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Theoretical note

Technical note on the joint receipt of quantities of a single good

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Abstract

The joint receipt of x and y is the fact of receiving them both. If x and y are objects that are valued, their joint receipt is valued as well. Assumming joint receipt is a binary operation that satisfies the conditions of extensive measurement, there is a numerical representation that is additive over joint receipt. We consider the case where x and y are quantities of the same infinitely divisible good. Different sets of assumptions are explored. Invariance with respect to multiplication proves to be interesting. Invariance with respect to addition yields a linear form. A relaxation of the latter yields an approximately linear form. Finally, we consider a non-commutative but bisymmetric joint-receipt operation with a representation arising from preferences over gambles.

1. Introduction

The joint receipt of x and y is the fact of having both x and y and is written as $x \oplus y$. Tversky and Kahneman (1992) assumed that, if $x, y \in \mathbb{R}^+$ (the non-negative real numbers) are sums of money, then $x \oplus y$, the joint receipt of x and y, is equal to $x + y$. It can easily be shown (Luce, 2000, Section 4.3.3) that this, together with some other assumptions (the axioms of extensive measurement), implies that there exists a measure $V(x) = \alpha x$ over $\mathbb{R}^+$ and it has strong consequences for the utility of binary gambles. Luce (2000, Section 4.5.2) suggests that it may be wise to consider "addition" rules different from $x \oplus y = x + y$. In this paper, we explore different sets of assumptions about $\oplus$, trying to obtain for $V(x)$ a more flexible form than the linear one.

First, we introduce some notations and recall a classical result of extensive measurement. Then, we turn to two different kinds of invariance conditions: displaced multiplicative and translation invariance. The former generalizes a condition that has already been investigated in Luce (2000) and proved to be quite interesting. The latter motivates us to assume, in Section 4, that $x \oplus y = x + y + r$. In Section 5, we relax the assumption that $x \oplus y = x + y + r$ by letting the equality be approximately true. In Sections 2–5, $\oplus$ is commutative. In Section 6, we consider a non-commutative but bisymmetric joint-receipt operation with a representation U such that $U(x \oplus y) = \rho U(x) + U(y)$. This representation arises from preferences over gambles (Aczél, Luce, & Ng, 2003 submitted). The last section is devoted to the proofs.

2. Additive value function

Let $\langle \mathscr{D}, \succsim, \oplus, \varepsilon \rangle$ denote a structure of valued objects, where $\mathscr{D}$ is the set of objects, $\varepsilon \in \mathscr{D}$, $\succsim$ is an ordering and $\oplus$ is a binary operation defined for all x and y in $\mathscr{D}$. The interpretations are, respectively, $\succsim$ is a preference order of objects, $x \oplus y$ denotes a composite object consisting of x and y together, and ε denotes no change in the status quo.[1]

The set of gains is defined as $\mathscr{D}_\varepsilon^+ = \{x\colon x \in \mathscr{D}, x \succsim \varepsilon\}$. Similarly, we define the set of the losses as $\mathscr{D}_\varepsilon^- = \{x\colon x \in \mathscr{D}, x \precsim \varepsilon\}$. In some cases, $\mathscr{D}$ will be assumed to be

[1] Luce (2002) has reinterpreted the mathematics of Luce (2000) and of Aczél et al. (2003) (submitted) in terms of the psychophysics of signal intensity where the threshold plays the role of the status quo. The latter paper explores the case of non-commutative joint receipt. In the former paper, Luce makes use of Theorem 4.

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the set of the real numbers (denoted by $\mathbb{R}$) or the set of the rational numbers (denoted by $\mathbb{Q}$). To ensure the existence of an additive measure over $\mathscr{D}_\varepsilon^+, \mathscr{D}_\varepsilon^-$ or $\mathscr{D}$, some axioms are needed. Let $\mathscr{S}$ denote either $\mathscr{D}$, $\mathscr{D}_\varepsilon^+$ or $\mathscr{D}_\varepsilon^-$.

A 1. *Closure*: The operation $\oplus$ is closed over $\mathscr{S}$.

A 2. *Weak order*: $\succsim$ is a weak order (transitive and connected).

For each x, y, z in $\mathscr{S}$,

A 3. *Weak commutativity*: $x \oplus y \sim y \oplus x$.

A 4. *Weak associativity*: $(x \oplus y) \oplus z \sim x \oplus (y \oplus z)$.

A 5. *Left weak monotonicity*: $x \succsim y \Leftrightarrow x \oplus z \succsim y \oplus z$.

A 6. *Left weak identity*: $\varepsilon \oplus x \sim x$.

To introduce the next axiom, we need a new notation. Let n be a positive integer and let $x(n)$ denote the joint receipt of n copies of x. Formally, $x(1) = x$ and $x(n) = x(n-1) \oplus x$.

A 7. *Archimedeanness*: For all x, y, w, z, with $x \succ y$, there exists an integer n such that

$$x(n) \oplus w \succsim y(n) \oplus z.$$

Note that weak commutativity together with left weak monotonicity or left weak identity imply, respectively, the following:

A 8. *Right weak monotonicity*: $x \succsim y \Leftrightarrow z \oplus x \succsim z \oplus y$.

A 9. *Right weak identity*: $x \oplus \varepsilon \sim x$.

