where $0 \leq S_i(\vec{\mathbf{C}}_n)$ and $S_i(\vec{\mathbf{C}}_n) = 0$ iff $C_i = \emptyset$, such that, for (2) holding,

$$U(\ldots; x_i, C_i; \ldots) = \sum_{i=1}^{n} U(x_i) S_i(\vec{\mathbf{C}}_n). \tag{6}$$

If the ranking is immaterial, it is called *weighted utility (WU).*

Note that the multiplicative weights of (6) depend both on i and on the entire ordered partition $\vec{\mathbf{C}}_n$. We study several representations that place restrictions on the latter dependence.

Rank weighted utility, (6), is the special case of RAU for which the functions $L_i(\cdot, \vec{\mathbf{C}}_n)$ are all linear, i.e., $L_i(Z, \vec{\mathbf{C}}_n) = Z S_i(\vec{\mathbf{C}}_n)$, where $S_i(\vec{\mathbf{C}}_n) = 0$ iff $C_i = \emptyset$. For $C_i \neq \emptyset$, then

$S_i(\vec{\mathbf{C}}_n) > 0, i = 1, \ldots, n$, which means that the representation is strictly increasing in each consequence so long as the rank ordering is maintained.

We discuss an axiomatization of the RWU form in Section 2.2.

Consider a gamble with $y_1 = \cdots = y_i = \cdots = y_n = y$ and consider the following property:

Definition 3. *Idempotence* of gambles is satisfied iff, for every $y \in X$ and every ordered event partition $(C_1, \ldots, C_i, \ldots, C_n)$,

$$(y, C_1; \ldots; y, C_i; \ldots; y, C_n) \sim y. \tag{7}$$

If we assume idempotence along with RWU, we see that

$$\sum_{i=1}^{n} S_i(\vec{\mathbf{C}}_n) = 1. \tag{8}$$

Most theories of utility, including the RWU one described in Marley and Luce (2001), have either explicitly or implicitly assumed idempotence. Luce and Marley (2000) present some theoretical results for non-idempotent binary gambles. To our knowledge, no related theoretical work has been done on failures of idempotence in general gambles except for the largely unknown, but most interesting, article of Meginniss (1976).

Here we will, insofar as we know how, define concepts without assuming idempotence, and then we impose explicitly that property separately.

We demonstrate next that the RWU representation can be rewritten in a form that shows its relation to the rank-dependent form described in Section 1.4, and in another way that shows its relation to the increasing utility increment form discussed in Section 4.

Proposition 4.

1. *The following forms are equivalent when* (2) *holds*:

 (i) *Ranked-weighted utility*, (6).

 (ii)

$$U(\ldots; x_i, C_i; \ldots) = \sum_{i=1}^{n} U(x_i)[W_i(\vec{\mathbf{C}}_n) - W_{i-1}(\vec{\mathbf{C}}_n)], \tag{9}$$

 where

$$W_i(\vec{\mathbf{C}}_n) = \begin{cases} 0, & i = 0 \\ W_{i-1}(\vec{\mathbf{C}}_n) + S_i(\vec{\mathbf{C}}_n), & 0 < i \le n \end{cases}.$$

2. *The following are equivalent*:

(i) *Idempotent RWU, i.e., (6) satisfying* (8).
(ii)

$$U(\dots;x_i,C_i;\dots)-U(x_n)=\sum_{i=1}^{n-1}[U(x_i)-U(x_n)]\,S_i(\vec{\mathbf{C}}_n) \tag{10}$$

(iii)

$$U(\dots;x_i,C_i;\dots)-U(x_n)=\sum_{i=1}^{n-1}[U(x_i)-U(x_{i+1})]\,W_i(\vec{\mathbf{C}}_n). \tag{11}$$

The equivalence of idempotent RWU, (6) and (8), to the form of (9) was established in Marley and Luce (2001) but it is easily generalized to the non-idempotent case. The equivalence of idempotent RWU and (10) follows by a simple calculation using (8). And the equivalence of idempotent RWU and (11) is also a simple calculation. The forms (9) and (10) are, respectively, related to the representations of rank-dependent utility and increasing utility increments discussed in Section 4.

1.4. Rank-dependent utility

The RWU representation, (6) is of interest because it encompasses several models in the literature including the standard rank-dependent model. In the RDU model the weights are expressed only in terms of weights from the binary case, in which case all of the weights are of the form $W_{C(n)}(C(i), C(n)\backslash C(i))$, which we choose to write as $W_{C(n)}(C(i)) := W_{C(n)}(C(i), C(n)\backslash C(i))$, $i = 1, 2, \dots, n$, because knowing $C(n)$ and $C(i)$ for each $i = 1, 2, \dots, n$ tells us what the partition is. This notation is similar to that often used. For example, when $n = 3$, and (C, D, E) is an ordered partition of $C \cup D \cup E$, the RDU model, which is assumed to be idempotent, gives

$$\begin{aligned}U(x,C;y,D;z,E) = {} & U(x)W_{C\cup D\cup E}(C)+U(y)[W_{C\cup D\cup E}(C\cup D)-W_{C\cup D\cup E}(C)]\\ & +U(z)[1-W_{C\cup D\cup E}(C\cup D)].\end{aligned}$$

Define

$$C(i) := \bigcup_{j=1}^{i} C_j. \tag{12}$$

Definition 5. Rank-dependent utility[2] *(RDU)* is the special case of RWU in the form (9) with weights of the form

$$W_i(\vec{\mathbf{C}}_n)=\begin{cases}0, & i=0\\ W_{C(n)}(C(i)), & 0<i<n.\\ 1, & i=n\end{cases}$$

Given this definition, RDU is idempotent. Because in any idempotent case we have $1 = \sum_{i=1}^{n} S_i(\vec{\mathbf{C}}_n)$, then in the RDU case we have

$$1=\sum_{i=1}^{n}\left[W_{C(n)}(C(i)) - W_{C(n)}(C(i-1))\right] = W_{C(n)}(C(n)).$$

One can examine a non-idempotent generalization by replacing 1 for $i = n$ by $W_{C(n)}(C(n))$. We do not pursue this generalization.

Given the RWU representation, a natural next question is what restriction is equivalent to the RDU form. The result, Theorem 11 below, was proved in Luce (1998), but the proof we include here is simpler.

1.5. Gains-decomposition utility

A second class of idempotent RWU models was proposed in Marley and Luce (2001).

Definition 6. Within the domain of second-order gambles of gains, a *gains-decomposition utility (GDU)* representation holds iff RWU holds for a family of binary weights $W_{C(i)}, i = 1, \ldots, n$, with $C(n)$ the universal event, and with the weights $W_i(\vec{\mathbf{C}}_n)$ in (9) given by:

$$W_i(\vec{\mathbf{C}}_n) = \begin{cases} 0, & i = 0 \\ \prod_{j=i}^{n-1} W_{C(j+1)}(C(j)), & 1 \le i < n. \\ 1, & i = n \end{cases} \tag{13}$$

As we will see in axiomatizing this representation, we may sometimes want explicitly to call it lower GDU. As stated, the GDU representation is idempotent. One can consider the non-idempotent generalization mentioned in connection with RDU.

This representation, although it arises quite naturally as we shall see, has received hardly any attention. Because RDU is almost certainly not adequate descriptively (see Section 2.3), gains decomposition, Section 2.4, bears more examination as was begun in Marley and Luce (2004).

1.6. Simple weighted and subjective expected utility

The second part of the following definition is the special case of RWU called subjective expected utility (SEU). It is perhaps the most thoroughly explored and used utility representation, both normatively and prescriptively, despite the fact it is neither descriptive nor the only one that can reasonably contend for the normative title “rational”.

Definition 7. *Simple weighted utility (SWU)* is the special case of idempotent WU where

$$S_i(\vec{\mathbf{C}}_n) = W_{C(n)}(C_i). \tag{14}$$

Subjective expected utility (SEU) is the special case of SWU when the weights $W_{C(n)}$ are finitely additive, i.e., for $C, D \subseteq C(n)$, $C \cap D = \varnothing$,

$$W_{C(n)}(C \cup D) = W_{C(n)}(C) + W_{C(n)}(D). \tag{15}$$

The SEU utility form was first made very prominent by Savage (1954) who, however, assumed a common universal set (state space) for all gambles (or *acts* as he called them) (see Section 2.6). Many subsequent axiomatizations have been given; for a summary see Fishburn (1988). We give another quite simple one below. The SWU form appeared, for example, in the original prospect theory of Kahneman and Tversky (1979).

1.7. On alternative approaches

A substantial part of the literature follows the tack taken by Anscombe and Aumann (1963) who augmented the underlying structure by supposing that there exists a dense set of events with known probabilities. Indeed, for each $\lambda \in]0, 1]$ and each non-null event C, one assumes there is an event $C_\lambda \subseteq C$ such that $\Pr(C_\lambda \mid C)$ is defined and is equal to λ. In particular, the second-order binary lotteries so generated all exist and are denoted $\lambda f + (1 - \lambda)g := (f, C_\lambda; g, C\backslash C_\lambda)$. Most of the assumptions are formulated within this structure with little direct emphasis on the uncertain or vague events, but the theory induces indirectly weights on such events. Some of the axiomatizations lead to forms we have mentioned, such as rank-dependent (Choquet expected) utility. Others do not. Perhaps the most important variants are those for which the weights are of a minmax character: one assigns to each event a weight that corresponds to the minimum value of the probabilities of that weight from the several distributions in some family of probability distributions. The idea is that the decision maker behaves as if every distribution in the family is possible, selects the one that assigns the smallest probability, and uses that probability as the weight. Some of the relevant papers are Casadesus-Masanell, Klibanoff, Ozdenoren (2000), Ghirardato and Marinacci (2002), Ghirardato et al. (2003), Gilboa and Schmeidler (1989), and Schmeidler (1989).

It may be useful to compare briefly the differences in domains postulated in the several approaches to decision making under risk and uncertainty. The seminal work of von Neumann and Morgenstern (1947) axiomatized explicitly the representation of binary gambles, but included an assumption about compound lotteries that extended the representation to lotteries of any size. Also, they assumed events with known probabilities—risk. They used those probabilities to construct a utility function that satisfied the EU representation, which is SEU where $W_{C(n)}(C_i) = \Pr(C_i \mid C(n))$. The SEU representation of Savage (1954) assumed a (huge) state space for which probabilities were not specified although he was led to assume what amounted to very fine partitions into nearly equal subjective probabilities. The state space was assumed to encompass any conceivable uncertain events over which one might construct gambles. Those following the tradition of Anscombe and Aumann (1963) retained the Savage state space but augmented it with some of the probability structure assumed by von Neumann and Morgenstern (1947). Using that they inferred subjective probabilities on the other events. The present approach as well as that described in Luce (2000) deals with what amounts to conditional gambles constructed from the event space of a specific "experiment," not on the fixed space of states of nature. This is similar in spirit to von Neumann and Morgenstern (1947), but without assuming that chance events have assigned probabilities. We feel that conditional gambles are far more easily identified with empirical situations than are universal state spaces, and avoiding determinate probabilities means that the theory encompasses the many decision makers who do not have a very firm grip on classical probability in any context or lack the relevant information to estimate probabilities.

2. Results on representations of gambles

2.1. Need for axiomatization of rank additive utility (RAU)

We do not have an axiomatization of the RAU representation, Definition 1. We thought initially that this representation follows from a simple application of Wakker's (1991) results on additive conjoint measurement on rank-ordered domains, and, in the unranked case, from the somewhat simpler unranked version of additive conjoint measurement (Krantz et al., 1971). However, we realize now that an application of Wakker's results leads, at most, to a ranked representation of the form:

$$L(U(x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n), \vec{\mathbf{C}}_n) = \sum_{i=1}^{n} L_i(U(x_i), \vec{\mathbf{C}}_n),$$

with the RAU representation being the special case where $L(\cdot, \vec{\mathbf{C}}_n)$ is the identity function for all $\vec{\mathbf{C}}_n$. Thus, an important open problem is to axiomatize the RAU form. Also, it is likely of interest to explore the properties of the above more general form.

2.2. Axiomatization of rank weighted utility (RWU)

2.2.1. Event commutativity. Let us suppose that the domain $\mathcal{D}_+$ is extended to include second-order compound gambles (see Section 1.1). Within that extended domain, one major property of binary gambles that has played a fairly key role is the concept of *binary event commutativity*: For all events C, C', D, D' with $C \cap C' = D \cap D' = \emptyset$, and $x \succsim y \succsim e$

$$((x, D; y, D'), C; y, C') \sim ((x, C; y, C'), D; y, D'). \tag{16}$$

One sees that if one reduces this to a first-order gamble, it amounts to saying that on each side x is the consequence if C and D both occur, and otherwise y is the consequence, the only difference being the order in which the experiments (C, C') and (D, D') are conducted. The binary RDU model satisfies this property. If one has $y = e$ in (16) it is called *status-quo, binary event commutativity*. The question is how this concept might be generalized for gambles with more than two consequences. We explore one possible generalization that gives rise to the RWU representation.

