2 Rationality in Choice Under Certainty and Uncertainty

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ABSTRACT

Since the time of Savage (1954) it has been accepted that subjective expected utility (SEU) embodies the concept of rational individual behavior under uncertainty. If, however, one alters the domain formulation in two ways, by distinguishing gains from losses and by adding a binary operation of joint receipt, then equally rational arguments lead in the case of binary mixed gambles to predictions quite different from those of SEU. A question, raised but not really answered, is whether there is a rational argument for choosing one domain formulation over the other.

CONTENTS

The Conventional Wisdom: SEU 65
- SEU Implies Accounting Rationality 66
- SEU Implies Preference Rationality 67
- These Properties Go a Long Way Toward Implying Binary SEU 68

A Peculiarity of the Conventional Wisdom 69

Additional Primitives 70
- Status Quo, Gains, and Losses 70
- Joint Receipt 70
- Linking Joint Receipt and Gambles of Gains: Segregation 74

Nonbilinear Utility of Mixed Gambles 76
- Utility of Mixed Joint Receipt 76
- Linking Joint Receipt and Gambles: General Segregation 76

This research was supported in part by National Science Foundation Grants SBR-9520107 and SBR-9808057 to the University of California, Irvine. I thank Ward Edwards and an anonymous referee for useful, if sometimes challenging, comments.

Linking Joint Receipt and Gambles: Duplex Decomposition 78
Six Related Observations 79

The Conventional Wisdom: SEU

Among those who study individual decisions under risk or uncertainty, fairly wide agreement seems to exist that a rational person will abide by expected utility (EU) under risk and subjective expected utility (SEU) under uncertainty. Various axiomatizations of EU and SEU can be found that are all based on behavioral properties that are viewed as locally rational. Probably the one mentioned most often is Savage (1954). In one way or another, the representation arrived at is as follows. Suppose that $x_i, i = 1, \dots, k$, are sure consequences and $\{E_1, \dots, E_k\}$ forms an event partition of a universal set E arising from some chance experiment **E**; then a gamble – which is brief for an uncertain alternative – can be represented as vector $(x_1, E_1; \dots; x_k, E_k)$ of event–outcome pairs $(x_i, E_i), i = 1, \dots, k$. This is interpreted to mean that the consequence is x_i if, when the experiment is run, the outcome lies in E_i. It is shown that preferences among such gambles can be represented by an order-preserving utility function U over consequences and gambles and a finitely additive probability function S over events such that

$$U(x_1, E_1; \dots; x_k, E_k) = \sum_{i=1}^{k} U(x_i)S(E_i) \tag{1}$$

is order preserving of preferences.

A good place to see how widely this model is accepted as the normative rule for rational behavior is Edwards (1992). The model involves a number of locally rational behavioral features that I describe in this section. *Locally rational* simply means that the behavioral assertion meets some criterion of rationality. It is to be distinguished from the *global rationality* that some people associate with the EU or SEU representations and all of their behavioral implications.

Locally rational properties, which are of two major types (given in the following two subsections), are normative for decision making in much the same way that principles of logic are normative for reasoning and the postulates of probability theory are for characterizing risk.

SEU Implies Accounting Rationality

Because of its bilinear form, Eq. (1) has some strong behavioral implications. One example occurs when we look at compound gambles in which one or more of the x_i are themselves gambles whose underlying experiments are independent of **E**; then one can show for Eq. (1) that all that matters in evaluating the compound gamble is the set of certain consequences plus the listing of chance event combinations needed to give rise to each. Moreover, the order in which the events occur is immaterial. This is a form of unbounded rationality against which Simon (1956) warned.

I refer to assumptions of this type as *accounting indifferences* because they involve alternative formulations of situations that have the same "bottom line" for the decision maker, and so a rational person should be indifferent among the several formulations.

To keep the issues as focused as possible, let us confine attention to binary gambles, that is, those with just two consequences. For such a binary gamble g, let $C = E_1, \bar{C} = E \backslash C$, that is, all elements of E that are not in C, and $x_1 = x, x_2 = y$, so $g = (x, C; y, \bar{C})$. We also admit the possibility of one level of compound gambles. So, for example, if f and h are gambles, then so is $(f, C; h, \bar{C})$. We assume there is preference order $\succsim$ over such gambles, and preference indifference is denoted by $\sim$.

A few important special binary cases make the idea of accounting indifferences clear.

Some Elementary Accounting Equivalences

Idempotence:

$$(f, C; f, \bar{C}) \sim f.$$

This simply says that if f occurs no matter what the outcome of experiment **E** is, then the gamble is indifferent to f.