Definition 1. Let $\mathscr{S}$ denote either $\mathscr{D}, \mathscr{D}_\varepsilon^+$ or $\mathscr{D}_\varepsilon^-$. Then $\langle \mathscr{S}, \varepsilon, \succsim, \oplus \rangle$ is a *joint-receipt preference structure* iff the following conditions hold for all elements in $\mathscr{S}$: closure, weak order, weak commutativity, weak associativity, left (or right) weak monotonicity, left (or right) weak identity. If, in addition, Archimedeanness holds, then we have an *Archimedean joint-receipt preference structure.*

In the sequel, $\mathbb{R}_0^+$ denotes the non-negative real numbers and $\mathbb{R}^+$, the positive real numbers.

Proposition 1 (Roberts & Luce, 1968). *The following are equivalent*:

(i) $\langle \mathscr{D}_\varepsilon^+, \varepsilon, \succsim, \oplus \rangle$ *is an Archimedean joint-receipt preference structure.*

(ii) *There is a representation* $V : \mathscr{D}_\varepsilon^+ \to \mathbb{R}_0^+$ *such that*

$$x \succsim y \Leftrightarrow V(x) \geqslant V(y), \tag{1}$$

$$V(x \oplus y) = V(x) + V(y), \tag{2}$$

$$V(\varepsilon) = 0 \tag{3}$$

This representation is unique up to a multiplication by a positive constant. The same holds for $\mathscr{D}_\varepsilon^-$, *with* $V : \mathscr{D}_\varepsilon^- \to \mathbb{R}_0^-$ *and for* $\mathscr{D}$, *with* $V : \mathscr{D} \to \mathbb{R}$.

When a representation satisfies the conditions (1) and (2) of Proposition 1, we call it an additive representation. Additive representations will be used throughout the paper.

3. Invariance

Let the elements of $\mathscr{D}$ be quantities of the same infinitely divisible good (for example, idealized money), i.e. $\mathscr{D} \subseteq \mathbb{R}$. Then addition and multiplication are defined on $\mathscr{D}$. In many places, we will consider $\succsim$ to be the same as $\geqslant$. One exception is Proposition 3. We now turn to two different kinds of invariance conditions: the first one is a very general form of invariance vis-à-vis multiplication, which is itself a generalization of homogeneity; the second one, invariance vis-à-vis addition, is called translation invariance.

3.1. Displaced multiplicative invariance

A 10. *Displaced multiplicative invariance*: There exists a function f from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that, for some real numbers r and s and for all $x, y, z \in \mathscr{D}$, with $z > 0$,

$$[z(x-r)+r] \oplus [z(y-r)+r] = f(z)[(x \oplus y) - s] + s.$$

In order to motivate this condition, let us consider several special cases. If $r = s = 0$, we obtain the classical multiplicative invariance.

A 11. *Multiplicative invariance*: There exists a function f from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that, for all $x, y, z \in \mathscr{D}$, with $z > 0$,

$$zx \oplus zy = f(z)(x \oplus y).$$

If $r = s = 0$ and $f(z) = z^k$, then $\oplus$ is said to be homogeneous of degree k. When $\oplus$ is homogeneous of degree 1, we are free to change all of the variables by any positive factor. Such numerical factors can either reflect systematic changes of the structure that are captured as its automorphisms or as changes of unit such as going from US\$ to Belgian francs. This case has been worked out, with $\mathscr{D} = \mathbb{R}$, in Luce (2000, Section 4.5 3).

One more condition is needed in order to present our first theorem.

A 12. *Continuity*: $\oplus$, considered as a function of two variables, is continuous in each variable.

Theorem 1. *Suppose that* $\langle \mathscr{D}, \geqslant, \oplus \rangle$, $\mathscr{D} \subseteq \mathbb{R}$, *satisfies the following conditions*: *weak order*, *weak associativity*, *weak monotonicity*, *continuity*, *and displaced multiplicative invariance. If* $\mathscr{D} = \mathbb{R}$ *or* $]r, \infty[$, *then* $s = r$ *and* $f(z) = z^k$ *with* $k = 1$ *or* 2.

(i) *If* $\mathscr{D} = \mathbb{R}$, *then* $k = 1$ *and, for some constants* $\beta > 0$ *and* $\gamma < 0, x \oplus y$ *is given by*

$$x \oplus y = \begin{cases} [(x-r)^\beta + (y-r)^\beta]^{1/\beta} + r, \\ \quad x \geqslant r,\ y \geqslant r, \\ [(x-r)^\beta + \frac{1}{\gamma}(r-y)^\beta]^{1/\beta} + r, \\ \quad x \geqslant r > y,\ x \oplus y \geqslant r, \\ r - [\gamma(x-r)^\beta + (r-y)^\beta]^{1/\beta}, \\ \quad x \geqslant r > y,\ x \oplus y < r, \\ r - [(r-x)^\beta + (r-y)^\beta]^{1/\beta}, \\ \quad x \leqslant r,\ y \leqslant r. \end{cases} \quad (4)$$

(ii) *If* $\mathscr{D} =]r, \infty[$, *then either* $k = 1$ *in which case, for some* $\beta > 0$,

$$x \oplus y = [(x-r)^\beta + (y-r)^\beta]^{1/\beta} + r, \quad (5)$$

or $k = 2$, *in which case, for some* $\eta > 0$,

$$x \oplus y = \eta(x-r)(y-r) + r. \quad (6)$$

(iii) *If* (4) *or* (5) *hold, then there is an additive value function defined by*

$$V(x) = \begin{cases} \alpha(x-r)^\beta, & x \geqslant r, \\ \alpha'(r-x)^\beta, & x < r, \end{cases} \quad (7)$$

where $\alpha > 0$ *and* $\alpha' = \alpha/\gamma$; *whereas, if* (6) *holds, then there is an additive value function defined by*

$V(x) = c \ln[\eta(x-r)]$,

with $c > 0$.