Definition 8. Suppose that $\vec{C}_n, \vec{D}_m$ are any two ordered event partitions and that $x_1 \succsim x_2 \succsim \ldots \succsim x_n \succsim e$. Then *event commutativity* is satisfied iff

$$
\begin{aligned}
&((x_1, D_1; e, D_2; \ldots), C_1; (x_2, D_1; e, D_2; \ldots), C_2; \ldots; (x_n, D_1; e, D_2; \ldots), C_n) \\
&\quad \sim ((x_1, C_1; x_2, C_2; \ldots; x_n, C_n), D_1; e, D_2; \ldots; e, D_m). \qquad (17)
\end{aligned}
$$

where all of the events D_k, $k = 2, \ldots, m$, have e attached to them.

As in the binary case, both sides give rise to x_i iff both C_i and D_1 occur. This seems just as rational as binary event commutativity. This general property has not been studied empirically. Binary status-quo event commutativity has received empirical attention, where it seems to be supported (Luce, 2000, pp. 74–76).

A property called *timing indifference,* which is closely related to event commutativity, was formulated by Wang (2003). Because time is not an explicit variable in the theory, we prefer our term.

2.2.2. The results.

Theorem 9. *Consider a structure $\langle D_+, \succsim \rangle$ for gambles with $n \geq 2$ which has a ranked additive utility (RAU) representation, Definition 1, with the functions $L_i(, \vec{\mathbf{C}}_n)$, $i = 1, 2, \ldots, n$, having derivatives at* 0. *Then the following statements are equivalent*:

1. *Event commutativity is satisfied*, (17).
2. *The RAU representation is a RWU representation*, *Definition* 2.

All proofs are in the appendix.

2.3. *Axiomatization of rank dependent utility (RDU)*

2.3.1. Coalescing. The RDU representation exhibits the following property:

Definition 10. Coalescing is satisfied iff for all ordered consequences, (2), and corresponding ordered partitions with $n > 2$ and with $x_{k+1} = x_k$, $k < n$,

$$
\begin{aligned}
&(x_1, C_1; \ldots; x_k, C_k; x_k, C_{k+1} \ldots; x_n, C_n) \\
&\quad \sim (x_1, C_1; \ldots; x_k, C_k \cup C_{k+1}; \ldots; x_n, C_n) \quad (k = 1, \ldots, n-1). \qquad (18)
\end{aligned}
$$

Note that the gamble on the left has n branches with $n-1$ distinct consequences whereas the one on the right has $n-1$ branches as well as $n-1$ distinct consequences. Also, observe that if RDU obtains and simplifying the notation to $W := W_{C(n)}$, (18) follows from the fact that

$$\begin{aligned} &W(C(k+1)) - W(C(k)) + W(C(k)) - W(C(k-1)) \\ &\quad = W(C(k+1)) - W(C(k-1)) \\ &\quad = W(C_{k+1} \cup C_k \cup C(k-1)) - W(C(k-1)). \end{aligned}$$

2.3.2. The result. The next result shows that coalescing is the key to RDU.

Theorem 11. *For $n > 2$, the following statements are equivalent*:

1. *RWU*, (10), *idempotence, Definition* 3, *and coalescing*, (18), *all hold.*
2. *RDU*, *Definition* 5, *holds.*

This was proved by Luce (1998); we give a somewhat simpler proof.

There is a large literature on ways to arrive at idempotent RDU (see, e.g., Luce, 2000, and Quiggin, 1993, for further discussion and references). As shown above, given an RAU representation, RDU follows from our assumptions of transitivity, idempotence, status-quo event commutativity, and coalescing. Somewhat related is Köbberling and Wakker (2003) who invoke weak ordering, weak monotonicity, co-monotonic Archimedean, and co-monotonic trade-off consistency axioms. Co-monotonic changes are those that do not alter the ranking of the pure consequences of gambles. Their co-monotonic properties are quite different from anything in this article; replacing the co-monotonic requirement in their case by monotonicity forces SEU.

2.3.3. Evidence against coalescing. We begin with one of the famous Ellsberg (1961) paradoxes. Recall that it arises from two pair of choices: A vs. B and A' vs. B' where

	Events				Events		
	R	G	Y		R	G	Y
A :	100	0	0	A' :	100	0	100
B :	0	100	0	B' :	0	100	100

and $\Pr(R) = 1/3$, $\Pr(G \cup Y) = 2/3$. The probability of G, and so of Y, is not specified. People typically pick A over B and B' over A'. The usual argument suggesting that this result is paradoxical is to note that one can (rationally) ignore event Y because the consequences in each pair are identical given Y. Once the event Y is ignored, the remaining structure in each pair is identical.

We show now that for the separable form with $e = 0$, $U(x, C; 0, D) = U(x)W_{C\cup D}(C)$, if choices conform to the Ellsberg paradox, then coalescing and a form of event monotonicity cannot both hold. Using coalescing,

$$\begin{aligned} A &\sim (100, R; 0, Y \cup G) \equiv (100, 1/3; 0, 2/3) \\ B &\sim (100, G; 0, R \cup Y) \\ A' &\sim (100, R \cup Y; 0, G) \\ B' &\sim (100, G \cup Y; 0, R) \equiv (100, 2/3; 0, 1/3). \end{aligned}$$

Using the separable form and suppressing the subscript on $W_{R\cup Y\cup G}$,

$$U(A) \gtreqless U(B) \quad \text{iff } W(R) \gtreqless W(G)$$

and

$$U(A') \gtreqless U(B') \quad \text{iff } W(R \cup C) \gtreqless W(G \cup C).$$

Thus, no paradox is equivalent to:

$$W(R) \gtreqless W(G) \quad \text{iff } W(R \cup Y) \gtreqless W(G \cup Y), \tag{19}$$

which is a form of event monotonicity. Thus, as stated above, given the separable form, the Ellsberg paradox means that coalescing, (18), and event monotonicity, (19), cannot both hold. The usual interpretation has been that event monotonicity does not hold, which it need not in some RDU models (of course it does hold in the SEU model because W is finitely additive.) If the paradox continues to hold when the gambles are presented in coalesced form, then either separability or event monotonicity is at fault. We are not aware of such data.

Until recently, no serious attention has been paid to the alternative that coalescing might be the culprit. But in the past few years, Birnbaum has provided empirical evidence pointing to the strong possibility that coalescing is not descriptive. For instance, Birnbaum (2000) provides evidence that its failure underlies the famous Allais paradox. As a further example, Birnbaum (1999) considered the ternary lotteries:

$$\begin{aligned} g &= (96, .90; 14, .05; 12, .05) \\ h &= (96, .85; 90, .05; 12, .10) \end{aligned}$$

Of 100 undergraduates, 70 chose h over g.

Now consider

$$\begin{aligned} g' &\sim (96, .85; 96, .05; 14, .05; 12, .05), \\ h' &\sim (96, .85; 90; .05; 12, .05; 12, .05). \end{aligned}$$

Then assuming consequence monotonicity, transitivity, and coalescing, we have $g' \sim g$ and $h' \sim h$. Clearly, g' dominates h'. When g' and h' were presented, 85 of the 100 undergraduates chose g' over h'. No two-way tabulation was given. Thus, if consequence monotonicity and transitivity hold, then coalescing fails.

Birnbaum (2004) explores this sort of violation more thoroughly. For an overall summary of his various results and the predictions of the several models, see Marley and Luce (2004), who conclude that the evidence is overwhelmingly against RDU.

Coalescing may be, psychologically, asymmetric in the following sense (Luce, 2000). Given two branches (x, C_k) and (x, C_{k+1}) there is but one way to coalesce them. But given a branch (x, D_k), usually there are many ways to partition it into (x, D) and $(x, D_k\backslash D)$ where $D \subset D_k$. The Ellsberg paradox uses coalescing in the easy direction, whereas many of Birnbaum's examples, such as the one described, involve the more diffuse direction, often called *event splitting*. But, conceptually, an event split from, say, a gamble g to a gamble g', is equivalent to coalescing g' to g. This can be seen in the g, g' pair of the Birnbaum example above

Taken together, these results suggest that coalescing may not be descriptively true thereby falsifying the class of RDU models. Note that this includes as special cases the popular Cumulative Prospect Theory (Tversky and Kahneman, 1992) and Subjective Expected Utility (Savage, 1954). These observations suggest an increasing focus on forms of RWU that do not require coalescing. The next subsection discusses one example.

2.4. *Axiomatization of gains decomposition utility*

We repeat, without proof, a main finding about the GDU representation, Definition 6 above, as given in Marley and Luce (2001) and provide some new ones as well. To that end we need another property. Suppose $g_{\vec{\mathbf{C}}_n}$ is a gamble with $n > 2$ consequences that is based on the ordered partition $\vec{\mathbf{C}}_n$ of $C(n) = \bigcup_{i=1}^n C_i$. Here, as in Section 2.2, we must extend the domain $\mathcal{D}_+$ to include second-order gambles involving as consequences first-order ones as well as pure consequences. Define the following subgamble of $g_{\vec{\mathbf{C}}_n}$:

$$g_{\vec{\mathbf{C}}_n\backslash C_i} := (x_1, C_1; \ldots ; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \ldots ; x_n, C_n). \tag{20}$$

Note that $g_{\vec{\mathbf{C}}_n\backslash C_i}$ is based on the sub-experiment with the universal event $C(n)\backslash C_i$ (see (12)).

The following definition modifies slightly the terminology used in Luce (2000, Chap. 5) which, in turn, generalized to more general events a property introduced in Liu (1995) for known probabilities:

Definition 12. Within the domain of second-order compound gambles of gains, *gains decomposition of type i*, $i = 1, \ldots, n$, holds iff for $g_{\vec{\mathbf{C}}_n}$ satisfying (2)

$$g_{\vec{\mathbf{C}}_n} \sim \begin{cases} (g_{\vec{\mathbf{C}}_n\backslash C_i}, C(n)\backslash C_i; x_i, C_i), & g_{\vec{\mathbf{C}}_n\backslash C_i} \succsim x_i \\ (x_i, C_i; g_{\vec{\mathbf{C}}_n\backslash C_i}, C(n)\backslash C_i), & g_{\vec{\mathbf{C}}_n\backslash C_i} \prec x_i \end{cases}, \tag{21}$$

where $(g_{\vec{\mathbf{C}}_n \backslash C_i}, C(n)\backslash C_i; x_i, C_i)$ and $(x_i, C_i; g_{\vec{\mathbf{C}}_n \backslash C_i}, C(n)\backslash C_i)$ are compound binary gambles.

The right side is the compound gamble with universal event $C(n)$. If the outcome lies in the event C_i, then the consequence is x_i. If, however, the outcome lies in $C(n)\backslash C_i$, then the subgamble $g_{\vec{\mathbf{C}}_n \backslash C_i}$ is the consequence attached to it, and so the experiment with universal event $C(n)\backslash C_i$ is next run to determine which consequence x_i, $i = 1, \dots, n-1$, is received.

In the earlier work, only the case $i = n$ was studied, and it was simply called gains decomposition.

These are additional accounting indifferences and so are rational properties in the sense that the bottom lines associated with the two sides are identical, with the difference being whether one or two chance phenomena are carried out. In general, gains decomposition is not consistent with coalescing, although each property seems rational in its own right. Gains decomposition of any type has not, to our knowledge, received any empirical study.

Theorem 13. *Suppose that there is a representation U of second-order compound gambles of gains with* $n > 2$. *Then*:

1. *The following are equivalent*:
(i) *Binary RWU*, *idempotence*, *Definition* 3, *and gains decomposition of type* n, *Definition* 2, *all hold.*
(ii) *GDU for gains*, *Definition* 6, *holds.*
2. *Any two of the following imply the third*:
(i) *RDU is satisfied.*
(ii) *GDU is satisfied.*
(iii) *The binary weights satisfy the following* choice property: *For all events with* $C \subseteq D \subseteq E$,

$$W_{\mathbf{E}}(C) = W_{\mathbf{D}}(C)W_{\mathbf{E}}(D), \tag{22}$$

This result follows immediately from the proof of Theorem 3, Luce and Marley (2001), although we failed to notice there that the full statement of part 2, above, follows from that proof.

