A MODEL OF RATIO PRODUCTION AND ESTIMATION AND SOME BEHAVIORAL PREDICTIONS

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Abstract

Within the context of a global psychophsical theory (Luce, 2002, 2003), a simple representation is given for ratio productions and estimations. Assuming that the estimations depend on the ratio of the signals presented but not on either one individually, the psychophysical function can be shown to be a power function but that the production and estimation functions cannot themselves also be power functions without leading to a behaviorally incorrect prediction. A behavioral condition equivalent to a numerical distortion function is described and will be evaluated empirically. Assuming that function is correct, we arrive at a model for magnitude estimations that is somewhat like extant models for sequential effects except that it is non-linear.

Our focus in this talk is on ratio productions, ratio estimations, and their relations. It relates closely to classical experimental results in the tradition of S.S. Stevens (1975).

The theory involves two primitive notions each involving the following type of stimuli. Let x and u denote the signal intensities, less their respective threshold, of two pure tones of the same frequency and phase presented to the left and right ears, respectively. Let (x,u) denote their simultaneous presentation. We assume that these intensities lie in the real interval $\Re = [0,\infty[$. Thus, *0* represents sub-threshold intensities in each ear. Many other interpretations of the formalizm are possible. For example, Zimmer, Luce, and Ellemeier (2001) report experiments based on successive presentations of brief tones of different intensities so little separated in time that they are heard as a single intensity.

The first primitive is a binary ordering of loudness beween presentations. In particular, assume that the two stimuli (x,u) and (y,v) are presented. If (x,u) is judged as at least as loud as presentation (y,v), we denote that by $(x,u) \succsim (y,v)$. Assume that this ordering is reflected by a numerical mapping $\Psi: \Re \times \Re \rightarrow \Re$ that is strictly increasing in each variable and that is order preserving, i.e.,

$$(x,u) \succsim (y,v) \Leftrightarrow \Psi(x,u) \geq \Psi(y,v), \quad \Psi(0,0) = 0.$$

The second primitive is judgments of ratios. Let p be a given positive number and let x,y, where $x > y > 0$, be given signals. The respondent is asked to provide the signal $z = z(x,y,p)$ that makes the apparent ratio of the psychological "interval" from (y,y) to (z,z) to the "interval" from (y,y) to (x,x) seem to be p.

A Theoretical Representation of Ratio Productions

The theory of Luce (2002, 2003), based on properties of summation, production, and their interactions, arrives at forms for $\Psi(x,u)$ in terms of $\Psi(x,0)$ and $\Psi(0,u)$, which we do not need here, and for $\Psi[z(x,y,p)],z(x,y,p)]$, which we abbreviate as $\psi[z(x,y,p)]$. That form is:

$$W(p) = \frac{\psi[\mathrm{z(x,y,p)}] - \psi(\mathrm{y})}{\psi(\mathrm{x}) - \psi(\mathrm{y})}$$

Here we focus only on the most common case of ratio production where $y = 0$ and so

$$\psi[z(x, p)] = \psi(x)W(p), \tag{1}$$

where in a slight abuse of notation we have appreviated $z(x,0,p)$ by $z(x,p)$. Equation (1) is fundamental to all that follows. We refer to the function ψ as the *psychophysical function* and the function W as the *distortion function.* Note that there is no assumption at this point about how either ψ or W is to be measured.

Ratio Estimations

Next consider ratio estimations as well as productions. Assuming that (1) holds, the following seems to be a natural interpretation of ratio estimation. Instead of the respondent producing $z(x,p)$ such that it stands in the ratio p to the experimenter prescribed standard intensity x, the respondent is asked to state the value of p that corresponds to the subjective ratio of z to a standard intensity x. If we change variables by defining $t: = z/x$, then p is a function of both the signal ratio t and x, i.e., $p = p(t,x)$. So, according to (1), we have the equation

$$W[p(t, x)] = \frac{\psi(tx)}{\psi(x)}. \tag{2}$$

The literature seems to have assumed that the ratio estimate $p(t,x)$ depends only on t and not on the reference signal x, i.e.,