Certainty:

$$(f, E; h, \emptyset) \sim f,$$

where $\theta = \bar{E} = E \backslash E$. In this case, the gamble gives rise to f with certainty, so the gamble is indifferent to f.

Complementarity:

$$(f, C; g, \bar{C}) \sim (g, \bar{C}; f, C).$$

Here the order in which the outcome–event pairs are written is immaterial.

These are viewed as so simple and transparent that they are little discussed in the literature. A slightly more complex accounting indifference involves a first-order compounding of binary gambles.

Event Commutativity. For all consequences x, y, independently run experiments **E**, **F**, and events $C \subseteq E$, and $D \subseteq F$,

$$((x, C; y, \bar{C}), D; y, \bar{D}) \sim ((x, D; y, \bar{D}), C; y, \bar{C}). \qquad (2)$$

The argument for the rationality of event commutativity is that one receives x if both C and D occur, which in the formulation to the left of $\sim$ happens with the events arising in the order D, C and on the right side in the order C, D; otherwise, the consequence is y.

SEU Implies Preference Rationality

In addition to accounting indifferences, SEU corresponds to certain important patterns of preference that all seem highly rational in an intuitive sense. Recall that $f \succsim h$ symbolizes that gamble f is preferred or indifferent to gamble h.

Transitivity. For all gambles f, g, and h,

$$f \succsim g \quad \text{and} \quad g \succsim h \Longrightarrow f \succsim h.$$

Here one can think of replacing g in the preference inequality $g \succsim h$ by f, which is preferred to g, and so f should also be preferred to h. Some, most notably Fishburn (1982), have questioned the rationality of transitivity, but for the most part, decision analysts agree that a rational decision maker should be transitive. For strict preference, one argument is that an intransitive person can be made into a money pump in the following sense. Suppose g is owned and f is preferred to g; then for some sufficiently small amount of money $\varepsilon > 0$, the decision maker will trade ε and g to get f. But with an intransitive triple, money is extracted from each transaction, presumably, indefinitely or until the decision maker revises his or her preferences so as to be transitive. The most striking violation of transitivity, known as the *preference reversal phenomenon*, has been shown to arise from mixing judgment and choice procedures. This means not a violation of transitivity so much as the need for distinct theories of choice and judgment (see Luce, 2000, pp. 39–44).

Consequence Monotonicity. For all non-null events $C, \bar{C}$ and gambles f, g, h, then,

$$g \succsim h \Longleftrightarrow (f, C; g, \bar{C}) \succsim (f, C; h, \bar{C}).$$

Both consequence monotonicity and transitivity are examples of a basic principle of rationality, namely, that it is desirable to replace something by something else that the decision maker prefers when all else is fixed.

Event Montonicity. For all experiments **E**, events $A, B, C \subseteq E$ with $A \cap C = B \cap C = \theta$, and gambles f, g, with $f \succ g$,

$$(f, A; g) \succsim (f, B; g) \Longleftrightarrow (f, A \cup C; g) \succsim (f, B \cup C; g). \tag{3}$$

De Finetti (1931) first formulated this condition (called, in translation, *additivity in qualitative probability*) as a central feature of his qualitative probability theory (see Fine, 1973, and Fishburn, 1986). The rationality lying behind event monotonicity is clear. The decision maker prefers the gamble that makes the preferred gamble f more likely than g to be the consequence, and so we infer from $(f, A; g) \succsim (f, B; g)$ that A is perceived to be at least as likely to occur as B. But if that is true, then augmenting both events by the same disjoint C should maintain that likelihood order, and so the preference on the right should hold (Luce & Marley, 2000; Marley & Luce, 2002).

These Properties Go a Long Way Toward Implying Binary SEU

Although I do not state a precise theorem for deriving the binary SEU representation, the locally rational properties I have listed – idempotence, certainty, complementarity, event commutativity, transitivity, consequence monotonicity, and event monotonicity – play a very crucial role.

As would be expected, there has been extensive discussion of the degree to which these properties, and so SEU, is an adequate description of behavior. I do not attempt to describe the details of these somewhat complex, arcane, and controversial debates. My conclusion, based on a careful examination of the literature, is that a major source of descriptive failure in the binary case lies in two places: event monotonicity (Ellsberg, 1961) and the assumption of unlimited accounting indifferences (Luce, 1992, 2000).