The reader will find the proofs of this theorem and all subsequent theorems in the last section.

Corollary 1. *Suppose that the structure* $\langle \mathscr{D}, \varepsilon, \geqslant, \oplus \rangle$ *is such that* $\langle \mathscr{D}, \geqslant, \oplus \rangle$ *satisfies the conditions of Theorem* 1, *and* ε *is a weak identity. If* $k = 1$, *then* $r = \varepsilon$; *and if* $k = 2$, *then* $\eta = 1/(\varepsilon - r)$.

Note that, under the conditions of Theorem 1, $r \oplus r = r$. If $k = 1$, this is not surprising because r is then a weak identity but, if $k = 2$, this is perhaps not what we intuitively expect.

It is possible to arrive at a special case of (4) without imposing either continuity or displaced multiplicative invariance but we then need two additional conditions, namely solvability and weak positivity.

A 13. *Solvability*: For all x, y, there exists z such that $z \oplus x \sim y$.

A 14. *Weak positivity*: $x \oplus x \succ x$, for all $x \succ e$.

Proposition 2. *Suppose that* $\langle \mathscr{D}, \varepsilon, \geqslant, \oplus \rangle$ *is a weakly positive joint-receipt preference structure* (*with* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{Q}$). *If* $\oplus$ *is solvable and homogeneous of degree* 1, *then* (4) *holds and there is an additive representation defined by* (7), *with* $r = 0$.

3.2. Translation invariance

An alternative assumption to A 10 is

A 15. *Translation invariance*: *There exists a function* $l : \mathscr{D} \to \mathbb{R}$ *such that, for all* x, y, z *in* $\mathscr{D}$, $(z + x) \oplus (z + y) = l(z) + (x \oplus y)$.

Theorem 2. *Suppose that* $\langle \mathscr{D}, \geqslant, \oplus \rangle$ *satisfies the following conditions: weak order, weak associativity, weak monotonicity, continuity and translation invariance, with* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{R}^+$. *Then*

(i) $l(z) = bz$ *with* $b = 1$ *or* 2.

(ii) *For* $b = 2$,

$$x \oplus y = x + y + r \quad (8)$$

and for $b = 1$,

$$x \oplus y = \frac{1}{k} \ln[e^{kx} + e^{ky}], \quad (9)$$

with $k > 0$.

(iii) *If* (8) *holds, then there is an additive representation given by*

$V(x) = a(x + r)$,

where a is strictly positive. If (9) *holds, then there is an additive representation given by*

$V(x) = \lambda e^{kx}$,

where λ *is strictly positive.*

Corollary 2. *Suppose that the structure* $\langle \mathscr{D}, \varepsilon, \geqslant, \oplus \rangle$ *is such that* $\langle \mathscr{D}, \geqslant, \oplus \rangle$ *satisfies the conditions of Theorem* 2 *and* ε *is a weak identity. Then,* $b = 2, r = -\varepsilon$ *and, so,*

$$x \oplus y = x + y - \varepsilon. \quad (10)$$

It is possible to arrive at (10) without imposing continuity but with a stronger condition than translation invariance, as stated in the next proposition. Here, we do not assume that $\succsim$ is the same as $\geqslant$.

Proposition 3. *Suppose that* $\langle \mathscr{D}, \varepsilon, \succsim, \oplus \rangle$ *satisfies weak identity and* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{Q}$ *or the set of the integers. If translation invariance holds with* $l(z) = 2z$, *then* (10) *holds.*

4. The case of $x \oplus y = x + y + r$

It may be assumed that receiving $x \oplus y$ (x and y) is valued differently from receiving the single amount $x +$

y because there is some intrinsic value attached to the fact of receiving. Receiving $x+y$ is the receipt of a single thing, whereas receiving $x \oplus y$ is the receipt of two distinct things. A possible formalization of this idea is $x \oplus y = x + y + r$, where r represents the extra value associated with receiving two rather than one formally equal objects. Note that this form for the joint-receipt operator is also motivated by Theorem 2 and Proposition 3, dealing with translation invariance. In this section, we examine the consequences of this hypothesis in various cases and it is assumed that $\mathscr{D}$ is the set of the real numbers or the set of the rationals.

In the two first results, we apply the condition $x \oplus y = x + y + r$ separately to gains and losses because people may process gains differently from losses. Some support for this can be found in Luce (2000).

Proposition 4. *Suppose that* $\langle \mathscr{D}_\varepsilon^+, \varepsilon, \succcurlyeq, \oplus \rangle$ *is an Archimedean joint-receipt preference structure (with* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{Q}$*) and* $x \oplus y = x + y + r$ *for all* $x, y > \varepsilon$*. Then* $r + \varepsilon \geqslant 0$ *and there is an additive representation* V *over* $\mathscr{D}_\varepsilon^+$ *as in Proposition* 1 *such that*

$$V(x) = br + bx, \quad \forall x > \varepsilon, \tag{11}$$

where $b > 0$*. It is unique up to a multiplication by a positive constant.*

If $\langle \mathscr{D}_\varepsilon^-, \varepsilon, \succcurlyeq, \oplus \rangle$ *is also an Archimedean joint-receipt preference structure and* $x \oplus y = x + y - s$ *for all* $x, y < \varepsilon$*, then* $s - \varepsilon \geqslant 0$ *and* V *is a representation over* $\mathscr{D}$ *with*

$$V(x) = -ds + dx, \quad \forall x < \varepsilon, \tag{12}$$

where $d > 0$*. It is unique up to multiplication by a positive constant.*

The constants for transforming V for positive and negative domains are chosen independently.