Equation (22) is just the choice axiom investigated by Luce (1959) under the assumption that the weights are finitely additive probabilities. As noted by Luce (2000, p. 78), the choice property simply says that the weights act somewhat like (subjective) conditional probabilities of the experiment **E**. This would be fully the case if there were a universal set Ω that includes all experiments as subevents and on which there exists a function W: $\Omega \rightarrow [0, 1]$ with W being the weighting function in the representation of gambles over partitions of Ω. For then (22) reduces to: for all $C \subseteq D \subseteq \Omega$,

$$W_{\mathbf{D}}(C) = \frac{W(C)}{W(D)}. \tag{23}$$

Luce (2000, p. 77) noted also that, in the presence of binary RWU, the choice property (22) is equivalent to the behavioral condition

$$((x, C; e, D\backslash C), D; e, E\backslash D) \sim (x, C; e, E\backslash C), \tag{24}$$

which is called *conditionalization.* Although conditionalization has a close family resemblance to gains decomposition, it does not follow from it unless coalescing holds.

We now show that the Ellsberg paradox, discussed relative to RDU and SEU, is surely not a problem for GDU. Applying gains decomposition of type 3 to the four 3-component gambles of the Ellsberg paradox gives:

$$\begin{aligned} A &\sim ((100, R; 0, G), R \cup G; 0, Y) \\ B &\sim ((100, G; 0, R), R \cup G; 0, Y) \\ A' &\sim ((100, R, 100, Y), R \cup Y; 0, G) \\ B' &\sim ((100, G; 100, Y), G \cup Y; 0, R). \end{aligned}$$

By consequence monotonicity and GDU,

$$\begin{aligned} A \succsim B \quad &\text{iff} \quad (100, R; 0, G) \succsim (100, G; 0, R) \\ &\text{iff} \quad U(100)W_{R\cup G}(R) \geq U(100)W_{R\cup G}(G) \\ &\text{iff} \quad W_{R\cup G}(R) \geq W_{R\cup G}(G). \end{aligned}$$

And, using gains decomposition,

$$\begin{aligned} A' \succsim B' \quad &\text{iff } U(A') \geq U(B') \\ &\text{iff} \quad U(100, R; 100, Y)W_{R\cup G\cup Y}(RUY) \\ &\qquad \geq U(100, G; 100, Y)W_{R\cup G\cup Y}(GUY) \\ &\text{iff} \quad W_{R\cup G\cup Y}(R \cup Y) \geq W_{R\cup G\cup Y}(GUY). \end{aligned}$$

Thus, the paradox does not occur provided

$$W_{R\cup G}(R) \geq W_{R\cup G}(G)$$

iff

$$W_{R\cup G\cup Y}(R \cup Y) \geq W_{R\cup G\cup Y}(G \cup Y).$$

Violation of this condition is not the same as violating (19). One can easily imagine these inequalities might be reversed because the weights are over different conditioning events. So the Ellsberg paradox does not automatically reject GDU.

2.5. *Axiomatization of subjective expected utility and of the choice property*

This subsection includes two results relevant to the SEU representation. The first gives an axiomatization of SEU starting with RDU. The second gives an axiomatization of both the choice property and SEU starting with RWU and some gains decomposition accounting indifferences.

Theorem 14. *The following two statements are equivalent*:

1. *The RDU representation holds and for any non-trivial partition* $\{C, D, E\}$ *there exist* x, x', y, z, z' *with* $x \succ y \succ x'$, $y \succ z$, $x' \succ z'$ *and such that*

$$(x, C; y, D; z, E) \sim (y, D; x', C; z', E). \tag{25}$$

2. *SEU is satisfied.*

This axiomatization is not fully in the spirit of this article because (25) is an existence assertion, not an accounting indifference. Put another way, the two sides do not have the same bottom line. We have failed to find an accounting indifference that coupled with RDU is necessary and sufficient for SEU.

The next result involves only accounting indifferences, but is unsatisfactory for another reason—it gets more than SEU, namely, the choice property.

Theorem 15.
1. *Suppose that binary idempotent RWU is satisfied and that gains decomposition of type* i, *Definition* 12, *holds for* $i = 1, 3$ *in gambles of size* $n = 3$. *Then the choice property*, (22), *is satisfied.*
2. *If in addition to* 1, *gains decomposition of type* 2 *holds in gambles of size* $n = 3$, *then the binary weighting functions are finitely additive.*
3. *If RDU is satisfied for all gambles, then it reduces to SEU*, *Definition* 7, *iff finite additivity holds.*

A further open issue is how to axiomatize SWU beginning with either RWU or WU.[3]

2.6. *On Savage's acts over states of nature*

Many theoretical economists are quite content to use Savage's (1954) formulation of the space of decision alternatives. This views uncertain alternatives—acts—as functions, with finite support, from a universal set of elementary "states of nature" into a set of pure consequences. And some have been quite critical of the approach taken by most psychologists—including us, for example—in which each decision alternative or gamble is defined over its own event structure. Earlier we mentioned the fact that any realistic set of decision alternatives tends to result in very large—billions of—states of nature. Here we make a further and key point.

As indicated above, in the Savage formulation, augmented in our work with a "natural" zero element e, all acts are simply a list of state-consequence pairs, with each state being atomic. In such a representation, there is no distinction between the two sides of the expressions that occur in an accounting indifference, such as event commutativity, coalescing, gains decomposition of type n, and idempotence, and so they are all automatically satisfied in that formulation. So, provided the RAU representation and the above four conditions hold, our results force RDU (Definition 5), and GDU (Definition 6). This is because event commutativity then forces a RWU representation (Theorem 9); coalescing forces a RDU representation (Theorem 11); and gains decomposition of type n forces a GDU representation (Theorem 13.1). Beginning with RDU we showed that SEU is equivalent to a simple indifference which, however, is an existence statement, not an accounting indifference. In getting from RWU to SEU, we found that gains decomposition of types 1, 2, and 3 holding on gambles of size 3 force both the choice property and finite additivity. It has not been previously recognized that the Savage framework leads to the restrictive choice property. To our knowledge, gains decomposition has not been studied empirically.

These facts strongly argue against Savage's description of decision situations unless one is content to arrive at SEU and the choice property. Anyone interested in descriptive theories that deviate from these both holding should either shun the state-space formulation or admit some violations of the underlying axioms for the additive conjoint representation.

2.7. *Summary*

Figures 1 and 2 present a summary of what we know using just properties of gambles.

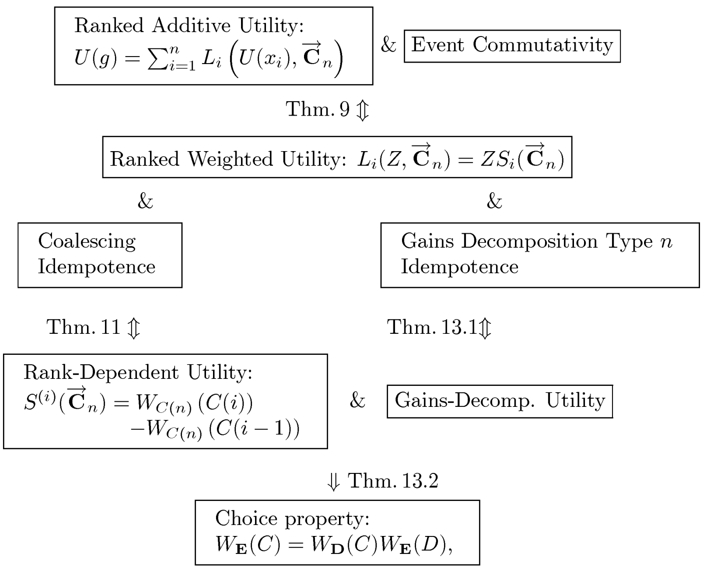

Figure 1. The network of theorems having to do with the ranked additive representations of gambles (Section 2) excluding SEU. The background assumptions are an order-preserving utility representation U, gambles are strictly increasing in the consequences, and joint receipts are strictly increasing in each variable. Conventions: When a pointed end of an arrow points to a box, all of its implications are in that box, and when it points to an & then both properties are implied.

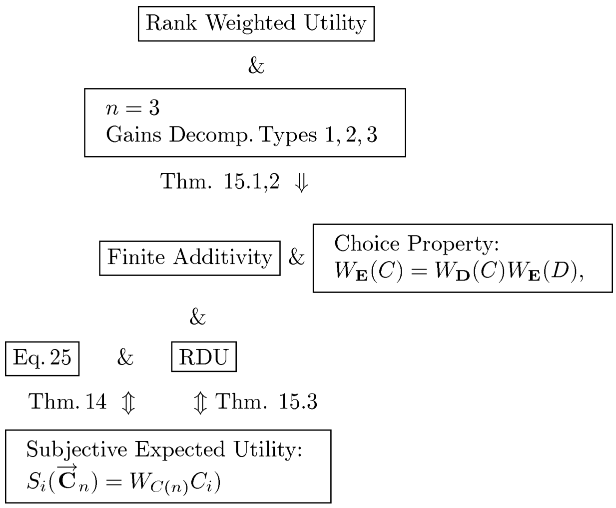

Figure 2. The results of Section 2 having to do with finite additivity and SEU. The conventions are as in Figure 1.

3. Joint receipts

In this section we extend the domain $\mathcal{D}_+$ to include the joint receipt of pure consequences and gambles. With X the set of pure consequences, for $x, y \in X$, $x \oplus y \in X$ represents receiving or having both x and y. When X denotes money, many authors assume that $x \oplus y = x + y$, but as discussed in Luce (2000) this is certainly not necessary and may well be false. When f, g are gambles, then $f \oplus g$ means having or receiving both gambles. In this article, we assume $\oplus$ to be a commutative operator, strictly increasing in each variable, with e its identity. Within the psychophysical context, Luce (2002, 2004) has studied the non-commutative case.

The following concept of generalized additivity is familiar from the functional equation literature:

Definition 16. The operation $\oplus$ has a *generalized additive representation* $U\colon \mathcal{D}_+ \overset{\text{onto}}{\longrightarrow} \mathbb{R}_+ := [0, \infty[$ iff (3), (4), and there exists a strictly increasing function φ such that

$$U(f \oplus g) = \varphi^{-1}(\varphi(U(f)) + \varphi(U(g))). \tag{26}$$

It is called *additive* if φ is the identity.

Note that $V = \varphi(U)$ is additive. Observe also that if $\oplus$ satisfies (26), then it is both commutative and associative and so

$$U(f_1 \oplus \cdots \oplus f_m) = \varphi^{-1}\left(\sum_{i=1}^{m} \varphi(U(f_i))\right)$$

iff

$$V(f_1 \oplus \cdots \oplus f_m) = \sum_{i=1}^{m} V(f_i).$$

An important special case is when φ is the identity in which case U is additive over $\oplus$. This is, of course, a strong property. For example, if for money consequences $x \oplus y = x + y$, then additive U implies $U(x) = \alpha x$. For at least modest amounts of money–"pocket money"–this may not be unrealistic, as M. H. Birnbaum and collaborators[4] have argued by fitting data.

Another important special case of generalized additivity is: for some $\delta \neq 0$,

$$U(f \oplus g) = U(f) + U(g) + \delta U(f)U(g), \tag{27}$$

which form has been termed *p-additivity*. This corresponds to the mapping $\varphi(z) = sgn(\delta)$ $\ln(1 + \delta z)$ from U to V (see, for instance, Luce, 2000).

3.1. *Independent and dependent gambles*

In previous work involving joint receipt, it has been assumed, sometimes implicitly, that the gambles involved are based on, in some sense, *independent* realizations of an underlying experiment. So if $\vec{\mathbf{C}}_n$ and $\vec{\mathbf{D}}_m$ are two ordered event partitions, and

$$f = (\ldots; x_i, C_i; \ldots), \quad g = (\ldots; y_j, D_j; \ldots), \tag{28}$$

then it is assumed when speaking of $f \oplus g$ that the experiments underlying $\vec{\mathbf{C}}_n$ and $\vec{\mathbf{D}}_m$ are run independently. With experiments whose statistical properties are not available, there is no easy formalization of exactly what "run independently" means. However, if what is relevant is the decision maker's behavioral manifestation of independence, then we have the formalization described below (Marley and Luce, 2001).