Indeed, with respect to the latter, there seems to be a sharp separation between event commutativity, which seems to hold descriptively, and a slightly more complex accounting equivalence called *autodistributivity*, which does not seem to hold. This boundary corresponds to the distinction between SEU and what is called *rank-dependent utility* (RDU) (Luce & Fishburn, 1991, 1995; Luce & von Winterfeldt, 1994, appendix; Quiggin, 1993; Tversky & Kahneman, 1992). In the latter model the binary SEU expression is modified in two ways: The weights are not probabilities (i.e., are not finitely additive), and the weighted utility representation of $(x, C; y, \bar{C})$ depends on whether $x \succsim y$ or $x \prec y$. Adding the autodistributivity property forces RDU to become SEU. Thus, RDU exhibits a form of bounded rationality of the sort urged by Simon (1956).

Still, RDU, as typically stated for gambles with three or more alternatives, does not appear to be an adequate descriptive theory. Work by, among others, Birnbaum and Chavez (1997), Birnbaum and Navarrete (1998), and Wu (1994) makes it clear that RDU is not descriptive beyond the binary case (see Luce, 2000). And so those who wish to develop descriptive theories have additional work to do. But that is not my issue in this chapter.

A Peculiarity of the Conventional Wisdom

Everyone dances around an oddity of conventional SEU. That theory simply does not distinguish between what are perceived as incremental gains and as losses. One tack is to say that one decides among choices not in terms of the alternatives, as one usually thinks of them, but rather in terms of one's total asset position. In choosing among risky alternatives, such as stocks, one supposedly thinks not about the alternatives, but about one's total financial situation in each case.

A second tack is to say that the utility function is shaped differently for gains and for losses. Often utility is assumed to be concave for monetary gains and convex for losses. However, this way of treating it seems very artificial within the context of standard SEU theory because the representation is invariant under positive linear transformations of utility, and so the kink can occur at any numerical level. That means, for all but one choice of utility function, either that the utility of some gains is assigned negative values or that the utility of some losses is assigned positive ones. That seems counterintuitive.

When providing advice to decision makers, it must take skill to convince them to ignore the distinction between gains and losses. And I know of no experiment that is presented in terms other than incremental gains and losses.

From a descriptive point of view, there is something deeply wrong with SEU even in the binary case when applied to gambles with mixed gains and losses. Attempts to apply utility ideas in practical decision-making situations, including certain business decisions and medical diagnosis and treatment where there are decided gains and losses, have encountered inconsistencies of estimates of the utilities for the domains of gains, losses, and mixed gambles (Fishburn & Kochenberger, 1979; Hershey, Kunreuther, & Schoemaker, 1982; von Winterfeldt & Edwards, 1986).

These failures go far beyond those involving complex accounting equivalences, and they raise the question of whether SEU really does capture what most people mean by rationality. Serious reevaluation seems called for.

Additional Primitives

To clearly distinguish gains from losses, one needs to introduce certain new primitives that have not been a part of the traditional axiomatization.

Status Quo, Gains, and Losses

The first primitive is the concept of *no change from the status quo,* which I will abbreviate to *status quo.* One is tempted to call this the *zero consequence,* but that too readily suggests that the consequences are money, and the model is far more general than that. I use the traditional mathematical symbol, e, of abstract algebra for an identity element, which is what no change from the status quo turns out to be in a well-defined sense (see Axiom JR5 later). The structure that we provide forces $U(e) = 0$, and so the representation is no longer unique up to positive linear transformations but only up to positive similarity ones; that is, U and U' are utility functions for the same situation if and only if for some constant $\alpha > 0$, $U' = \alpha U$.

Any consequence or gamble with $g \succ e$ is said to be perceived as a *gain* and any with $g \prec e$ as a *loss*. Thus, the utility of a gain is always a positive number and that of a loss is always a negative number.

Joint Receipt

The second additional primitive is what I call *joint receipt*. Mathematically, it is simply a binary operation $\oplus$ over consequences and gambles. The behavioral interpretation of $f \oplus g$ is that one receives both f and

g. So, for example, in the gamble $(f \oplus g, C; h, \bar{C})$ the interpretation is that if event C occurs, the consequence is the receipt of both f and g, whereas if $\bar{C}$ occurs, the consequence is the singleton h. Clearly, joint receipt is a common occurrence of everyday life. Indeed, more often than not, one deals simultaneously, or nearly so, with two or more goods, two or more bads, or mixes of both goods and bads. As we shall see, the general case of the joint receipt of more than two entities devolves to the binary one under the assumptions made.

The traditional way to deal with joint receipts in economics has been as vectors called *commodity bundles*. That approach strikes me as somewhat more artificial than treating it as an operation. Which formulation is used matters because they lead to quite different mathematical structures – vector spaces versus ordered algebras.