Note that this proposition does not tell us what the value of $x \oplus y$ is when one of them is a gain and the other one a loss. This is due to the fact that we did not require that $\mathscr{D}$ be closed under $\oplus$. The joint receipt of a gain and a loss is not necessarily in $\mathscr{D}$. Nonetheless, it seems natural to explore the consequences of assuming that $\mathscr{D}$ is closed.

Theorem 3. *Let the set* $\mathscr{D}$ *be closed under* $\oplus$*. Suppose that* $\langle \mathscr{D}_\varepsilon^+, \varepsilon, \succcurlyeq, \oplus \rangle$ *and* $\langle \mathscr{D}_\varepsilon^-, \varepsilon, \succcurlyeq, \oplus \rangle$ *are two Archimedean joint-receipt preference structures (with* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{Q}$*). Suppose, in addition, that* $x \oplus y = x + y + r$ *for all* $x, y > \varepsilon$ *and* $x \oplus y = x + y - s$ *for all* $x, y < \varepsilon$*. Then* $r = -s = -\varepsilon$ *and there is an additive representation* V *over* $\mathscr{D}$ *as in Proposition* 1 *such that*

$$V(x) = b(x - \varepsilon), \quad \forall x > \varepsilon, \tag{13}$$

and

$$V(x) = d(x - \varepsilon), \quad \forall x < \varepsilon, \tag{14}$$

where $b, d > 0$*. It is unique up to multiplication by independent positive constants for the positive and negative domains.*

But what happens when we impose some very weak condition linking gains and losses? We explore this issue in the next corollary.

Corollary 3. *Suppose that* $\langle \mathscr{D}, \varepsilon, \succcurlyeq, \oplus \rangle$ *is an Archimedean joint-receipt preference structure (with* $\mathscr{D} = \mathbb{R}$ *or* $\mathbb{Q}$*). Suppose, in addition, that* $x \oplus y = x + y + r$ *for all* x, y*. Then* $r = -\varepsilon$ *and there is an additive representation* V *as in Proposition* 1 *such that*

$$V(x) = b(x - \varepsilon), \quad \forall x, \tag{15}$$

where $b > 0$*. It is unique up to a multiplication by a positive constant.*

5. Approximately linear value function

In Section 4, assuming $x \oplus y = x + y + r$, we obtained different forms for V but all of them were piecewise linear. A possible way to avoid this linearity might be to relax the assumption $x \oplus y = x + y + r$. We could impose the less demanding condition that, for some $\epsilon > 0$ (not to be confused with ε) and all x, y, $|(x \oplus y) - (x + y + r)| \leqslant \epsilon$.

Proposition 5. *Suppose that* $\langle \mathscr{D}, \varepsilon, \succcurlyeq, \oplus \rangle$ *is an Archimedean joint-receipt preference structure with* $\mathscr{D} = \mathbb{R}$ *and an additive representation* V *(Proposition* 1*). Suppose that, for some real* r*, some* $\epsilon > 0$ *and all* x, y*,* $|(x \oplus y) - (x + y + r)| \leqslant \epsilon$ *and* $\oplus$ *is continuous. Then,*

$$\max\{0, ax + s - \delta\} \leqslant V(x) \leqslant ax + s + \delta, \quad x > \varepsilon, \tag{16}$$

$$ax + s - \delta \leqslant V(x) \leqslant \min\{0, ax + s + \delta\}, \quad x < \varepsilon, \tag{17}$$

where

- $s = \frac{V(-2\epsilon) + V(2\epsilon)}{2}$,
- *a and* $\delta = V(2\epsilon) - s$ *are strictly positive,*
- $|\varepsilon + r| \leqslant \epsilon$ *and* $|a\varepsilon + s| \leqslant \delta$.

If $\epsilon = \sup_{x,y} |(x \oplus y) - (x + y + r)|$ *then,* $a\epsilon \leqslant 3\delta + s$.

As shown in Fig. 1, the representation obtained in Proposition 5 is in some sense approximately linear: it is restricted between two parallel straight lines.

6. A non-commutative case

In the previous section, $\oplus$ was always commutative; it was one of our assumptions (Sections 2, 4 and 5) or it was a consequence of our assumptions (Section 3). In

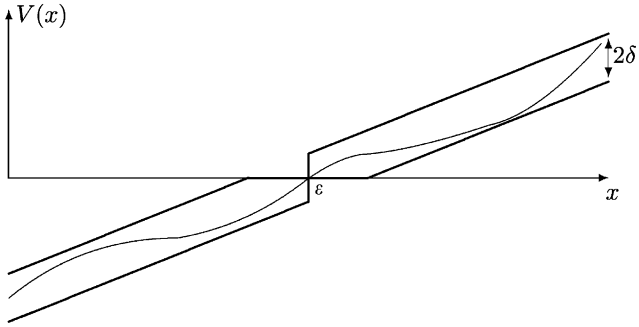

Fig. 1. An approximately linear value function.

most previous work on joint receipt in utility theory, it is also assumed that $\oplus$ is commutative. This assumption may well be unrealistic. For instance, when we receive mail, we open the envelopes in some order. Receiving a \$1000 check from a lottery followed by a −\$1000 correction on one's income tax versus the correction followed by the check might produce a different overall reaction. The first one is likely to seem a disappointment whereas the second will give a sense of relief.