$$p(t,x) = p(t). \tag{3}$$

The only relevant data on this that we have uncovered are in Hellman and Zwislocki (1961). They had 9 participants give ratio estimates in response to each of five different standards x_0 = 40,60,70,80,90 dB. For each standard, the experimenters assigned the scale value of 10. Their Fig. 6 has the following features. If one replots these data so that all of the standard pairs $(x_0,10)$ are superimposed, then for values above the standard, the curves do not seem to depend on x_0 but only on t. Things are not so favorable for (3) holding for values below the standard. The three lower standards seem to agree, but the two higher ones have a shallower slope. Experience suggests that many people are uneasy about working with numbers less than 1, somehow not recognizing that the region between 0 and 1 is just as spacious as the numerical region above one—n and $1/n$. For the higher standards and with the value of 10 assigned, it is easy to see that one should work with numbers less than 1, and to avoid doing so would necessarily lead to the lower slopes. One would like to see the study redone with the standard scale set at, say, 100. We hope to be able to report such data at the ISPP meeting in October.

Why ψ Should Be a Power Function

Combining (2) and (3) yields

$$\psi(tx) = W[p(t)]\psi(x),$$

which is a Pexider functional equation. It solutions for strictly increasing functions are well known (Aczél, 1966, p. 144) to be for $t \geq 0$,

$$\psi(t) = \alpha t^{\beta}, \tag{4}$$

$$W[p(t)] = t^{\beta}, \tag{5}$$

where $\alpha > 0$, $\beta > 0$. Thus, the psychophysical function is predicted to be a power function. Its exponent β will appear as a non-identifiable parameter in the behavioral equation (8) below.

A Behavioral Equivalent To ψ Being A Power Function

Define the operation $\oplus$ in terms of the following solution that is assumed to exist:

$$(x \oplus u, x \oplus u) \sim (x, u).$$

Using the form for $\Psi(x,u)$ derived in Luce (2003), one can show (Luce, 2000; Marchant and Luce, 2003) that $\psi(t) = \alpha t^{\beta}$ holds if, and only if, (iff) the following condition is satisfied :

Multiplicative Invariance: For any $\lambda > 0$ and all $x > 0$, $u > 0$,

$$\lambda x \oplus \lambda u = \lambda(x \oplus u).$$

For those who fail reduction invariance, at least one of (2) and (3) is wrong. We will report data where some respondents satisfy multiplicative invariance but some do not.

Why Ratio Productions and Estimations Cannot Be Power Functions

Assume that ψ is a power function, and consider the possibility that, as Stevens (1975) and others have claimed, that the estimation function $p(t)$ is also a power function: there are positive constants ρ,η such that

$$p(t) = \rho t^{\eta} \qquad (t > 0). \tag{6}$$

Assuming that (5) holds, then it is easy to see that (6) holds iff the distortion function $W(p)$ is also a power function with exponent β/η, i.e., for $p > 0$

$$W(p) = \left(\frac{p}{\rho}\right)^{\beta/\eta}.$$

This form has the following implication. Consider the property of *threshold subjective-production commutativity* which was supported empirically by Ellermeier and Faulhammer (2000):

$$z(z(x,p),q) = z(z(x,q),p) = z(x,s).$$

By (1) this means that

$$W(p)W(q) = W(s).$$

This together with the fact that W is strictly increasing yields, using the Cauchy functional equation (Aczél, 1966, p. 41), that W is a power function if, and only if,

$$s = pq.$$

This condition, which was also discussed by Narens (1996), was tested empirically by Ellermeier and Faulhammer (2000) and was unambiguously rejected.

Thus, the assumptions that ratio estimation is independent of the reference point and has a power function form is equivalent to *W* being a power function, which is empirically wrong. So we conclude that if the psychophysical function is a power function, then the estimation cannot also be a power function. This simple observation seems to have been overlooked in the literature.

A similar argument obtains if the production function *t(p)* is a power function.

A Possible Weighting Function *W*

Following ideas of Prelec (1998), Luce (2001) showed that, in the presence of the representation for productions, the following two statements are equivalent:

1) W is of the form (called the Prelec function)

$$W(p) = \begin{cases} \exp[-\lambda(-\ln p)^{\mu}, & (0 < p \le 1) \\ \exp[-\lambda'(\ln p)^{\mu}, & 1 < p) \end{cases},$$

where λ, λ', μ are positive constants, if, and only if, the following condition is satisfied:

2) *Reduction Invariance:*

$$z(z(x,p),q) = z(x,s)$$

implies for *N=2,3*

$$z\left(z\left(x,p^{N}\right),q^{N}\right) = z\left(x,s^{N}\right).$$

An experiment to test reduction invariance is under way and we hope to be able to report data at the ISPP meeting in October.