Once one considers such an operation, it is clear that it can be studied without reference to uncertainty. It looks very much like a classical measurement situation from physics. The relevant structure is $\mathfrak{D} = \langle \mathcal{D}, e, \succsim, \oplus \rangle$, where $\mathcal{D}$ is the domain of entities under consideration, which may or may not include gambles; e in $\mathcal{D}$ is the status quo; $\succsim$ is a preference order over $\mathcal{D}$; and $\oplus$ is a binary operation over $\mathcal{D}$. This looks a great deal like the kinds of structures that arose early on in physical measurement, for example, of mass except for the fact that we have elements of $\mathcal{D}$ both better than and worse than e. Nonetheless, the situation is very similar.

Additive Representation of Joint Receipt. The question is, what properties are reasonable to assume for $\mathfrak{D}$, and where do they lead in terms of a representation? We state the latter first. There will be sufficient axioms to prove the existence of $V: \mathcal{D} \to \mathbb{R}$, that is the set of real numbers, such that for all f, g in $\mathcal{D}$

$$f \succsim g \Longleftrightarrow V(f) \geq V(g), \tag{4}$$

$$V(f \oplus g) = V(f) + V(g), \tag{5}$$

$$V(e) = 0. \tag{6}$$

Because of Eq. (5), this representation is called *additive* over $\oplus$.

It is usually dismissed as a theory of value as soon as it is stated. Among the arguments given are the following:

- *Complementary goods:* The claim is that some goods have value only together. Examples are shoes in pairs (for people with both feet), guns and appropriate bullets, pairs of earrings (at least prior to about 1968), and so on. This is easily bypassed simply by defining the elementary goods as consisting of those groupings that are relevant to the decision maker, which, of course, is what stores usually do. Note, however, that what one considers as unitary may differ whether one is a buyer or seller or even by the type of seller. For example, an automobile dealer typically treats vehicles as unitary goods, whereas a repair supply shop deals with parts of automobiles such as valves and brake drums.

- *Incompatible goods:* For well-known practical reasons, some valued goods should not be placed in close proximity if, for example, an explosion is to be avoided. This is no less a problem in mass measurement, and obvious precautions are taken.
- *Leads to utility that is proportional to money:* Consider the domain of money consequences. If one supposes that $x \oplus y = x + y$ and that V maps onto a real interval, then it is easy to show from Eqs. (4) and (5) that for some constant $\alpha > 0$, $V(x) = \alpha x$. But everyone knows that utility of money is not linear with money (e.g., the St. Petersburg paradox). This argument has two weaknesses. First, it is not at all clear that $x \oplus y = x + y$ is actually correct (Luce, 2000; Thaler, 1985). Second, and more important, no reason has been provided to suppose that V is actually proportional to U, where U is the utility function determined from gambles. This is explored more fully in the next section of this chapter.

Axioms Underlying the Additive Representation. We turn now to the assumptions that are well known to give rise to this representation (Hölder, 1901; Krantz, Luce, Suppes, & Tversky, 1971). I will simply list them and then discuss them from a rational perspective.

Definition. $\langle \mathcal{D}, e, \succsim, \oplus \rangle$ is said to form a *joint-receipt preference structure* if and only if the following five conditions holds for all f, g, h in $\mathcal{D}$:

Axiom JR1. Weak Order:

$\succsim$ is transitive and connected.

Axiom JR2. Weak Commutativity:

$$f \oplus g \sim g \oplus f.$$

Axiom JR3. Weak Associativity:

$$f \oplus (g \oplus h) \sim (f \oplus g) \oplus h.$$

Axiom JR4. Weak Monotonicity:

$$f \succsim g \Longleftrightarrow f \oplus h \succsim g \oplus h.$$

Axiom JR5. Weak Identity:

$$f \oplus e \sim f.$$

The adjective *weak* in each case refers to the fact that the property holds for indifference, $\sim$, not just for equality, $=$. (Given axiom JR1, we know that $\sim$ is an equivalence relation, and so if one works with the equivalence classes of $\sim$, all of these become transformed into the usual conditions involving $=$ rather than $\sim$.)

The rationality of weak transitivity is unchanged from the gambling context. The rationalities of weak commutativity and associativity are species of accounting indifference. Both simply require that neither the order in which joint receipt is written nor the grouping into binary pairs is material as far as preference is concerned. It is important not to misinterpret what this means. Preference for the goods can easily be confounded by other considerations. For example, suppose that you receive a shipment of three goods, x, y, z, but when you open them you find that x' is included instead of the x you ordered. Suppose further that they were packed in two cartons. Do you care if x' is in its own carton and y and z are in another or that x' and y are in one carton and z is in a separate one? Most of us do, because with x' in a separate container, it is easier to return it in exchange for x. This preference concerns not how you feel about the goods as such, but convenience in dealing with the error.