A theory of non-commutative joint receipts has been worked out in two papers (Aczél et al., 2003 submitted; Ng, Luce, & Aczél, 2002). Under some relatively plausible assumptions concerning not only $\oplus$ but also the utility of gambles composed of joint receipts, they show that the utility U of a joint receipt can have one of the three following forms. Either

$$U(x\oplus y) = U(x) + U(y) - \delta U(x)U(y), \tag{18}$$

$$U(x\oplus y) = \rho U(x) + U(y), \quad \rho > 1 \tag{19}$$

or

$$U(x\oplus y) = U(x) + \rho' U(y), \quad \rho' > 1. \tag{20}$$

Here we use U instead of V to make clear that U is derived from preferences on gambles while V, in this paper, is derived from preferences on joint receipts. In a given situation, it might be possible to derive a V and a U that are not identical.

The first of the three forms (18) is commutative, the other two (19 and 20) are not; they are not associative either but they are bisymmetric, in the following sense.

A 16. *Bisymmetry*: $(x\oplus y)\oplus(z\oplus w) \sim (x\oplus z)\oplus(y\oplus w)$.

The first non-commutative form (19) has a left weak identity ε while the second one (20) has a right weak identity.

In this section, we examine the effect of Displaced Multiplicative Invariance and Translation Invariance when combined with one of the two non-commutative forms (19 and 20).

Theorem 4. *Suppose that $\langle \mathscr{D}, \succsim, \oplus \rangle$, $\mathscr{D} \subseteq \mathbb{R}$, satisfies displaced multiplicative invariance and has a closed, non-commutative, continuous, order-preserving representation as in* (19). *If $\mathscr{D} = \mathbb{R}$ or $]r, \infty[$, then $s = r$ and $f(z) = z^k$ with $k = 1$ or $1 + \rho$.*

(i) *If $\mathscr{D} = \mathbb{R}$, then $k = 1$ and, for some constants $\beta > 0$ and $\gamma < 0$, $x\oplus y$ is given by*

$$x\oplus y = \begin{cases} [\rho(x-r)^\beta + (y-r)^\beta]^{1/\beta} + r, \\ \quad x \geqslant r,\ y \geqslant r, \\ [\rho(x-r)^\beta + \frac{1}{\gamma}(r-y)^\beta]^{1/\beta} + r, \\ \quad x \geqslant r > y,\ x\oplus y \geqslant r, \\ r - \rho[\gamma(x-r)^\beta + (r-y)^\beta]^{1/\beta}, \\ \quad x \geqslant r > y,\ x\oplus y < r, \\ r - [\rho(r-x)^\beta + (r-y)^\beta]^{1/\beta}, \\ \quad x \leqslant r,\ y \leqslant r. \end{cases} \tag{21}$$

(ii) *If $\mathscr{D} =]r, \infty[$, then either $k = 1$ in which case, for some $\beta > 0$,*

$$[(x\oplus y) - r]^\beta = \rho(x-r)^\beta + (y-r)^\beta, \tag{22}$$

or $k = 1 + \rho$, in which case, for some $\eta > 0$,

$$x\oplus y = \eta(x-r)^\rho(y-r) + r. \tag{23}$$

(iii) *If* (21) *or* (22) *hold, then*

$$U(x) = \begin{cases} \alpha(x-r)^\beta, & x \geqslant r, \\ \alpha'(r-x)^\beta, & x < r, \end{cases} \tag{24}$$

where $\alpha > 0$ and $\alpha' = \alpha/\gamma$; whereas, if (23) *holds, then $U(x) = c\ln[\eta(x-r)]$, with $c > 0$.*

Note that the forms obtained for $\oplus$ in Theorem 4 are generalizations of the forms obtained in Theorem 1. But it would be misleading to think that this is due to a weakening of the assumptions. The assumptions in Theorem 1 do not imply those of Theorem 4. The assumptions of Theorem 4 are about $\oplus$ and the utility representation arising from preferences over gambles whereas the assumptions of Theorem 1 are just about $\oplus$ and have nothing to do with gambles.

Theorem 5. *Suppose that $\langle \mathscr{D}, \succsim, \oplus \rangle$ satisfies translation invariance and has a closed, non-commutative, continuous, order-preserving representation as in* (19), *with $\mathscr{D} = \mathbb{R}$ or $\mathbb{R}^+$. Then*

(i) *$l(z) = bz$ with $b = 1$ or 2.*
(ii) *For $b = 2$,*

$$x\oplus y = \rho x + y + r \tag{25}$$

and for $b = 1$,

$$x\oplus y = \frac{1}{k}\ln[\rho e^{kx} + e^{ky}], \tag{26}$$

with $k > 0$.

(iii) *If* (8) *holds, then*
$U(x) = a(x + r)$,
where a is strictly positive. If (9) *holds, then*
$U(x) = \lambda e^{kx}$,
where λ is strictly positive.

The forms obtained for $\oplus$ in Theorem 5 are also generalizations of those obtained in Theorem 2. The comments about the relation of Theorems 1 and 4 to their assumptions apply equally well to the relation between Theorems 2 and 5.

If, in Theorems 4 and 5, we use representation (20) instead of (19), the theorems still hold with the obvious needed changes.

7. Conclusion

We explored different sets of assumptions about $\oplus$ and, with most of them, we ended up with some restrictive forms for $\oplus$. The case of displaced multiplicative invariance, combined with a commutative or non-commutative representation, leads to very interesting forms (see Theorem 1, Proposition 2 and Theorem 4).