3.1.1. Behaviorally independent gambles. In the world of objective probabilities two gambles (in the form of random variables) are independent if, in the distribution of the sum of the random variables, $x_i + y_j$ arises with probability $\Pr(C_i)\Pr(D_j)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. In statistics this distribution is called the convolution[5] of the other two. Within the domain of uncertain alternatives, something else must be used. Marley and Luce (2001) took this up and studied the conditions RWU must satisfy for the following to be satisfied. Suppose that $f = (x_1, C_1; \ldots; x_n, C_m)$ and $g = (y_1, D_1; \ldots; y_m, D_m)$ are two gambles with, respectively, universal sets $C(n) = \bigcup_{i=1}^{n} C_i$ and $D(m) = \bigcup_{j=1}^{m} D_j$. Then consider when $f \oplus g \sim (\ldots; x_i \oplus y_j, (C_i, D_j); \ldots)$, where the universal set of the first-order gamble on the right is $C(n) \times D(m)$. They found (Theorem 5) necessary and sufficient conditions that are different depending on whether U over $\oplus$ is additive or p-additive, i.e., $\delta = 0$ or $\delta \neq 0$.

3.1.2. Dependent gambles and component summing. There is at the opposite extreme the idea of experimentally totally dependent gambles. This arises when $m = n$ and $\vec{\mathbf{D}}_n \equiv \vec{\mathbf{C}}_n$, and it is assumed that a single, commonly realized experiment underlies all three gambles f, g, $f \oplus g$. An example involves two tickets for the same lottery realization, with the tickets possibly differing in their consequences on at least one event of the lottery. In such a case, we define a concept of component-wise summing when the partitions underlying f and g are identical:

Definition 17. Let f and g be of the form (28) but with $m = n$ and $\vec{\mathbf{C}}_n = \vec{\mathbf{D}}_n$ for a single realization of the underlying experiment. Then *component summing*, denoted $f \circledast g$, of the gambles f and g is defined by:

$$f \circledast g := (x_1 \oplus y_1, C_1; \dots ; x_i \oplus y_i, C_i; \dots ; x_n \oplus y_n, C_n). \tag{29}$$

A natural question is when is $\oplus \approx \circledast$ true? This indifference is a rational property if the chance experiment is run just once for all three gambles, f, g, and $f \circledast g$. Indeed, it almost seems to be a triviality and, with money consequences, is often invoked, without explicit comment, using simple addition. However, as we shall see in Theorem 19 and its corollary, it is in fact a powerful assumption.

The property of *duplex decomposition*[6]

$$(x, C; y, C') \sim (x, C; e, C') \oplus (e, C; y, C') \quad (x \succsim e \succsim y, C \cap C' = \emptyset), \tag{30}$$

which has been applied to binary gambles where $x \succsim e \succsim y$ in Luce (2000), is also a special case of component summing if we assume that all three gambles involve the ordered partition (C, C'), and not independent replications, and we extend the definition to cover losses as well as gains.

A version of segregation, Section 4.3 below, and also of duplex decomposition for mixed binary gambles was studied experimentally in Cho, Luce, and Truong (2002). They interpreted the ordered partitions as representing independent realizations of the chance events in the sense that they established certainty equivalents for each term and then asked if, for example,

$$CE(x, C; y, C') = CE(x, C; e, C') + CE(e, C; y, C') \quad (x \succsim e \succsim y).$$

At most half of their respondents satisfied either duplex decomposition or segregation generalized to the mixed case. Perhaps the results would have been different if Cho, Luce, and Truong (2002) had held the events C, C' fixed and asked whether the certainty equivalents satisfy

$$CE(x, C; y, C') = CE(x, C; e, C') \oplus CE(e, C; y, C') \quad (x \succsim e \succsim y).$$

with C, C' fixed in the three terms instead of arising from independent experiments in the three terms.

3.1.3. Decomposability of joint receipts. Most scientific models of utility assume that the utility U is decomposable in certain ways. For example, all gamble representations that are special cases of RAU require that the x_i enter only through $U(x_i)$ and the ranking. We will need a comparable concept for joint receipts.

Definition 18. U is said to be *decomposable over joint receipt* iff there is a function F: $I \times I \xrightarrow{\text{onto}} I$, with F strictly increasing in each variable, such that

$$U(f \oplus g) = F(U(f), U(h)) \quad (f, g \in \mathcal{D}_+). \tag{31}$$

All this definition says is that the utility of a joint receipt of gambles depends on the consequences via their individual utilities $U(f)$ and $U(g)$. This is true of most extant theories of utility. In particular, it is true for the concept of generalized additivity given by (26).

Because we are assuming that $\oplus$ is commutative, we have $F(X, Y) = F(Y, X)$, and because e is an identity of $\oplus$, $F(X, 0) = F(0, X) = X$.

3.2. RAU and component summing

Theorem 19. *Suppose that U is a RAU order-preserving representation on the non-negative real interval* $[0, \infty[$, *and that joint receipts are strictly increasing in each variable. If U is decomposable over joint receipts*, *Definition* 18, *e is an identity of* $\oplus$, *and for gambles with the same event partition* $\oplus \approx \circledast$ *is satisfied*, *where* $\circledast$ *denotes component summing, Definition* 17, *then* $\oplus$ *has a generalized additive form*, *Definition* 16.

Note that this means $\oplus$ is commutative and associative.

The following corollary says that if one assumes AU rather than RAU, then the result is much stronger.[7]

Corollary to Theorem 19. *Suppose that U is an AU order-preserving representation on the non-negative real interval* $[0, \infty[$, *that joint receipts are strictly increasing in each variable*, *and that e is an identity of* $\oplus$. *Then*, *the following are equivalent*:

1. *U is decomposable over joint receipts*, *Definition* 18, *and for gambles with the same event partition* $\oplus \approx \circledast$ *is satisfied*, *where* $\circledast$ *denotes component summing*, *Definition* 17.
2. *There exists* U^* *that both forms a WU representation*, *Definition* 2, *of gambles and is additive over* $\oplus$, *Definition* 16, *i.e.*, φ *is the identity.*

This corollary is surprisingly strong and seems at first a bit disquieting. The property of component summing of gambles on the same ordered partition seems, on its face, highly innocent. Yet, in the presence of decomposability of $\oplus$ and $\oplus \approx \circledast$, it implies not only WU, which is fine, but also additive joint receipts, which is not so fine. Much of the empirical literature, with the exception of some of Birnbaum's model fitting, strongly suggests that with money gambles and assuming that WU holds, then U is non-linear with money. But if for money $x \oplus y = x + y$, then U has to be proportional to money. This result seems inconsistent with various interpretations of empirical data unless we are prepared to abandon one of the assumptions.

Our conclusion is that the corollary argues against assuming AU and favors RAU—at least if we cannot strengthen the Theorem itself to forcing the additive representation of $\oplus$.

3.3. Summary

The results of Theorem 19 and its corollary are summarized in Figure 3.

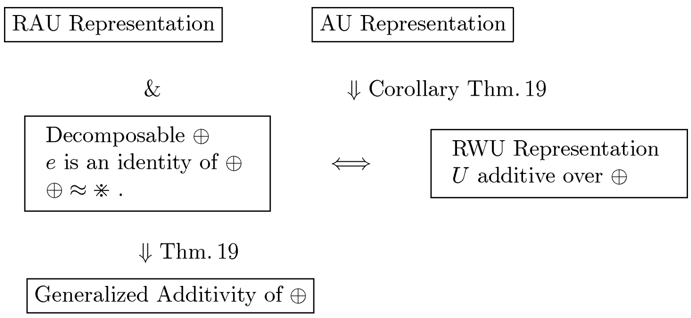

Figure 3. A graphical representation of Theorem 19. The notational conventions are those of Figure 1.

4. Increasing utility increments

4.1. Definition

A great deal of data (for a summary, see Birnbaum, 1997) suggest that at least binary gambles of the form $(x, C; e, D)$ are dealt with differently from $(x, C; y, D)$, $y \succ e$, in the sense that the utility of the latter does not approach that of the former monotonically as $U(y)$ approaches 0 (see the example below). Also Slovic et al. (2002) have made a similar observation. In particular, this rules out using segregation (see Section 4.3 below) as a tool in arriving at $U(x, C; y, D)$, $y \succ e$. Luce (2003) has shown for binary gambles how to bypass segregation by using the property of binary distribution (see Section 4.3). This gives rise to a representation involving differences of utility which he called increasing utility increments. The natural generalization of that form to general gambles is:

Definition 20. An order-preserving utility representation U is an *increasing utility increment (IUI)* one iff there exist functions $M_i(\cdot, \vec{\mathbf{C}}_n)$ that are strictly increasing with $M_i(0, \vec{\mathbf{C}}_n) = 0$ such that, for all $x_i \succsim x_n \succsim e$,

$$U(\ldots; x_i, C_i; \ldots) - U(x_n) = \sum_{i=1}^{n-1} M_i(U(x_i) - U(x_n), \vec{\mathbf{C}}_n), \tag{32}$$

and $U(\ldots; x_i, C_i; \ldots; x_n, C_n)$ is not constant on any interval of x_n with the other terms x_i, $i \neq n$, fixed. It is RIUI if restricted to the ordered case, (2).

Note that the IUI representation implies that idempotence must hold.

So far as we know, this representation is new and not a great deal is yet known about it. It arose naturally in the binary case in a psychophysical interpretation of the primitives (Luce, 2004), and the current form is a straightforward generalization of that case. Note that for each $i < n$, the gambles must be strictly co-monotonically increasing (i.e., increasing so long as the consequence ranking is preserved) because, by the fact that U is strictly increasing, $x_{i-1} \succsim x_i' \succsim x_i \succ x_n$ is equivalent to $U(x_i') - U(x_n) \geq U(x_i) - U(x_n)$ and the ranking is unchanged which in turn makes the gamble strictly increasing because each $m_i(\bullet, \vec{\mathbf{C}}_n)$ is, by assumption, strictly increasing. However, the dependence on the nth component need not be strictly increasing because $U(x_n)$ appears with both positive and negative signs. In the psychophysical context with $n = 2$ it was useful to assume that the representation is not constant with respect to the 2nd component over any non-trivial interval—a concept that is coyly referred to in the functional equation literature as "philandering."

We illustrate conditions under which violations of monotonicity hold in the RIUI representation. Consider the binary RIUI form with $\vec{\mathbf{C}}_2 = (C, D)$, and

$$U(x_1, C; x_2, D) = M_1[U(x_1) - U(x_2), \vec{\mathbf{C}}_2] + U(x_2) \quad (x_1 \succsim x_2 \succsim e). \tag{33}$$

Suppose that $M_1(\cdot, \vec{\mathbf{C}}_2)$ is differentiable with derivative $\dot{M}_1(\cdot, \vec{\mathbf{C}}_2)$, then the derivative of (33) relative to $U(x_2)$ is $\dot{M}_1(U(x_1) - U(x_2), \vec{\mathbf{C}}_2)(-1) + 1$. The necessary and sufficient condition for this to be < 0, which means non-monotonicity in x_2, with $x_1 \succsim x_2 \succsim e$, is

$$\dot{M}_1(U(x_1) - U(x_2), \vec{\mathbf{C}}_2) > 1 \tag{34}$$

So if $\dot{M}_1(\cdot, \vec{\mathbf{C}}_2)$ changes rapidly enough, one will have a non-monotonicity.

In Birnbaum's data, non-monotonicity was found to hold for events with $\Pr(C) \geq 0.85$ and for x_1 comparatively large relative to x_2. If, for example, $M_1(z, \vec{\mathbf{C}}_2) = W(C)z^b$, we have from (34) that

$$W(C)bz^{b-1} > 1 \Leftrightarrow z > \left(\frac{1}{W(C)b}\right)^{\frac{1}{b-1}}.$$

Assuming $b > 1$, the smaller $W(C)$ is the larger z has to be to achieve a non-monotonicity.

Two other generalizations of the RWU form (Proposition 4) may also be worth considering. The first generalizes (9) and assumes that

$$U(\ldots; x_i, C_i; \ldots) = \sum_{i=2}^{n} P_i(W_i(\vec{\mathbf{C}}_n) - W_{i-1}(\vec{\mathbf{C}}_n), \mathbf{X}),$$

where $\mathbf{X} = (U(x_1), \ldots, U(x_n))$ and the functions $P_i(\cdot, \mathbf{X})$ are strictly increasing with $P_i(\cdot, \mathbf{0}) = 0$. The second generalizes (11) as follows:

$$U(\ldots; x_i, C_i; \ldots) - U(x_n) = \sum_{i=1}^{n-1} Q_i(U(x_i) - U(x_{i+1}), \vec{\mathbf{C}}_n),$$

where the functions $Q_i(\cdot, \vec{\mathbf{C}}_n)$ are strictly increasing with $Q_i(0, \vec{\mathbf{C}}_n) = 0$. Neither of these has been studied to our knowledge, and we do not do so here.