Weak monotonicity has the same compelling rational flavor it always has. However, the following example has been suggested (in a review of one of my grant proposals) as casting doubt upon it. Suppose that g is a lottery and x is a modest sum of money such that $x \succ g$ but $x < EV(g)$, where EV denotes expected value. Now suppose that z is a very large sum of money – \$1 million (\$1M) for most people will do. The claim is that $x \oplus z$ may be less preferred than $g \oplus z$ because with \$1M in the bank, one need not be so risk averse as without it. I am not certain that people change their stance about gambles so easily, nor from a rational perspective do I see why they should.

Weak identity simply asserts that adding no change from the status quo to any valued alternative has no impact on the preference pattern. This has more to do with the meaning of terms than it does with anything that is empirically testable.

There are two additional axioms of a different character. The first is an Archimedean one that can be stated as follows. For any gamble f and any integer n, let $f(n)$ denote the joint receipt of n copies of f. Formally,

$$f(1) = f,$$

$$f(n) = f(n-1) \oplus f.$$

Axiom JR6. Archimedean. If $f \succ e$ and any g, there exists an integer n such that

$$f(n) \succ g.$$

This property says that enough of any one gain exceeds in preference any other gain (and certainly any loss). For this to make sense, one has to have trading of some sort as an implicit background postulate because most of us do not care to have huge amounts of one good.

The next axiom is a structural one that says that for each gain there is a compensating loss and for each loss there is a compensating gain. This assumption no doubt limits the applicability of the theory. For example, many of us think that there are personal disasters, among them death or total incapacitation, for which no gain can compensate. Accepting that limitation, we assume:

Axiom JR7. Inverse. For each $f \in \mathcal{D}$, there exists $f^{-1} \in \mathcal{D}$ such that

$$f \oplus f^{-1} \sim e. \qquad (7)$$

These seven conditions are well known to be sufficient to yield the additive representation stated in Eqs. (4) to (6), and that representation implies that Axioms JR1 to JR6 must hold (Hölder, 1901).

Linking Joint Receipt and Gambles of Gains: Segregation

Now, given that we have some idea of how to derive utility U from gambles and value V from joint receipt, an immediate question is: How do these two structures relate, and in particular, how does U relate to V? Clearly, over the same domain, they must be strictly increasing functions of each other because they both preserve the same preference order. The issue is what other restrictions exist.

We continue to confine ourselves to gains for the moment. For this purpose, let $\mathcal{B}$ denote the set of binary gambles closed under joint receipt, and let $\mathcal{B}^+ \subset \mathcal{B}$ be such that $g \in \mathcal{B}^+$ if and only if both $g \in \mathcal{B}$ and $g \succsim e$. Consider the following possible link between the structures:

Definition. (Binary) segregation is said to hold if and only if for all f, g in B^+,

$$(f \oplus g, C; g, \bar{C}) \sim (f, C; e, \bar{C}) \oplus g. \tag{8}$$

The gamble to the left of $\sim$ yields $f \oplus g$ when the event C occurs and g otherwise, and the joint receipt on the right side yields g as well as f when C occurs and e otherwise. So, in reality, they are two different ways of expressing exactly the same situation, and it seems as locally rational as any of the other accounting indifferences we have encountered.

Segregation, weak commutativity, binary RDU (which, of course, includes binary SEU for gains as a special case), and assuming the utility function is onto an interval and the weighting function is onto $[0, 1]$, are sufficient to show that for some real constant δ (with the dimension $1/U$)

$$U(f \oplus g) = U(f) + U(g) - \delta U(f)U(g). \tag{9}$$

From this it follows that joint receipt has an additive representation V of the form shown in Eq. (5). We call V a *value function.* In fact, Eq. (9) is known to be the only polynomial form in the two variables $U(f)$ and $U(g)$ with $U(e) = 0$ that can be transformed into an additive one, and for that reason, I call this representation *p-additive.*

Depending on the sign of δ, there are three distinct relations between U and V:

$$\delta = 0 \Longrightarrow \text{for some } \alpha > 0, \quad U = \alpha V, \tag{10}$$

$$\delta > 0 \Longrightarrow \text{for some } \kappa > 0, \quad |\delta|\, U = 1 - e^{-\kappa V}, \tag{11}$$

$$\delta < 0 \Longrightarrow \text{for some } \kappa > 0, \quad |\delta|\, U = e^{\kappa V} - 1. \tag{12}$$

Note that Eq. (11) is necessarily concave and bounded in V, whereas Eq. (12) is convex and unbounded in V.