The property $x \oplus y = x + y + r$ arose as a consequence of Theorem 2. One of its possible relaxations or generalization is

$$|(x \oplus y) - (x + y + r)| \leqslant \epsilon.$$

In Section 5, we combined this relaxation with an additive representation. It might be interesting to relax other conditions, such as (22), (23), (25) or (26). But, currently, we do not know how to solve the functional "inequations" corresponding to these cases.

8. Proofs

Proof of Theorem 1. Denote $x \oplus y = F(x, y)$ and so displaced multiplicative invariance becomes

$$F[z(x-r)+r, z(y-r)+r] = f(z)[F(x,y)-s]+s, \quad z>0. \tag{27}$$

By (27),

$$F[zw(x-r)+r, zw(y-r)+r] = f(zw)[F(x,y)-s]+s.$$

Now, observe that if we set $u = w(x-r)+r, v = w(y-r)+r$, using (27) twice,

$$\begin{aligned} F[zw(x-r)+r, zw(y-r)+r] &= F[z(u-r)+r, z(v-r)+r] \\ &= f(z)[F(u,v)-s]+s \\ &= f(z)[F(w(x-r+r, w(y-r)+r)-s]+s \\ &= f(z)f(w)[F(x,y)-s]+s. \end{aligned}$$

So, equating these, we see that,

$$f(zw) = f(z)f(w), \quad z, w>0. \tag{28}$$

As F is strictly increasing, so then is f, and thus the functional equation (28) has as its only solution $f(z) = z^k$, with $k>0$. Hence,

$$F[z(x-r)+r, z(y-r)+r] = z^k[F(x,y)-s]+s, \quad z>0. \tag{29}$$

Because F is continuous, strictly increasing and associative, we can apply a well-known result (Aczél, 1966, p. 256) that asserts there exists a continuous and strictly monotonic function V such that

$$F(x,y) = V^{-1}[V(x)+V(y)]. \tag{30}$$

Eq. (29) becomes

$$z^k(V^{-1}[V(x)+V(y)]-s)+s = V^{-1}[V(z[x-r]+r)+V(z[y-r]+r)], \tag{31}$$

for $z>0$. Keeping z fixed, with the notations $H(x) = V(z(x-r)+r)$, $J(x) = V(z^k(x-s)+s)$, $a = V(x)$ and $b = V(y)$, we obtain

$$J[V^{-1}(a+b)] = H[V^{-1}(a)] + H[V^{-1}(b)].$$

The general solution to this Pexider equation is well known to be

$$H[V^{-1}(u)] = Au + B, \quad J[V^{-1}(u)] = Au + 2B.$$

Letting z vary again, we have

$$\begin{aligned} H[V^{-1}(a)] &= V(z(x-r)+r) \\ &= A(z)V(x) + B(z), \quad z>0 \end{aligned} \tag{32}$$

and

$$\begin{aligned} J[V^{-1}(a)] &= V(z^k(x-s)+s) \\ &= A(z)V(x) + B(z) + B'(z), \quad z>0. \end{aligned} \tag{33}$$

There are now two cases:

1. *A is identically* 1. Then, $V(z(x-r)+r) = V(x) + B(z)$, for all $z>0$. Let $u = x-r$ and $W(x) = V(x+r)$. We obtain

   $$W(zu) = W(u) + B(z).$$

   For positive values of u, this Pexider equation has a unique solution $W(u) = c \ln u + d$, where c, d are constants with $c>0$, and $B(z) = c \ln z$. Therefore, $V(x) = c \ln(x-r) + d$, for $x>r$. If we replace V in (31), we observe that $k = 2$ and $r = s$. Finally, setting $\eta = e^{d/c}$,

   $$F(x,y) = \eta(x-r)(y-r)+r, \quad x, y>r.$$

   It is clear that this solution cannot be extended to accommodate values of x and y smaller than r.
2. *A is not identically* 1. Then, $V(z(x-r)+r) = A(z)V(x) + B(z)$, for all $z>0$. Consider any x larger than r and introduce the notations $x - r = e^p, z = e^q, W^*(x) = V(e^x + r), A^*(x) = A(e^x)$ and $B^*(x) =$

$B(e^x)$. We obtain

$$W^*(p+q) = A^*(q)W^*(p) + B^*(q), \quad p, q \in \mathbb{R}. \tag{34}$$

Given its definition, A^* cannot be identically 1 because, by assumption, A is not identically 1. Therefore, the unique solution of (34) is given by

$$W^*(p) = \alpha e^{\beta p} + d, \quad p \in \mathbb{R}, \tag{35}$$

with $\alpha, \beta \neq 0$ (Aczél, 1966, p. 150). Hence, $V(x) = \alpha(x-r)^\beta + d$ for all $x > r$. If we replace V in (31), we observe that $d = 0$, $k = 1$ and $r = s$, i.e.

$$V(x) = \alpha(x-r)^\beta, \quad x > r. \tag{36}$$

Similarly, $B(z) = 0$. Let us now extend this solution to values of x and y smaller than r. V being continuous, it is obvious that $V(r) = 0$. Let us set $x = r+1$ in $V(z(x-r)+r) = A(z)V(x)$ and we obtain $V(z+r) = \alpha A(z)$. Hence, $V(z(x-r)+r) = V(z+r)V(x)/\alpha$. Setting $x = r-1$, yields $V(-z+r) = V(z+r)V(r-1)/\alpha = V(r-1)z^\beta = \alpha' z^\beta, \quad z > 0$. This is equivalent to

$$V(x) = \alpha'(r-x)^\beta, \quad x < r, \tag{37}$$

with $\alpha' = V(r-1) < 0$. If x and y are larger than r, then

$$F(x,y) = [(x-r)^\beta + (y-r)^\beta]^{1/\beta} + r.$$

The other cases are treated in the same fashion, taking into account the position of x, y and $x \oplus y$ with respect to r. □