Relation of Ratio Estimates and Productions

We know that if $p(t,x) = p(t)$, then

$$W[p(t)] = \frac{\psi(tx)}{\psi(x)} = t^{\beta}.$$

Thus,

$$t(p) = W(p)^{1/\beta} \qquad (p \quad given),$$

$$p(t) = W^{-1}\left(t^{\beta}\right) \qquad (t \quad given).$$

It is convenient to convert all of the quantities involved into dB form: $p_{dB} = 10 \log p$, etc., because that is the way data are usually presented. Putting the Prelec form for W into the above expressions, doing a bit of algebra, and defining the constant $\rho := \beta(10 \log e)^{\mu - 1}$ yields the following forms for $t(p)_{dB}$ and for $p(t)_{dB}$, respectively,

$$t(p)_{dB} = \frac{1}{\rho}\begin{cases} -\lambda(-p_{dB})^{\mu}, & (0 < p \le 1) \\ \lambda'(p_{dB})^{\mu}, & (1 < p) \end{cases},$$

$$p(t)_{dB} = \rho^{1/\mu}\begin{cases} -\left(-\dfrac{\beta}{\lambda}t_{dB}\right)^{1/\mu}, & (0 < t \le 1) \\ \left(\dfrac{\beta}{\lambda'}t_{dB}\right)^{1/\mu}, & (1 < t) \end{cases}. \quad (7)$$

For plausible choices of the parameters, e.g., μ = 0.8333, β/λ = 0.5, β/λ' between 0.3 and 0.7, these plots are similar to Hellman and Zwislocki.

Magnitude Estimation

Only a fraction of the global psychophysical literature has examined ratio production and estimation; much more prevelant has been the Stevens (1975) methods of magnitude production and estimation. We focus on magnitude estimation. Here the experimenter provides neither a standard signal nor any particular response value, but instructs the respondents to capture signal ratios in their response patterns. There seem to be at least two possible strategies for a respondent. One is to fix a personal standard (x,k) and to use it throughout the experiment. In that case, we simply have ratio estimation. A second possible behavior is to treat the signal presented on the previous trial as the reference signal and to construct the succesive responses so as to maintain the ratio of the current trial to that of the previous one. Thus, in this case, the magnitude estimate is the response on trial n in dB, 10logR_n, and it depends on the present signal in dB, 10logS_n, the previous one, 10log S_{-n-1}, the previous response 10logR_{n-1}, and, in some models, also on 10logS_{n-2} . A number of authors (see DeCarlo, 2003, for references) have formulated linear models involving these variables, thus leading to what are called sequential effects.

In the present model, let

$$t_n = \frac{S_n}{S_{n-1}}, p_n = \frac{R_n}{R_{n-1}},$$
$$S_{n,dB} = 10\log S_n, R_{n,dB} = 10\log R_n.$$

Then the above result for $p(t)_{dB}$ yields

$$R_{n,dB} = R_{n-1,dB} + \rho^{1/\mu}\begin{cases} -\left[-\dfrac{\beta}{\lambda}\left(S_{n,dB} - S_{n-1,dB}\right)\right]^{1/\mu}, & (S_n \le S_{n-1}) \\ \left[\dfrac{\beta}{\lambda'}(S_{n,dB} - S_{n-1,dB}\right]^{1/\mu}, & (S_n > S_{n-1}) \end{cases}. \quad (8)$$

Note that $\mu \ne 1$, else W is a power function. This model has yet to be fit to data.

Summary and Conclusions

A number of fairly strong predictions and conclusions follow from a very simple, intuitive assumption about the representation of ratio productions formulated in terms of a psychophysical function ψ and numerical distortion function W in (1) and the assumption that ratio estimations depend on the signal ratio but are independent of the experimenter standard signal. First, ψ must be a power function. Second, within the theoretical framework of Luce (2002, 2003), the power function property is equivalent to a testable multiplicative invariance condition. Third, the estimation function cannot also be a power function without contradicting an empirical finding. Fourth, a plausible form for W, called the Prelec function, is shown to be equivalent to a testable condition called reduction invariance. We hope to have data on that by the time of the meeting. Fifth, if that is sustained, then a simple model for magnitude estimation is provided which is similar to other sequential effect models except that it is non-linear in the signals in dB. This needs to be compared with existing data.

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