A similar development holds for all losses, but with different constants, $\alpha' > 0$, $\delta' > 0$, and $\kappa' > 0$. For simplicity, and with very little real loss of generality, we will assume that $\kappa' = \kappa$.

In what follows, it is simpler to state things in terms of

$$U_+(x) = |\delta|\, U(x), \qquad x \succsim e, \tag{13}$$

$$U_-(x) = |\delta'|\, U(x), \qquad x \prec e. \tag{14}$$

Nonbilinear Utility of Mixed Gambles

Utility of Mixed Joint Receipt

I will not give the full details for the mixed case, which can be found in Luce (1997, 2000), but only for the basic ideas. We assume that V is additive throughout the entire domain of alternatives.

Consider mixed joint receipts. The results are described in detail only for the case where the joint receipt is seen as a gain, that is, $f_+ \succ e \succ g_-$ and $f_+ \oplus g_- \succsim e$. When U is proportional to V we have

$$U(f_+ \oplus g_-) = U(f_+) + \frac{\alpha}{\alpha'} U(g_-).$$

The more interesting cases are the two exponential cases, Eqs. (11) and (12). What is surprising about them is that from the additive representation of V and the p-additive representation of U over gains and separately over losses, we find that the U representation of mixed joint receipt is nonlinear. The key to deriving this is the very well known, indeed defining, property of exponentials, namely, $e^{V(x \oplus y)} = e^{V(x)+V(y)} = e^{V(x)} e^{V(y)}$. Taking into account the sign of V and Eqs. (11) and (12), one can show for concave gains that

$$U_+(f_+ \oplus g_-) = \frac{U_+(f_+) - U_+(g_-^{-1})}{1 - U_+(g_-^{-1})}. \tag{15}$$

When U_+ is convex in V for gains, the formula is the same except that the minus sign in the denominator is changed to a plus sign. Somewhat similar formulas arise when the joint receipt is seen as a loss (see Luce, 2000, pp. 240–241).

So, to this point, we see that locally rational assumptions about gambles and joint receipt lead to an unanticipated nonlinearity. The question is: What does this imply about the utility of mixed gambles? Recall that this is where the traditional bilinear theories seem to have encountered serious descriptive difficulties and their normative use has proved somewhat unsatisfactory.

Linking Joint Receipt and Gambles: General Segregation

The question now becomes: Given that we have the representation of the utility of mixed joint receipts, what can we say about the utility of mixed gambles? The most obvious, locally rational link between the two structures is a generalization of segregation. To that end, define the "subtraction" corresponding to $\oplus$:

$$f \ominus g \sim h \iff f \sim g \oplus h.$$

We assume that the element h exists, which for money alternatives is no issue.

Definition. General segregation holds if for binary gambles f, g, with $f \succsim g$, and underlying experiment **E** with event C,

$$(f, C; g, \bar{C}) \sim \begin{cases} (f \ominus g, C; e, \bar{C}) \oplus g, & \text{if } (f, C; g, \bar{C}) \succ e \\ (e, C; g \ominus f, \bar{C}) \oplus f, & \text{if } (f, C; g, \bar{C}) \prec e \end{cases}. \tag{16}$$

As with segregation, this condition is entirely rational, provided that one is willing to accept the idea that one simplifies the given gamble differently, depending on whether it is seen as a gain or a loss.

We assume both general segregation and a separable representation in the following sense. There are weighting functions $W_{\mathbf{E}}^{+}$ and $W_{\mathbf{E}}^{-}$ and a utility function U such that for gambles of the form $(f_+, C; e, \bar{C})$, $U W_{\mathbf{E}}^{+}$ forms a separable representation in the sense that

$$(f_+, C; e, \bar{C}) \succsim (g_+, D; e, \bar{D}) \iff U(f_+) W_{\mathbf{E}}^{+}(C) \geq U(g_+) W_{\mathbf{E}}^{+}(D),$$

and for gambles of the form $(e, C; g_-, \bar{C})$, $U W_{\mathbf{E}}^{-}$ forms a separable representation. Under these conditions and dropping the **E** subscript on the W's, we have the following conclusions:

- If U is proportional to V, then for $(f_+, C; g_-, \bar{C}) \succ e$,

  $$U_+(f_+, C; g_-, \bar{C}) = U_+(f_+) W^+(C) + \frac{\alpha}{\alpha'} U_-(g_-)[1 - W^+(C)]; \tag{17}$$

  and for $(f_+, C; g_-, \bar{C}) \prec e$,

  $$U_-(f_+, C; g_-, \bar{C}) = \frac{\alpha'}{\alpha} U_+(f_+)[1 - W^-(\bar{C})] + U_-(g_-) W^-(\bar{C}). \tag{18}$$

  This, of course, is bilinear and so consistent with SEU.
- Suppose that U is exponentially related to V and is concave for gains and convex for losses (for the convex/concave case the signs are in parentheses). Then for $f_+ \succsim e \succsim g_-$ with $(f_+, C; g_-, \bar{C}) \succ e$,

  $$U_+(f_+, C; g_-, \bar{C}) = U_+(f_+) W^+(C) + \frac{U_-(g_-)}{1 + (-)U_-(g_-)}[1 - W^+(C)] \tag{19}$$

and for $(f_+, C; g_-, \bar{C}) \prec e$,

$$U_-(f_+, C; g_-) = \frac{U_+(f_+)}{1 - (+)U_+(f_+)}[1 - W^-(\bar{C})] + U_-(g_-)W^-(\bar{C}). \tag{20}$$

- Suppose that U in terms of V is either concave for both gains and losses or convex for both. Then for $(f_+, C; g_-, \bar{C}) \succ e$,

$$U_+(f_+, C; g_-, \bar{C}) = U_+(f_+)W^+(C) + U_-(g_-)[1 - W^+(C)] \tag{21}$$

and for $(f_+, C; g_-, \bar{C}) \prec e$,

$$U_-(f_+, C; g_-, \bar{C}) = U_+(f_+)[1 - W^-(\bar{C})] + U_-(g_-)W^-(\bar{C}). \tag{22}$$

The last case, like the proportional one, is bilinear, similar to SEU. The most interesting case is the middle one, Eqs. (19) and (20), where the utility function is concave in one region and convex in the other. This leads to a nonbilinear form different from that of SEU. What happens is that for a gamble perceived as a gain, the weight assigned to the loss term is increased (decreased) by an amount dependent on the utility of the loss in the concave/convex (convex/concave) case. When the gamble is seen as a loss, the change is in the weight assigned to the gain term; it is increased (decreased) in the concave/convex (convex/concave) case.

Linking Joint Receipt and Gambles: Duplex Decomposition

Luce (1997) and Luce and Fishburn (1991, 1995) have studied another link – a nonrational one – between joint receipt and gambles that is called *duplex decomposition*. It first appeared in the empirical work of Slovic and Lichtenstein (1968). It asserts that for all f_+ a gain, g_- a loss, and C and event of experiment **E**

$$(f_+, C; g_-, \bar{C}) \sim (f_+, C'; e, \bar{C}') \oplus (e, C''; g_-, \bar{C}''), \tag{23}$$

where, on the right side of $\sim$, events C' and C'' mean occurrences of event C arising from two independent realizations of gambles.

There were three motives originally for studying duplex decomposition: its vague plausibility despite its nonrational character; the empirical data of Slovic and Lichtenstein (1968), which suggested it in the first place; and the mathematical fact that under certain assumptions it leads to the bilinear form of Kahneman and Tversky's (1979) and Tversky and Kahneman's (1992) cumulative prospect theory. Within the context of an additive value representation, it leads to a somewhat different nonbilinear form than does general segregation for the case of U concave with V in one region and convex in the other. For $(f_+, C; g_-, \bar{C}) \succsim e$, then for the case of concave gains and convex losses (convex gains and convex losses)

$$U_+(f_+, C; g_-, \bar{C}) = \frac{U_+(f_+)W^+(C) + U_-(g_-)W^-(\bar{C})}{1 + (-)U_-(g_-)W^-(\bar{C})}. \tag{24}$$

The result for $(f_+, C; g_-, \bar{C}) \prec e$ is similar with $1 - (+)U_+(f_+)W^+(C)$ in the denominator.

The net effect is to take the standard expression for cumulative prospect theory (CPT) and modify it. If the gamble is seen as a gain, then the CPT form is increased (decreased) in the case of concave gains and convex losses (convex/concave), whereas for a perceived loss it is decreased (increased).