Proof of Corollary 1. By part (iii) of Theorem 1, we know that if $k = 1$, then $V(r) = \alpha(\varepsilon - r)^\beta = 0 = V(\varepsilon)$ and so $r = \varepsilon$. And if $k = 2$, then $V(\varepsilon) = 0 = c \ln[\eta(\varepsilon - r)]$ and so $1 = \eta(\varepsilon - r)$. □

Proof of Proposition 2. First, we prove that the structure is Archimedean. By homogeneity of degree 1, $zx \oplus zy = z(x \oplus y)$. Setting $z = 1/y$ and $\phi(z) = z \oplus 1$, we find

$$x \oplus y = y\phi\left(\frac{x}{y}\right), \quad y \neq 0. \tag{38}$$

Let us set $\phi(1) = a$. Then,

$$x(2) = x \oplus x = xa, \quad x \neq 0. \tag{39}$$

Using repetitively (39), we find that $x(2^n) = xa^n$, $x \neq 0$. By weak positivity, $a > 1$ and $x(2^n) \neq x(2^{n+1})$, for $x \neq 0$. But, $\varepsilon(2^n) = \varepsilon(2^{n+1})$. Thus, 0 is the identity.

Take any x, y, w, z such that $x > y$. By solvability, we know that there are b and c such that $b \oplus y = x$ and $c \oplus w = z$. Because $x > y$, we see that $b > \varepsilon$ and so $b > 0$. Therefore, there is n such that $a^n b > c$. By monotonicity,

$$a^n b \oplus w > c \oplus w = z.$$

By monotonicity, associativity, commutativity and homogeneity,

$$a^n b \oplus a^n y \oplus w = a^n(b \oplus y) \oplus w = a^n x \oplus w > z \oplus a^n y.$$

Because, as was shown after (39), $x(2^n) = xa^n$, we get

$$x(2^n) \oplus w > y(2^n) \oplus z$$

and Archimedeanness is proved.

Since we have an Archimedean joint-receipt preference structure, we know that there is an additive representation V. By the assumption of homogeneity of degree 1, for $z > 0$,

$$V[z(x \oplus y)] = V(zx \oplus zy) = V(zx) + V(zy).$$

Setting $V_z(x) = V(zx)$, we see that

$$V_z(x \oplus y) = V_z(x) + V_z(y).$$

So, both V and V_z are additive representations of $\oplus$, whence, for some function θ,

$$V_z(x) = \theta(z)V(x) = V(zx). \tag{40}$$

So, for $x \geqslant 0$, it is well known that for some $\alpha > 0$ and $\beta > 0$,

$$V(x) = \alpha x^\beta.$$

For $x < 0$, we conclude from (40),

$$V(rx) = z^\beta V(x).$$

Set $x = -1$

$$V(-r) = z^\beta V(-1) = -\alpha' z^\beta,$$

where $\alpha' = V(-1) > 0$. Note that the function V is equivalent to the function V in the proof or Theorem 1. To complete the proof, simply substitute the expressions for $V(x)$ into

$$V(x \oplus y) = V(x) + V(y),$$

taking into account the sign of the three terms. □

Proof of Theorem 2. Because the proof is very similar to the proof of Theorem 1 we only sketch it. Applying translation invariance three times, we obtain the Cauchy equation

$$l(w+z) = l(w) + l(z)$$

whose solution is $l(z) = bz$, with $b > 0$. Then we use expression (30) and we find

$$V^{-1}[V(x+z) + V(y+z)] = bz + V^{-1}[V(x) + V(y)].$$

After some change of notations, we arrive at a Pexider equation admitting two solutions, namely (8) and (9). □

Proof of Corollary 2. By Theorem 2, we know that one of (8) or (9) holds. Expression (9) is immediately eliminated by weak identity and $r = -\varepsilon$ follows readily. □

Proof of Proposition 3. Translation invariance, in this particular case, is written as

$$(z+x)\oplus(z+y) = 2z + (x\oplus y).$$

Setting $y = -z$ and $f(x) = x\oplus 0$, we obtain

$$x\oplus y = f(x-y) + 2y. \quad (41)$$

If we replace y by ε, we observe that

$$x\oplus\varepsilon = x = f(x-\varepsilon) + 2\varepsilon.$$

Hence, $f(x) = x - \varepsilon$ and replacing in (41) yields (10). □

Proof of Proposition 4. Consider any $x, y > \varepsilon$. By Proposition 1, $V(x\oplus y) = V(x) + V(y) = V(x+y+r)$. We can rewrite this functional equation as $V[(x+r)+(y+r)-r] = V[(x+r)-r] + V[(y+r)-r]$. Let $W(x) = V(x-r)$, $x+r = u$ and $y+r = v$. The functional equation becomes $W(u+v) = W(u) + W(v)$. Because V is strictly increasing, so is W, and this well-known Cauchy equation admits only one solution (up to a multiplication by a constant): $W(u) = bu$. Therefore, $V(x-r) = bx$ and $V(x) = b(x+r)$ for any $x > \varepsilon$. To ensure that V is order preserving, b must be strictly positive.