4.2. *RAU and RIUI*

The next result shows that the simultaneous imposition of the general RAU and the RIUI representations force the RWU representation over all gambles satisfying (2).

Theorem 21. *Suppose that U and U^* are order-preserving, utility representations of $\langle \mathcal{D}_+, \succsim \rangle$. The following are equivalent.*

1. *U forms a RAU representation*, (5), *and U^* forms a RIUI representation*, (32).
2. *There exists an idempotent RWU representation.*

Corollary to Theorem 21. *The theorem is still valid if RAU, RIUI, and RWU are replaced, respectively, by AU, IUI, and WU.*

This corollary is a trivial consequence of the Theorem.

4.3. *Segregation and distribution*

Both Luce (2000) and earlier papers cited there made heavy use of the following segregation property in arriving at the RDU representation, and Luce (2003) has used the closely related distribution property to modify the representation to accommodate failures of monotonicity when $x_n = e$:

Definition 22. Consider the following accounting indifference: for every integer n, $x_i \succsim e$, $y \succsim e$, and every partition $\vec{\mathbf{C}}_n$ of $C(n)$,

$$\begin{aligned}(x_1, C_1; x_2, C_2; \ldots ; x_{n-1}, C_{n-1}; x_n, C_n) \oplus y \\ \sim (x_1 \oplus y, C_1; x_2 \oplus y, C_2; \ldots ; x_{n-1} \oplus y, C_{n-1}; x_n \oplus y, C_n).\end{aligned} \tag{35}$$

Segregation are those cases of (35) with $x_n = e$ and distribution are those with $x_n \succ e$.

Luce (2003) explored some of the differences between assuming segregation and distribution.

Proposition 23. *Idempotence, Definition* 3 *and* $\oplus \approx *$, *where* $\oplus$ *is joint receipt and* $*$ *is component summing*, *Definition* 17, *imply segregation and distribution.*

The proof is trivial because, for $y \succsim e$, using idempotence and component summing,

$$\begin{aligned}&(x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) \oplus y \\ &\sim (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) \oplus (y, C_1; \ldots; y, C_i; \ldots; y, C_n) \\ &\sim (x_1, C_1; \ldots; x_i, C_i; \ldots; x_n, C_n) * (y, C_1; \ldots; y, C_i; \ldots; y, C_n) \\ &\sim (x_1 \oplus y, C_1; \ldots; x_i \oplus y, C_i; \ldots; x_n \oplus y, C_n).\end{aligned}$$

An interesting property of IUI when U is additive over $\oplus$ is:

Theorem 24. *Suppose that U is additive over* $\oplus$, *Definition* 22, *and an order-preserving representation of gambles.*

1. *If RIUI*, *Definition* 20, *is satisfied*, *then distribution*, *Definition* 22, *holds.*
2. *Suppose U is an RAU representation*, *Definition* 1. *Then segregation holds iff U is an RIUI representation for* $x_n \succ e$. *Under these conditions*, *U is an idempotent RWU representation.*

4.4. *An axiomatization of RIUI*

We know of no purely gambling axiomatization of RIUI; it would be interesting to have one, but here we give one using joint receipts as well as gambles. As we noted earlier, not a great deal is known about the general case of RIUI although the binary one is better understood (Luce, 2003).

For the following, it is useful to have the qualitative concept of subtraction, $\ominus$, that is defined for $x \succsim y$ by $x \ominus y \sim z \Leftrightarrow x \sim z \oplus y$. As noted earlier, it appears difficult, if not impossible, to axiomatize RIUI without such a concept.

Theorem 25. *Suppose that a set of gambles have a RAU representation U*, *that* $\oplus$ *is a commutative*, *strictly increasing operation for which e is the identity and for which*

U is decomposable, Definition 18, with combining function F, and that segregation, Definition 22, holds. Then:

1. *There exist strictly increasing functions $L_i(\cdot, \vec{\mathbf{C}}_n)$ with $L_i(0, \vec{\mathbf{C}}_n) = 0$, $i = 1, \ldots, n-1$, such that for all gambles with $x_i \succsim x_n$,*

$$U(\ldots; x_i, C_i; \ldots) = F\left(\sum_{i=1}^{n-1} L_i(U(x_i \ominus x_n), \vec{\mathbf{C}}_n), U(x_n)\right).$$

2. *If the representation of $\oplus$ is generalized additive, Definition 3, this representation reduces, with $V = \varphi(U)$, to:*

$$V(\ldots; x_i, C_i; \ldots) = \varphi\left(\sum_{i=1}^{n-1} M_i(V(x_i) - V(x_n), \vec{\mathbf{C}}_n)\right) + V(x_n).$$

3. *If the representation of $\oplus$ is additive, i.e., φ in Part 2 is the identity, then U has the RIUI form.*

4.5. Summary

The three results of this section do not lend themselves particularly well to a compact graphical summary. Theorem 21 establishes that the assumptions that a structure has both a RAU and a RIUI representation, possibly with distinct utility functions, leads to the conclusion that they are, in fact, the same and that RWU is satisfied. Theorem 24 establishes that RIUI implies distribution and that under RAU, segregation is equivalent to RIUI and that these imply RWU. And finally, Theorem 25 considers the case of RAU and segregation and establishes several representational forms corresponding, respectively, to decomposability of $\oplus$, general additivity of $\oplus$, and finally under additive of $\oplus$ it leads to RIUI.

5. Conclusions

The article includes six new results plus slightly sharpened versions of two older ones about ranked additive representations of gambles. The general strategy was to assume that a ranked additive utility representation is given and to ask what conditions imposed on it lead to various familiar representations. To the extent that we know how, we limit ourselves to conditions that are accounting indifferences in which the gambles have the same bottom line on both sides of the indifference. The first new result, Theorem 9, is an axiomatization of ranked weighted utility (RWU) in terms of a form of event commutativity. Two old results, Theorems 11 and 13, give axiomatizations, respectively, of rank-dependent utility and gains-dependent utility. Another new result, Theorem 14, gives a new axiomatization of SEU. It is not entirely satisfactory because the condition added to the RDU representation is an existence assertion, not an accounting indifference. And Theorem 15 provides a new axiomatization of SEU starting with RWU and adding accounting indifferences. The above results are summarized in Figures 1 and 2 of Section 2.7. Another new result, Theorem 19 and its corollary in Section 3, present the consequences of assuming both RAU and component summing. In the AU case, using properties of joint receipt, we axiomatized weighted utility. These results are summarized in Figure 3 of Section 3.3. Section 4 includes three new results about structures with both ranked additive utility, RAU, and ranked increasing increment utility, RIUI, representations. Theorem 21 shows that they imply RWU. Theorem 24 explores some relations of RIUI with segregation and distribution. The last new result, Theorem 25, explores axioms leading to a RIUI representation and generalizations of it. These results do not lend themselves to being graphed.

Perhaps the most striking findings of the article are the following two. First, once the ranked-additive representation is assumed, the other additive utility representations, except for SEU and the missing development for SWU, involve just adding accounting indifferences. This fact led us to conclude that the Savage formulation of decision making in terms of acts over state space goes a very long way to driving RDU, GDU, and SEU. That formulation also leads to the choice property, which means that there is a natural way to encompass all of the gambles within a single universal event, as assumed in the Savage model. Second, is the remarkable fact of how strong component summing is in the presence of fairly weak axioms. For example, if AU is supposed and $\oplus$ is decomposable, then component summing forces not just WU but that U must also be additive over joint receipt. This flies in the face of the fact that component summing is often used with little or no comment when dealing with monetary gambles. It argues strongly for ranked models (but see note 6 regarding that case as well).

The following seven unsolved problems are worthy of note:

1. Axiomatize the ranked additive form.
2. What accounting indifference added to RDU is equivalent to SEU?
3. What accounting indifference added to RWU is equivalent to SWU?
4. Exactly what do we mean by independent realizations of gambles? In Marley and Luce (2001), we gave a definition that corresponds to subjective independence, but this is not what one usually intends.
5. Either prove that Theorem 19 implies $\oplus$ has an additive representation or provide a counter example. One way to do this is by finding the solutions of (47).
6. Find a behavioral property not involving joint receipt that in conjunction with the existence of a continuous utility representation is equivalent to an RIUI representation. We are not convinced that this is possible because of the presence of the terms $U(x_i)-U(x_n)$, which seem to be difficult to understand without the concept of joint receipt although they do arise in an alternative form for RWU, (10).
7. Find some conditions that are necessary and sufficient to reduce idempotent RWU to, at least, the special cases of the configural weight model that Birnbaum and his co-worker's have fit to data with some success (see Birnbaum, 1997, 1999). As mentioned in Section 1.2, we have shown that his general TAX model is equivalent to idempotent RWU. But, in practice, Birnbaum has worked with very special cases and these have yet to be axiomatized.

Appendix: Proofs

Theorem 9

Proof: 1 implies 2: Suppose that, for $n \geq 2, m \geq 2$, both RAU, Definition 1, and event commutativity (EC), Definition 9, hold with $x_1 = x$, $y_1 = y$, and $x_3 = x_4 = \ldots = x_n = e$. Apply RAU to EC to get:

$$\begin{aligned} &L_1(L_1(U(x), \vec{\mathbf{D}}_m), \vec{\mathbf{C}}_n) + L_2(L_1(U(y), \vec{\mathbf{D}}_m), \vec{\mathbf{C}}_n) \\ &\quad = L_{\mathbf{1}}(L_{\mathbf{1}}(U(x), \vec{\mathbf{C}}_n) + L_2(U(y), \vec{\mathbf{C}}_n), \vec{\mathbf{D}}_m) \quad (x \succsim y). \end{aligned}$$

Introduce the abbreviations $X = U(x)$, $Y = U(y)$, $\theta(\cdot) = L_1(\cdot, \vec{\mathbf{D}}_m)$, $\varphi(\cdot) = L_1(\cdot, \vec{\mathbf{C}}_n)$, $\psi(\cdot) = L_2(\cdot, \vec{\mathbf{C}}_n)$, then the functional equation becomes

$$\varphi(\theta(X)) + \psi(\theta(Y)) = \theta(\varphi(X) + \psi(Y)) \quad (X \geq Y \geq 0).$$

Observe that by setting $Y = 0$ we have $\varphi(\theta) = \theta(\varphi)$, and so

$$\theta(\varphi(X)) + \psi(\theta(Y)) = \theta(\varphi(X) + \psi(Y)) \quad (X \geq Y \geq 0).$$

Now, let $u = \varphi(X)$, $v = \psi(Y)$, $\xi = \psi\theta\psi^{-1}$, and this becomes[8]

$$\theta(u) + \xi(v) = \theta(u + v) \quad (u \geq \varphi\psi^{-1}(v) \geq 0).$$

This is the well known Pexider equation. Since both φ and ψ are strictly increasing, the equation is defined on a region (nonempty connected open set) and thus has a unique extension whose solutions for strictly increasing functions with $\theta(0) = \xi(0) = 0$ are known (Aczél, 1987) to be $\theta(u) = \xi(u) = au$, $(a > 0)$. Thus, using the definition of ξ, we have $\psi^{-1}(av) = a\psi^{-1}(v)$ $(a > 0)$. Because the derivative of ψ exists at 0, so does the derivative of ψ^{-1}. Thus, by Aczél and Kuczma (1991), the solution to this equation is $\psi(v) = cv$, $c > 0$.

To get the form for φ, simply interchange the roles of $\vec{\mathbf{C}}_n$ and $\vec{\mathbf{D}}_m$ in the definition of event commutativity. This produces the same functional equation but with φ playing the role of θ, so $\varphi(u) = bu$, $b > 0$.

Given the linearity for the first two terms, then setting $x_4 = e$, the RAU form and (17) yield

$$L_3(U(x_3)S_1(\vec{\mathbf{D}}_m), \vec{\mathbf{C}}_n) = L_3(U(x_3), \vec{\mathbf{C}}_n)S_1(\vec{\mathbf{D}}_m).$$

This is of the form $F(XC) = CF(X)$. So, given that $F(\cdot) = L_3(\cdot, \vec{\mathbf{C}}_n)$ has a derivative at 0, we know by Aczél and Kuczma (1991) that $L_3(t, \vec{\mathbf{C}}_n) = tS_3(\vec{\mathbf{C}}_n)$, $S_3(\vec{\mathbf{C}}_n) > 0$. By induction, all of the additive terms are of this form, yielding RWU.