For some purposes, this form may be more descriptive than that derived from general segregation. For example, Sneddon and Luce (2001) used certainty equivalence data from 144 respondents of Cho, Luce, and von Winterfeldt (1994) to compare the models. Using a minimum least squares measure, they found that the additive V models, where $\oplus$ is associative throughout the domain, do better than the additive U models (basically, CPT) for about 84% of the respondents; that the nonbilinear fits are better than the bilinear ones for about 67% of the respondents; and that the nonrational duplex decomposition is favored over the rational general segregation by about 3 to 1.

But my concern here is with rationality, not description, so I will not pursue duplex decomposition further. More details may be found in my monograph (Luce, 2000).

Six Related Observations

1. Adding the concepts of (no change from) the status quo and joint receipt has at least two happy features. First, they capture some very common features of decision situations, namely, the distinction between gains and losses and the simple fact that consequences and alternatives often involve several unitary, valued entities. Second, they lead to a simple theory of how to evaluate preferences among certain consequences that is independent of our descriptions of behavior under uncertainty. Note, however, the cautious "how to evaluate preferences" rather than

"the utility of preferences" because one may not automatically assume that the value function V arising from joint receipts is necessarily proportional to the utility function U arising from binary gambles of all gains and all losses. The relation between U and V must be investigated, which is done by discovering a qualitative behavioral connection linking the gambling structure to the joint receipt one. The one proposed, called *segregation*, is a type of accounting rationality. Moreover, if the utility function U is rank dependent (including SEU as a special case), then U is one of three functions of V: proportional, negative exponential, or exponential.

2. Of course, the classical SEU theory applies to the mixed case because it fails to distinguish gains from losses. But making that distinction, which I claim virtually everyone does, and accepting that SEU is a suitable rational theory for binary gambles of gains and separately for binary gambles of losses, then the representation of rational behavior in the mixed case is neither uniquely determined nor necessarily bilinear. For example, assuming that V is additive over joint receipt, then with U concave relative to V for one region and convex for the other one, the predictions of the rational general segregation for the mixed case are not bilinear. This nonbilinearity holds also for the link called *duplex decomposition*, which, although not rational, appears to be somewhat more descriptively accurate than general segregation.
3. This is an odd impasse. At least since the work of Savage (1954), it has been widely held that SEU is *the* rational theory for individual decision making under uncertainty. Now we see that there can be two equally rationally based theories with quite different predictions (or prescriptions) in cases of mixed gains and losses. Or put another way, the union of Savage's assumptions and the rational ones for joint receipt and its relations to gambles are consistent only in two cases. The first is when U is proportional to V, and the weights satisfy both finite additivity and $W^+(C) + W^-(\bar{C}) = 1$; see Eqs. (17) and (18). These requirements are strong. For example, if the domain of sure consequences includes amounts x, y of money and if, as seems rational, $x \oplus y = x + y$, then it is easy to prove that $U(x)$ is proportional to x. Thus, over lotteries of money consequences with known probabilities, this rational case reduces to expected

value, not EU, and over more general gambles it is subjective expected certainty equivalents. The second case is when the same restriction holds for the weights and U is either exponential or negative exponential in V for both gains and losses; see Eqs. (21) and (22). These restrictions on U and W are fairly severe and not descriptive of many people.

4. A question immediately comes to mind: Is there some deeper sense of rationality that permits one to select between the two formulations? Because the major distinction between the approaches is whether or not one distinguishes between gains and losses and whether or not the operation of joint receipt is included among the primitives, the question becomes: Is there a rational argument for either excluding or including the status quo and joint receipts? I know of none. There is a pragmatic argument that says that since almost everyone makes the gain–loss distinction, it probably should not be ignored in a rational theory. And there is also the pragmatic argument that because joint receipt is ubiquitous, it too probably should not be ignored by theorists. But I do not see either of these as rationally compelling, merely plausible. But equally, I do not see any rational or otherwise compelling reason for excluding them.
5. What does this development say to a decision analyst? The discoveries described are too new and insufficiently absorbed and criticized by the field for anyone – even their author – to have full confidence about them. So, at the moment, the only reasonable stance is caution in basing prescriptive recommendations solely on EU or SEU. I would like to see systematic efforts comparing, including decision makers comparatively evaluating, the recommendations from SEU and the present nonbilinear forms for mixed gains and losses, in particular Eqs. (19) and (20). Probably such studies should also include the nonrational, but apparently often descriptive, theory based on duplex decomposition, Eq. (24). Because the estimates for gains and losses separately agree in all versions, the focus is entirely on the mixed gain–loss case, which of course is the case of usual interest.
6. What does it say to the scientist? Perhaps that we simply have to live with the fact that what seems rational depends more on the formulation of the domain of study than we had previously acknowledged. These observations certainly need additional discussion by the field.

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