Let us consider the restrictions on r. Because V is order-preserving, we know that

$$0 = V(\varepsilon) \leqslant \limsup_{x\to\varepsilon} V(x) = br + b\varepsilon.$$

This is possible only if $b(r+\varepsilon) \geqslant 0$ which, because $b > 0$, is equivalent to $r + \varepsilon \geqslant 0$.

To prove the second part of this proposition, we just need to apply the same reasoning to $\mathscr{D}_\varepsilon^+$ and $\mathscr{D}_\varepsilon^-$, separately. □

Proof of Theorem 3. By Proposition 4, $V(x) = -ds + dx$ for all losses and $V(x) = br + bx$ for all gains, with d and b positive, $r + \varepsilon \geqslant 0$ and $s - \varepsilon \geqslant 0$. Suppose now that $b(r+\varepsilon) > 0$. Because V is linear except at ε and $\mathscr{D}$ is dense, we can choose x and y such that

$$0 = V(\varepsilon) < V(x) + V(y) < b(r+\varepsilon).$$

Because $V(x\oplus y) = V(x) + V(y)$, we see that $\varepsilon < x\oplus y$. But also

$$V(x\oplus y) = br + b(x\oplus y) < br + b\varepsilon$$

and so, because $b > 0$, $x\oplus y < \varepsilon$, which is a contradiction. So $r = -\varepsilon$. The same reasoning applies to $d(s-\varepsilon)$. Thus, $r = -s = -\varepsilon$. □

Proof of Corollary 3. We just have to solve the functional equation $V(x) + V(y) = V(x+y+r)$ on $\mathscr{D}$, as in Proposition 4. □

Proof of Proposition 5. Setting $y = u - r + \epsilon$ in $x\oplus y \geqslant x + y + r - \epsilon$, we have $x\oplus(u - r + \epsilon) \geqslant x + u$. Because V is additive, $V(x) + V(u - r + \epsilon) \geqslant V(x+u)$, that is

$$V(x+u) - V(x) \leqslant V(u - r + \epsilon). \quad (42)$$

Because $|(x\oplus y) - (x+y+r)| \leqslant \epsilon$ and V is strictly increasing, we know that

$$V(x\oplus y) - V(x+y) \leqslant V(x+y+r+\epsilon) - V(x+y)$$

which, using (42) with $x+y$ for x and $r+\epsilon$ for u, is equivalent to

$$V(x\oplus y) - V(x+y) \leqslant V(2\epsilon).$$

Setting now $y = u - r - \epsilon$ in $x\oplus y \leqslant x + y + r + \epsilon$, we easily follow the same reasoning as above and we come to the conclusion that

$$V(x\oplus y) - V(x+y) \geqslant V(-2\epsilon).$$

Let $s = \frac{V(-2\epsilon)+V(2\epsilon)}{2}$ and $\delta = V(2\epsilon) - s$. Then,

$$|V(x\oplus y) - V(x+y) - s| \leqslant \delta.$$

Let $G(u) = V(u) - s$. Then, $|G(x) + G(y) - G(x+y)| \leqslant \delta$ and it can be proved (Hyers, Isac, & Rassias, 1998, p. 13) that $|G(x) - ax| \leqslant \delta$ or, equivalently, $|V(x) - (ax+s)| \leqslant \delta$. As V is strictly increasing and $V(\varepsilon) = 0$, we easily obtain (16) and (17) with the condition on ε.

Let us consider the following absolute value: $|\varepsilon + r| = |x - x - \varepsilon - r| = |(x\oplus\varepsilon) - (x+\varepsilon+r)| \leqslant \epsilon$. Because $|V(x\oplus y) - V(x+y) - s| \leqslant \delta$, using (16) and (17), we find that $|(x\oplus y) - (x+y)|$ is necessarily smaller than $(3\delta + s)/a$. If $\epsilon = \sup_{x,y} |(x\oplus y) - (x+y+r)|$, then we find that $\epsilon \leqslant (3\delta + s)/a$. □

Proof of Theorem 4. The proof is very similar to that of Theorem 1. The main difference lies at the beginning. Instead of using continuity, monotonicity and associativity to derive the existence of an additive representation (30), we directly use the non-commutative representation. That is, combining (29) and (19), we obtain

$$z^k(U^{-1}[\rho U(x) + U(y)] - s) + s$$
$$= U^{-1}[\rho U(z[x-r]+r) + U(z[y-r]+r)]$$

and we go on as in Theorem 1. □

Proof of Theorem 5. The proof is very similar to those of Theorems 2 and 4. □

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References

Aczél, J. (1966). *Lectures on functional equations and their applications*. New York: Academic Press.

Aczél, J., Luce, R. D., & Ng, C. T. (2003). Functional equations arising in a theory of rank dependence and homogeneous joint receipt. Journal of Mathematical Psychology, in press.

Hyers, D. H., Isac, G., & Rassias, T. M. (1998). *Stability of functional equations in several variables*. Basel: Birkhäuser.

Luce, R. D. (2000). *Utility of gains and losses: measurement-theoretical and experimental approaches*. Mahwah, NJ: Lawrence Erlbaum.

Luce, R. D. (2002). A psychophysical theory of intensity proportions, joint presentations and matches. *Psychological Review*, *109*, 520–532.

Ng, C. T., Luce, R. D., & Aczél, J. (2002). Functional characterizations of basic properties of utility representations. *Monatshefte für Mathematik*, *135*, 305–319.

Roberts, F. S., & Luce, R. D. (1968). Axiomatic thermodynamics and extensive measurement. *Synthese*, *18*, 311–326.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk Uncertainty*, *5*, 204–217.