2 implies 1: This is a trivial calculation that rests entirely on the commutativity of multiplication. □

Theorem 11

Proof: 1 implies 2: Consider, first, $n = 3$, and so by coalescing we have

$$(x_1, C_1; x_1, C_2; x_3, C_3) \sim (x_1, C_1 \cup C_2; x_3, C_3)$$
$$(x_1, C_1; x_2, C_2; x_2, C_3) \sim (x_1, C_1; x_2, C_2 \cup C_3)$$

Thus, by RWU in the form of (9), and using coalescing on the first two outcomes, we have:

$$\begin{aligned} &U(x_1, C_1; x_1, C_2; x_3, C_3) \\ &\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_1)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)] \\ &\qquad + U(x_3)[1 - W_2(C_1, C_2, C_3)] \\ &\quad = U(x_1)W_2(C_1, C_2, C_3) + U(x_3)[1 - W_2(C_1, C_2, C_3)] \\ &\quad = U(x_1, C_1 \cup C_2; x_3, C_3) \\ &\quad = U(x_1)W_1(C_1 \cup C_2, C_3) + U(x_3)[1 - W_1(C_1 \cup C_2, C_3)]. \end{aligned}$$

Because $x_1 \geq x_3$ can be selected independently, it follows from the final three equations that we may define the binary weight

$$W_{C(3)}(C_1 \cup C_2) := W_1(C_1 \cup C_2, C_3) = W_2(C_1, C_2, C_3).$$

Again, by RWU in the form of (9), and using coalescing on the last two outcomes, we have

$$\begin{aligned} &U(x_1, C_1; x_2, C_2; x_2, C_3) \\ &\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)] \\ &\qquad + U(x_2)[1 - W_2(C_1, C_2, C_3)] \\ &\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[1 - W_1(C_1, C_2, C_3)] \\ &\quad = U(x_1, C_1; x_2, C_2 \cup C_3) \\ &\quad = U(x_1)W_1(C_1, C_2 \cup C_3) + U(x_2)[1 - W_1(C_1, C_2 \cup C_3)]. \end{aligned}$$

Thus, we introduce the following notation for the binary weights

$$W_{C(3)}(C_1) := W_1(C_1, C_2 \cup C_3) = W_1(C_1, C_2, C_3).$$

Therefore

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_2, C_3)\\
&\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)]\\
&\qquad + U(x_3)[1 - W_2(C_1, C_2, C_3)]\\
&\quad = U(x_1)W_{C(3)}(C_1) + U(x_2)[W_{C(3)}(C_1 \cup C_2) - W_{C(3)}(C_1)]\\
&\qquad + U(x_3)[1 - W_{C(3)}(C_1 \cup C_2)],
\end{aligned}$$

which is RDU for $n = 3$.

We proceed by induction. Suppose the result is true for $n - 1$, then by coalescing for $k = 1$,

$$(x_1, C_1; x_1, C_2; \ldots x_n, C_n) \sim (x_1, C_1 \cup C_2; \ldots ; x_n, C_n),$$

whence by the induction hypothesis

$$\begin{aligned}
0 &= U(x_1, C_1; x_1, C_2; \ldots x_n, C_n) - U(x_1, C_1 \cup C_2; \ldots ; x_n, C_n)\\
&= U(x_1)[W_1(\vec{\mathbf{C}}_n) + W_2(\vec{\mathbf{C}}_n) - W_1(\vec{\mathbf{C}}_n)] + \sum_{i=3}^{n} U(x_i)[W_i(\vec{\mathbf{C}}_n) - W_{i-1}(\vec{\mathbf{C}}_n)]\\
&\quad - U(x_1)W_{C(n)}(C_1 \cup C_2) - \sum_{i=3}^{n} U(x_i)(W_{C(n)}(C(i)) - W_{C(n)}(C(i-1))),
\end{aligned}$$

so, as in the case of $n = 3$, we have

$$\begin{aligned}
W_2(\vec{\mathbf{C}}_n) &= W_{C(n)}(C_1 \cup C_2),\\
W_i(\vec{\mathbf{C}}_n) - W_{i-1}(\vec{\mathbf{C}}_n) &= W_{C(n)}(C(i)) - W_{C(n)}(C(i-1)) \quad (i = 3, \ldots , n).
\end{aligned}$$

and so we can define

$$W_{C(n)}(C(i)) := W_i(\vec{\mathbf{C}}_n) = W_i(C_1, \ldots , C_n) \quad (i = 2, \ldots , n)$$

In like manner, using the coalescing

$$(x_1, C_1; \ldots ; x_1, C_{n-1}; x_n, C_n) \sim \left(x_1, \bigcup_{j=1}^{n-1} C_j; x_n\right)$$

allows us to define

$$W_{C(n)}(C_1) := W_1(\vec{\mathbf{C}}_n) = W_1(C_1, \ldots , C_n).$$

Thus, RDU holds.

2 implies 1: The proof is trivial in going from RDU. □

Theorem 14

Proof: 1. implies 2. Apply RDU to (25) where we abbreviate $W_{C\cup D\cup E}$ by just W:

$$\begin{aligned}&U(x)W(C) + U(y)[W(C \cup D) - W(C)] + U(z)[1 - W(C \cup D)]\\&\quad = U(y)W(D) + U(x')[W(C \cup D) - W(D)] + U(z')[1 - W(C \cup D)].\end{aligned}$$

Rewriting,

$$\begin{aligned}0 &= [U(y) - U(x')[W(C \cup D) - W(C) - W(D)]\\&\quad + [U(x) - U(x')]W(C) + [U(z) - U(z')[1 - W(C \cup D)].\end{aligned}$$

Because we may vary y in the interval (x, x'), we see that $W(C \cup D) - W(C) - W(D) = 0$, which is finite additivity. This applied to general RDU implies that SEU holds.

2. implies 1. Apply SEU to the two sides of (25) and we see that this equation holds provided we select $x \succ y \succ x'$ and z, z' such that $U(z') - U(z) = [U(x) - U(x')]W(C)/W(E)$. □

Theorem 15

Proof: 1. Suppose that idempotent binary RWU holds. Then, the utility of a typical binary gamble with $x \succsim y$ can be written

$$U(x, C; y, D) = U(x)W_{C\cup D}(C) + U(y)[1 - W_{C\cup D}(C)], \tag{36}$$

which is the same form as binary RDU. Consider $x \succsim y \succsim z$, then gains decomposition of type i, for $i = 1, 3$, implies

$$\begin{aligned}(x, C; y, D; z, E) &\sim ((x, C; y, D), C \cup D; z, E)\\&\sim (x, C; (y, D; z, E), D \cup E).\end{aligned}$$

Applying (36) to the binary gambles on the right and denoting $W := W_{C\cup D\cup E}$, we see that the following two expressions are equal:

$$\begin{aligned}&U(x)W_{C\cup D}(C)W(C \cup D) + U(y)[W(C \cup D) - W_{C\cup D}(C)W(C \cup D)]\\&\quad + U(z)[1 - W(C \cup D)] \qquad (37)\\&\quad = U(x)W(C) + U(y)W_{D\cup E}(D)[1 - W(C)]\\&\quad + U(z)[1 - W(C)][1 - W_{D\cup E}(D)]. \qquad (38)\end{aligned}$$

Subtracting (38) from (37) yields

$$\begin{aligned}0 &= U(x)[W_{C\cup D}(C)W(C \cup D) - W(C)]\\&\quad + U(y)[W(C \cup D) - W_{C\cup D}(C)W(C \cup D) - W_{D\cup E}(D)[1 - W(C)]]\\&\quad + U(z)[1 - W(C \cup D) - [1 - W(C)][1 - W_{D\cup E}(D)]].\end{aligned}$$

Because, holding y, z fixed, with $y \succ z$, we can vary x over some interval so long as $x \succsim y \succsim z$, and the only way that this expression can remain 0 is if the coefficient of $U(x)$ is 0, i.e.,

$$W_{C\cup D}(C)W(C \cup D) - W(C) = 0, \tag{39}$$

which is the choice property. Because the choices of the events are arbitrary so long as C and D are disjoint, we see that this always holds.

2. Now add gains decomposition of type 2 for gambles with $n = 3$. Choose x, y, z with $x \succ z$ such that $y \succsim (x, C; z, E)$. Using gains decomposition 2, we see that (37) must also equal

$$U(x)W_{C\cup E}(C)[1 - W(D)] + U(y)W(D) + U(z)[1 - W_{C\cup E}(C)][1 - W(D)]. \tag{40}$$

Subtracting (40) from (37):

$$\begin{aligned} 0 = {} & U(x)(W_{C\cup D}(C)W(C \cup D) - W_{C\cup E}(C)[1 - W(D)]) \\ & + U(y)[W(C \cup D) - W_{C\cup D}(C)W(C \cup D) - W(D)] \\ & + U(z)(1 - W(C \cup D) - [1 - W_{C\cup E}(C)][1 - W(D)]). \end{aligned}$$

Arguing as before, the weight coefficient of $U(y)$ must be zero, so with (39) we have

$$\begin{aligned} 0 &= W(C \cup D) - W_{C\cup D}(C)W(C \cup D) - W(D) \\ &= W(C \cup D) - W(C) - W(D), \end{aligned}$$

which is finite additivity.

3. If RDU holds, then because as we have shown finite additivity holds, SEU follows. □

Theorem 19

Proof: Let F be the decomposable function for $\oplus$, then by the RAU form applied to $\oplus \approx \circledast$

$$F\left(\sum_{i=1}^{n} L_i(U(x_i), \vec{\mathbf{C}}_n), \sum_{i=1}^{n} L_i(U(y_i), \vec{\mathbf{C}}_n)\right) = \sum_{i=1}^{n} L_i(F(U(x_i), U(y_i)), \vec{\mathbf{C}}_n. \tag{41}$$

Consider the special case $x_2 = y_2 = x_3 = y_3 = \cdots x_n = y_n = e$, abbreviate $L_1(\cdot, \vec{\mathbf{C}}_n)$ for the moment by $L_1(\cdot)$, denote $U(x_1) = r, U(y_1) = s$ and recall that $L_i(U(e), \vec{\mathbf{C}}_n) = L_i(0, \vec{\mathbf{C}}_n) = 0$. Thus, $F(L_1(r), L_1(s)) = L_1(F(r, s))$. This is Eq. (7) on p. 62 of Aczél (1966), and so the general solution is

$$F(t, z) = \varphi(\varphi^{-1}(t) + \varphi^{-1}(z)), L_1(t) = \varphi(k\varphi^{-1}(t)). \tag{42}$$

Thus, F is the generalized additive form and returning to the general notation, $L_1(t, \vec{\mathbf{C}}_n) = \varphi(S_1(\vec{\mathbf{C}}_n)\varphi^{-1}(t))$, where the constant k is written $S_1(\vec{\mathbf{C}}_n)$ to show the explicit dependence on $\vec{\mathbf{C}}_n$ and to conform to our earlier notation. □

Corollary to Theorem 19

Proof: 1 implies 2. Because in the case of AU, we have no constraint on the ordering of the x_i, we can use the same argument as in the proof of the theorem for any i with $x_j = y_j = e$ for $j \neq i$. Thus

$$L_i(t, \vec{\mathbf{C}}_n) = \varphi(S_i(\vec{\mathbf{C}}_n)\varphi^{-1}(t)).$$

Define $U^* = \varphi^{-1}(U)$ and so we have

$$U^*(f \oplus g) = \varphi^{-1}(U(f \oplus g)) = \varphi^{-1}(U(f)) + \varphi^{-1}(U(g)) = U^*(f) + U^*(g).$$

Now, let f_i denote the gamble with the branch (x_i, C_i) and all other consequences e, i.e., with branches (e, C_j), $j \neq i$. Then, by component summing $f \sim f_1 \oplus f_2 \oplus \ldots \oplus f_n$, so by induction and using (42),

$$\begin{aligned} U^*(f) &= \sum_{i=1}^{n} U^*(f_i) \\ &= \sum_{i=1}^{n} \varphi^{-1}(L_i(U(x_i), \vec{\mathbf{C}}_n) \\ &= \sum_{i=1}^{n} S_i(\vec{\mathbf{C}}_n)\varphi^{-1}(U(x_i) \\ &= \sum_{i=1}^{n} S_i(\vec{\mathbf{C}}_n)U^*(x_i), \end{aligned}$$

thus proving both the additivity of U^* over $\oplus$ and that gambles have the WU form in the utility function U^*.

2 implies 1. An additive $\oplus$ is, of course, decomposable. The calculation for $\oplus \approx \divideontimes$ is routine. □

Comment: In the proof of Theorem 19, we have the general Eq. (41). By focusing on just the first component we were led to

$$F(t, z) = \varphi\big(\varphi^{-1}(t) + \varphi^{-1}(z)\big), \quad L_1(t) = \varphi(k\varphi^{-1}(t)), \tag{43}$$

and returning to the general notation,

$$L_1(t) = L_1(t, \vec{\mathbf{C}}_n) = \varphi(S_1(\vec{\mathbf{C}}_n)\varphi^{-1}(t)), \tag{44}$$

where the constant k is written $S_1(\vec{\mathbf{C}}_n)$ to show the explicit dependence on $\vec{\mathbf{C}}_n$ and to conform to our earlier notation.

Next, consider the special case $x_3 = y_3 = x_4 = y_4 = \cdots = e$, and for $i = 1, 2$, abbreviate $L_i(\cdot, \vec{\mathbf{C}}_n)$ by $L_i(\cdot)$, and denote $U(x_i) = X_i$, $U(y_i) = Y_i$ and recall that $L_i(U(e), \vec{\mathbf{C}}_n) = L_i(0, \vec{\mathbf{C}}_n) = 0$. Then we have, with $X_1 \geq X_2$, $Y_1 \geq Y_2$,

$$F(L_1(X_1) + L_2(X_2), L_1(Y_1) + L_2(Y_2)) = L_{\mathbf{1}}(F(X_1, Y_1)) + L_{\mathbf{2}}(F(X_2, Y_2)). \tag{45}$$

The question is what are its solutions. If we now define G by

$$G(r, s) = L_1(r) + L_2(s), \tag{46}$$

then the equation becomes of the form

$$F(G(X_1, X_2), G(Y_1, Y_2)) = G(F(X_1, Y_1), F(X_2, Y_2)). \tag{47}$$

or, taking into account what we already know, i.e., (44) and (46), then (47) becomes

$$\begin{aligned}&\varphi(\varphi^{-1}(\varphi(k\varphi^{-1}(X_1)) + L_2(X_2)) + \varphi^{-1}(\varphi(k\varphi^{-1}(Y_1)) + L_2(Y_2)))\\ &\quad = (\varphi(k\varphi^{-1}(X_1) + \varphi^{-1}(Y_1)) + L_2(\varphi(\varphi^{-1}(X_2) + \varphi^{-1}(Y_2)).\end{aligned}$$

Without the restriction $X_1 \geq X_2$, $Y_1 \geq Y_2$, the general case of (47) has been dealt with by Maksa (1999), but so far the restricted case has not been solved by functional equation experts. It is left as an open problem.

Theorem 21

Proof: 1. implies 2. Throughout the proof, we suppose that (2) holds. Define f by $U = f(U^*)$. Note that f is strictly increasing. Now set $x_i = e$ for all $i = 2, \ldots, n$. Then with $X = U(x_1)$, (5) and (32) give

$$L_1(X, \vec{\mathbf{C}}_n) = f\big(M_1(f^{-1}(X)), \vec{\mathbf{C}}_n\big). \tag{48}$$

Now set $x_i = x_n$ for all $i = 3, \ldots, n$. Recall that $L_i(0, \vec{\mathbf{C}}_n) = M_i(0, \vec{\mathbf{C}}_n) = 0$, and let $X_1 = U^*(x_1)$, $X_2 = U^*(x_2)$, $X_n = U^*(x_n)$. Then because $U = f(U^*)$, the assumption that RAU and RIUI both hold yields the functional equation

$$\begin{aligned}&L_1(f(X_1), \vec{\mathbf{C}}_n) + L_2(f(X_2), \vec{\mathbf{C}}_n) + \sum_{i=3}^{n} L_i(f(X_n), \vec{\mathbf{C}}_n)\\ &\quad = f(M_1(X_1 - X_n, \vec{\mathbf{C}}_n) + M_2((X_2 - X_n) + X_n, \vec{\mathbf{C}}_n),\end{aligned} \tag{49}$$

Consider which is continuous because each $L_i(X_n, \vec{\mathbf{C}}_n)$ is strictly increasing on an interval. Consider first the case where $n > 2$. By idempotence in (49),

$$\sum_{i=3}^{n} L_i(f(X_n), \vec{\mathbf{C}}_n) = f(X_n) - L_1(f(X_n), \vec{\mathbf{C}}_n) - L_2(f(X_n), \vec{\mathbf{C}}_n),$$

so (49) becomes

$$\begin{aligned} &L_1(f(X_1), \vec{\mathbf{C}}_n) + L_2(f(X_2), \vec{\mathbf{C}}_n) + f(X_n) - L_1(f(X_n), \vec{\mathbf{C}}_n) - L_2(f(X_n), \vec{\mathbf{C}}_n) \\ &\quad = f(M_1(X_1 - X_n, \vec{\mathbf{C}}_n) + M_2((X_2 - X_n) + X_n, \vec{\mathbf{C}}_n). \end{aligned} \tag{50}$$

By (48),

$$L_1(X, \vec{\mathbf{C}}_n) = f(M_1(f^{-1}(X), \vec{\mathbf{C}}_n)) = f(m_1(f^{-1}(X))), \tag{51}$$

where $m_1(\cdot)$ stands for $M_1(\cdot, \vec{\mathbf{C}}_n)$. Abbreviate $X = X_1$, $Y = X_2$, $Z = X_n$ and let

$$g_2(Y) = L_2(f(Y), \vec{\mathbf{C}}_n). \tag{52}$$

Substituting (51) and (52) into (50),

$$\begin{aligned} &f(m_1(X)) + g_2(Y) + f(Z) - f(m_1(Z)) - g_2(Z) \\ &\quad = f(m_1(X - Z) + m_2(Y - Z) + Z). \end{aligned} \tag{53}$$

In this functional equation, set $Z = 0$ to get

$$f(m_1(X)) + g_2(Y) = f(m_1(X) + m_2(Y)) \quad (X \geq Y),$$

which is a Pexider equation with each of the unknown functions strictly increasing. So, using the same argument as in the Proof of Theorem 10, $f(R) = cR$ with $c > 0$. Using that in (53) and collecting terms yields

$$m_1(X - Z) - m_1(X) + m_1(Z) = -m_2(Y - Z) + \frac{1}{c}g_2(Y) - \frac{1}{c}g_2(Z).$$

Because X, Y may be chosen independently subject to $X \geq Y$, the common value is a function of Z only, say $K(Z)$. Then, setting $Y = Z$ shows that $K(Z) \equiv 0$. Thus, m_1 satisfies a Cauchy equation and so using (48) and reinstating the dependence on $\vec{\mathbf{C}}_n$, we have

$$M_1(X_1, \vec{\mathbf{C}}_n,) = m_1(X_1) = S_1(\vec{\mathbf{C}}_n)X_1 = L_i(X_1, \vec{\mathbf{C}}_n).$$

Also, m_2 and g_2 satisfy a Pexider equation, and so

$$M_2(X_2, \vec{\mathbf{C}}_n) = m_2(X_2) = S_2(\vec{\mathbf{C}}_n)X_2.$$

We have also from that Pexider equation that $g_2(Y) = cm_2(Y)$, which with (52) and the fact that $f(Y) = cY$ gives

$$L_2(X_2, \vec{\mathbf{C}}_n) = S_2(\vec{\mathbf{C}}_n)X_2.$$

One now proceeds by induction to show that for each $i \leq n - 1$

$$M_i(X_i, \vec{\mathbf{C}}_n) = m_i(X_i) = S_i(\vec{\mathbf{C}}_n)X_i = L_i(X_i, \vec{\mathbf{C}}_n).$$

Thus, we have that the RWU representation.

Next, consider $n = 2$. Then using the notation as in the case $n > 2$, the functional equation becomes

$$f(m_1(X)) + g_2(Y) = f(m_1(X - Y) + Y).$$

Setting $X = Y$, we see that

$$g_2(Y) = f(Y) - f(m_1(Y)), \tag{54}$$

and so the functional equation is

$$f(m_1(X)) - f(m_1(Y)) + f(Y) = f(m_1(X - Y) + Y) \quad (X \geq Y). \tag{55}$$

There are the following three classes of solutions[9] to (55):

Solution 1

$$m_1(z) = z, \; f \text{ arbitrary}. \tag{56}$$

is of no interest because it means there is no dependence on the variable y, which is the utility corresponding to consequences.

Solution 2 asserts that there exist constants r, s such that

$$f(y) = ry \quad (r > 0), \tag{57}$$

$$m_1(z) = sz \quad (s \in]0, 1]), \tag{58}$$

which, given that we have an RIUI representation, corresponds to U being a RWU representation.

Solution 3 asserts the existence of constants $\alpha \neq 0$, $\gamma > 0$, $a \in]0, 1]$ such that

$$f(y) = \frac{1}{\alpha\gamma}(e^{\alpha y} - 1), \tag{59}$$

$$m_1(z) = \frac{1}{\alpha} \ln\left(\frac{ae^{\alpha z} + 1}{a + 1}\right), \tag{60}$$

and in particular

$$f^{-1}(y) = \frac{1}{\alpha} \ln(1 + \alpha\gamma y) \tag{61}$$

Because m_1 actually depends on the ordered partition $\vec{\mathbf{C}}_2$, and f does not, we replace a by $a_{\vec{\mathbf{C}}_2}$. Now remember that $U = f(U^*)$ and substitute (59), (60), and (61) in (51), then a routine calculation yields $L_1(U(x), \vec{\mathbf{C}}_2) = \frac{a_{\vec{\mathbf{C}}_2}}{a_{\vec{\mathbf{C}}_2}+1} U(x)$. Similarly, using (52), (54), and the above forms for f and m_1, a routine calculation gives that that $L_2(U(x), \vec{\mathbf{C}}_2) = \frac{1}{a_{\vec{\mathbf{C}}_2}+1} U(x)$, and so U is a RWU representation.

2. implies 1. Trivial. □

Theorem 24

Proof: By assumption, U is additive over $\oplus$.

1. Suppose that RIUI holds and that $x_n \succ e$. Using the additivity of U freely,

$$\begin{aligned} U((\ldots; x_i, C_i; \ldots) \oplus y) &= U(\ldots; x_i, C_i; \ldots) + U(y) \\ &= \sum_{i=1}^{n-1} M_i(U(x_i) - U(x_n), \vec{\mathbf{C}}_n) + U(x_n) + U(y) \\ &= \sum_{i=1}^{n-1} M_i(U(x_i \oplus y) - U(x_n \oplus y), \vec{\mathbf{C}}_n) + U(x_n \oplus y) \\ &= U(\ldots; x_i \oplus y, C_i; \ldots), \end{aligned}$$

whence (35) with $x_n \succ e$, which is distribution.

2. Recall that we are assuming that U is additive both over joint receipt $\oplus$ and over gambles. For $x \succsim y$, define $x \ominus y \sim z$ iff $x \sim z \oplus y$. Then because U is additive over joint receipts we have $U(x \ominus y) = U(x) - U(y)$ and $(x \ominus y) \oplus y \sim x$. Thus,

$$U(\ldots; x_i, C_i; \ldots) = U(\ldots; (x_i \ominus x_n) \oplus x_n, C_i; \ldots; e \oplus x_n, C_n).$$

Now suppose that segregation holds. Then, from this expression and using that U forms an additive representation of gambles, (5),

$$\begin{aligned} U(\ldots; x_i, C_i; \ldots) &= U((\ldots; x_i \ominus x_n, C_i; \ldots; e, C_n) \oplus x_n) \\ &= U(\ldots; x_i \ominus x_n, C_i; \ldots; e, C_n) + U(x_n) \\ &= \sum_{i=1}^{n-1} L_{\mathbf{i}}(U(x_i) - U(x_n), \vec{\mathbf{C}}_n) + U(x_n), \end{aligned}$$

which is the RIUI form (20).

Conversely, suppose the RIUI form, then

$$\begin{aligned} U(\ldots; x_i \oplus x_n; \ldots; e \oplus x_n) &= \sum_{i=1}^{n-1} M_i(U(x_i \oplus x_n) - U(x_n), \vec{\mathbf{C}}_n) + U(x_n) \\ &= \sum_{i=1}^{n-1} M_i(U(x_i), \vec{\mathbf{C}}_n) + U(x_n) \\ &= U(\ldots; x_i, C_i; \ldots; e) + U(x_n) \\ &= U((\ldots; x_i, C_i; \ldots; e) \oplus x_n), \end{aligned}$$

whence, taking U^{-1}, segregation follows.

Thus, we have

$$\sum_{i=1}^{n} L_i(U(x_i), \vec{\mathbf{C}}_n) = \sum_{i=1}^{n-1} M_i(U(x_i) - U(x_n), \vec{\mathbf{C}}_n) + U(x_n),$$

i.e., there is both an additive and RIUI representation, and so Theorem 21 implies that idempotent RWU holds. □

Theorem 25

Proof: 1. By segregation, decomposability of joint receipt using F, the assumption that gambles have a RAU representation, and the fact that $x_i \succsim x_n = e$,

$$\begin{aligned} &U(\ldots; x_i, C_i; \ldots) \\ &\quad = U((x_1 \ominus x_n, C_1; \ldots; x_{n-1} \ominus x_n, C_{n-1}; e, C_n) \oplus x_n) \\ &\quad = F(U(x_1 \ominus x_n, C_1; \ldots; x_{n-1} \ominus x_n, C_{n-1}; e, C_n), U(x_n)) \\ &\quad = F\left(\sum_{i=1}^{n-1} L_i(U(x_i \ominus x_n), \vec{\mathbf{C}}_n), U(x_n)\right). \end{aligned}$$

2. Suppose that U is generalized additive over $\oplus$, (26), then:

$$
\begin{aligned}
U(\ldots;x_i,C_i;\ldots) &= F\left(\sum_{i=1}^{n-1} L_i(U(x_i \ominus x_n),\vec{\mathbf{C}}_n),U(x_n)\right)\\
&= \varphi^{-1}\left(\varphi\left(\sum_{i=1}^{n-1} L_i(U(x_i \ominus x_n),\vec{\mathbf{C}}_n)\right)+\varphi(U(x_n))\right)\\
&= \varphi^{-1}\left(\varphi\left(\sum_{i=1}^{n-1} L_i(\varphi^{-1}(\varphi(U(x_i)-\varphi U(x_n)),\vec{\mathbf{C}}_n)\right)+\varphi(U(x_n))\right),
\end{aligned}
$$

which, if we set $V=\varphi(U)$, becomes:

$$
\begin{aligned}
V(\ldots;x_i,C_i;\ldots) &= \varphi\left(\sum_{i=1}^{n-1} L_i(\varphi^{-1}(V(x_i)-V(x_n)),\vec{\mathbf{C}}_n)\right)+V(x_n)\\
&= \varphi\left(\sum_{i=1}^{n-1} M_i(V(x_i)-V(x_n),\vec{\mathbf{C}}_n)\right)+V(x_n)
\end{aligned}
$$

where $M_i(\cdot,\vec{\mathbf{C}}_n)=L_i(\varphi^{-1}(\cdot),\vec{\mathbf{C}}_n)$.

3. Suppose that φ is the identity. Then the representation U of Part 2 over gambles with reduces to the RIUI form, Definition 20. □

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Notes

1. Birnbaum has typically called them "configural weight models" but we feel this term is a bit more accurate.
2. In much of the literature since Gilboa and Schmeidler (1989) and Wakker (1990) this is called Choquet expected utility because the weights are of a form studied in Choquet (1953). John Quiggin, the originator of the representation in Quiggin (1982), which he originally called anticipated utility, came to call it rank-dependent expected utility in Quiggin (1993). Several others, including the authors, use the term rank-dependent utility without the adjective "expected." With certain additional assumptions it has also been called cumulative prospect theory for gains (Tversky and Kahneman, 1992).
3. Marley and Luce (2004) give a characterization of SWU beginning with RWU, but their added conditions are not accounting indifferences in the sense of the present paper.
4. For example, Birnbaum and Beeghley (1997), and Birnbaum et al. (1992) for judgment data, and Birnbaum and Chavez (1997) and Birnbaum and Navarrete (1998) for choice.

5. There is a theoretical literature that explores analogues of convolution for non-independent random variables. But independence usually underlies the concept of convolution used in statistics.
6. In the cited sources, the notation $\bar{C}$ is used instead of C'. This is probably unwise when there is no universal set from which the events are drawn.
7. After the proof of the corollary to Theorem 19, we discuss whether or not a stronger result might also hold under the ranked condition of the Theorem.
8. János Aczél (personal communication February 21, 2004) suggested this line of proof.
9. This was first solved by Aczél, Luce, and Marley (2003) on the assumption that f and m are once differentiable. Later Ng (2003) showed that the result is unchanged if differentiability is dropped: "Monotonic solutions of a functional equation arising from simulataneous utility representations." Submitted.

References

Aczél, J. (1966). *Lectures on Functional Equations and their Applications.* New York: Academic Press.

Aczél, J. (1987). *A Short Course on Functional Equations Based on Applications to the Behavioral and Social Sciences.* Dordrecht-Boston-Lancaster-Tokyo: Reidel-Kluwer.

Aczél, J. and M. Kuczma. (1991). "Generalizations of a 'Folk-Theorem' on Simple Functional Equations in a Single Variable," *Results in Mathematics* 19, 5–21.

Aczél, J., R.D. Luce, and A.A.J. Marley. (2003). "A Functional Equation Arising from Simultaneous Utility Representations," *Results Math.* 43, 193–197.

Anscombe, F.J. and R.J. Aumann. (1963). "A Definition of Subjective Probability," *Annals Math. Stat.* 34, 199–205.

Birnbaum, M.H. (1997). "Violations of Monotonicity in Judgment and Decision Making." In A.A.J. Marley (ed.). *Choice, Decision, and Measurement: Essays in Honor of R. Duncan Luce*. Mahwah, NJ: Erlbaum, pp. 73–100.

Birnbaum, M.H. (1999). "Paradoxes of Allais, Stochastic Dominance, and Decision Weights." In J. Shanteau, B. A. Mellers, and D.A. Schum (eds.), *Decision Science and Technology: Reflections on the Contributions of Ward Edwards*. Norwell, MA: Kluwer Academic Publishers, pp. 27–52.

Birnbaum, M.H. (2000). "Decision Making in the Lab and on the Web." In M. H. Birnbaum (ed.). *Psychological Experiments on the Internet*. San Diego, CA: Academic Press, pp. 3–34.

Birnbaum, M.H. (2004). "Tests of Rank-Dependent Utility and Cumulative Prospect Theory in Gambles Represented by Natural Frequencies: Effects of Format, Event Framing, and Branch Splitting," *Organizational Behavior and Human Decision Processes* 95, 40–65.

Birnbaum, M.H. and D. Beeghley. (1997). "Violations of Branch Independence in Judgments of the Value of Gambles," *Psychological Science* 8, 87–94.

Birnbaum, M.H. and A. Chavez. (1997). "Tests of Theories of Decision Making: Violations of Branch Independence and Distribution Independence," *Organizational Behavior and Human Decision Processes* 71, 161–194.

Birnbaum, M.H., G. Coffey, B.A. Mellers, and R. Weiss. (1992). "Utility Measurement: Configural-Weight Theory and the Judge's Point of View," *Journal of Experimental Psychology: Human Perception and Performance* 18, 331–346.

Birnbaum, M.H., and J. Navarrete. (1998). "Testing Descriptive Utility Theories: Violations of Stochastic Dominance and Cumulative Independence," *Journal of Risk and Uncertainty* 17, 49–78.

Casadesus-Masanell, R., P. Klibanoff, and E. Ozdenoren. (2000). "Maxmin Expected Utility over Savage Acts with a Set of Priors," *J. Econ. Theory* 92, 35–65.

Chew, S.H. (1983). "A Generalization of the Quasilinear Mean and Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox," *Econometrica* 51, 1065–1092.

Cho, Y.-H., R.D. Luce, and L. Truong. (2002). "Duplex Decomposition and General Segregation of Lotteries of a Gain and a Loss: An Empirical Evaluation," *Org. Beh. Hum. Dec. Making* 89, 1176–1193.

Choquet, G. (1953). "Theory of Capacities," *Annales Inst. Fourier* 5, 131–295.

Ellsberg, D. (1961). "Risk, Ambiguity, and the Savage Axioms," *Q. J. Econ*. 75, 643–669.

Fishburn, P.C. (1988). *Nonlinear Preference and Utility Theory*. Baltimore, MD: Johns Hopkins Press.

Ghirardato, P. and M. Marinacci. (2002). "Ambiguity Made Precise: A Comparative Foundation," *J. Econ. Theory* 102, 251–289.

Ghirardato, P., F. Maccheroni, M. Marinacci, and M. Siniscalchi. (2003). "A Subjective Spin on Roulette Wheels," *Econometrica* 71, 1897–1908.

Gilboa, I., and D. Schmeidler. (1989). "Maxmin Expected Utility with a Non-Unique Prior," *J. Math. Econ.* 18, 141–153.

Kahneman, D. and A. Tversky. (1979). "Prospect Theory: An Analysis of Decision Making Under Risk," *Econometrica* 47, 263–291.

Karmarkar, U.S. (1978). "Subjectively Weighted Utility: A Descriptive Extension of the Expected Utility Model," *Org. Beh. Human Performance* 21, 61–72.

Köbberling, V. and P.P. Wakker. (2003). "Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique," *Math. Op. Res.* 28, 395–423.

Krantz, D. H., R.D. Luce, P. Suppes, and A. Tversky. (1971). *Foundations of Measurement,* Vol. I. San Diego: Academic Press.

Liu, L. (1995). *A Theory of Coarse Utility and its Application to Portfolio Analysis*. University of Kansas, Ph.D. dissertation.

Luce, R.D. (1959). *Individual Choice Behavior: A Theoretical Analysis*. New York: Wiley.

Luce, R.D. (1990). "Rational versus Plausible Accounting Equivalences in Preference Judgments," *Psychol. Sci.* 1, 225–234.

Luce, R.D. (1998). "Coalescing, Event Commutativity, and Theories of Utility," *J. Risk Uncert.* 16, 87–114. Errata: 18, 1999, 99.

Luce, R.D. (2000). *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*. Mahwah, NJ: Erlbaum, Errata: see Luce's web page at http:www.socsci.uci.edu.

Luce, R.D. (2002). "A Psychophysical Theory of Intensity Proportions, Joint Presentations, and Matches," *Psych. Rev.* 109, 520–532.

Luce, R.D. (2003). "Increasing Increment Generalizations of Rank-Dependent Theories," *Theory Dec.* 55, 87–146.

Luce, R.D. (2004). "Symmetric and Asymmetric Matching of Joint Presentations," *Psych. Rev.* 111, 446–454.

Luce, R.D. and A.A.J. Marley. (2000). "Elements of Chance," *Theory Dec.* 49, 97–126.

Maksa, G. (1999). "Solution of Generalized Bisymmetry Type Equations Without Surjectivity Assumptions," *Aequationes Math.* 57, 50–74.

Marley, A.A.J. and R.D. Luce. (2001). "Ranked-Weighted Utility and Qualitative Convolution," *J. Risk Uncert.* 23, 135–163.

Marley, A.A.J. and R.D. Luce. (2004). "Independence Properties vis-à-vis Several Utility Representations," submitted.

Meginniss, J.R. (1976). "A New Class of Symmetric Utility Rules for Gambles, Subjective Marginal Probability Functions, and a Generalized Bayes' Rule," *Proceedings American Statistical Association, Business and Economic Statistics Sec.* 471–476.

Ng, C.T. (2003). "Monotonic Solutions of Functional Equation Arising from Simultaneous Utility Representations," *Results in Mathematics* 44, 340–361.

Quiggin, J. (1982). "A Theory of Anticipated Utility," *J. Econ. Beh. Org.* 3, 323–343.

Quiggin, J. (1993). *Generalized Expected Utility Theory: The Rank-Dependent Model.* Boston: Kluwer Academic Publishers.

Savage, L.J. (1954). *The Foundations of Statistics.* New York: Wiley.

Schmeidler, D. (1989). "Subjective Probability and Expected Utility Without Additivity," *Econometrica* 57, 571–587.

Slovic, P., M. Finucane, E. Peters, and D.G. MacGregor. (2002). "The Affect Heuristic." In T. Gilovich, D. Griffin, and D. Kahneman (eds.), *Heuristics and Biases: The Psychology of Intuitive Judgment.* New York: Cambridge University Press, pp. 397–420.

Tversky, A. and D. Kahneman. (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *J. Risk Uncert*. 5, 297–323.

von Neumann, J. and O.Morgenstern. (1947). *Theory of Games and Economic Behavior*, 2nd edition. Princeton, NJ: Princeton University Press.

Wakker, P. P. (1990). "Under Stochastic Dominance Choquet-Expected Utility and Anticipated Utility are Identical," *Theory and Decision* 119–132.

Wakker, P.P. (1991). "Additive Representations on Rank-Ordered Sets. I. The Algebraic Approach," *J. Math. Psych.* 35, 501–531.

Wang, T. (2003). "Conditional Preferences and Updating," *J. Econ. Theory* 108, 286–321.